Reduced Product in Abstract Interpretation

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- Results of one domain can refine the results of another domain.

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- Let $\langle A_1, \sqsubseteq_1 \rangle$ and $\langle A_2, \sqsubseteq_2 \rangle$ be abstract domains with their concretization functions γ_1 and γ_2, respectively. Their **Cartesian product** [1] is $\langle \boldsymbol{A}, \sqsubseteq \rangle$ where:
  - $\boldsymbol{A} = A_1 \times A_2$
  - $\langle p_1, p_2 \rangle \sqsubseteq \langle q_1, q_2 \rangle \iff p_1 \sqsubseteq_1 q_1 \wedge p_2 \sqsubseteq_2 q_2$
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- Such combined abstract domain does not provide more precise results than running the analyses with each abstract domain independently [2].
- The **reduced product** is $\langle \boldsymbol{A}/_{\equiv}, \sqsubseteq \rangle$ where $P \equiv Q \iff \gamma_{\boldsymbol{A}}(P) = \gamma_{\boldsymbol{A}}(Q)$ and $\gamma_{\boldsymbol{A}}$ and $\sqsubseteq$ are extended to the equivalence classes of $\equiv$.

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- Finding the equivalence class of an abstract context can be seen as using a **reduction function** $\sigma : \boldsymbol{A} \rightarrow \boldsymbol{A}$ such that $\sigma(\langle p_1, p_2 \rangle) = \langle \alpha_1(\gamma_{\boldsymbol{A}}(\langle p_1, p_2 \rangle)), \alpha_2(\gamma_{\boldsymbol{A}}(\langle p_1, p_2 \rangle)) \rangle$

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- For example, in a reduced product of interval and parity domain, $\langle [1, 9], \mathit{even}\rangle \equiv \langle [2, 8], \mathit{even}\rangle$:

$$\begin{aligned}
\boldsymbol{\gamma_A}(\langle [1,9], \mathit{even}\rangle) &= \gamma_1([1,9]) \cap \gamma_2(\mathit{even}) \\
&= \{1, 2, \ldots, 9\} \cap \{0, 2, 4, \ldots\} \\
&= \{2, 4, 6, 8\} \\
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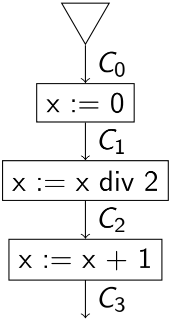
- In practice, analyzers compute an over-approximation of the reduction using some rules (concretization is not feasible).
- Typically, messages are exchanged between domains, each domain implements refinement based on a received message. The message format varies (e.g. various logics).

Full example [3]

- Consider the parity and sign domains.



- $A_1 = \{\perp, odd, even, \top\}$
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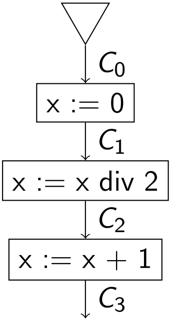

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C_2		
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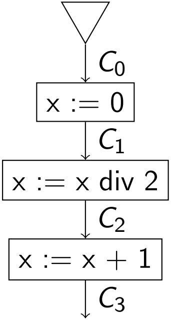

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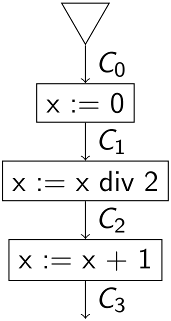

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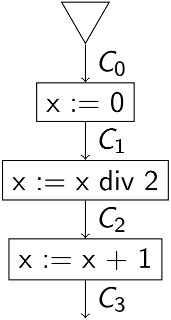

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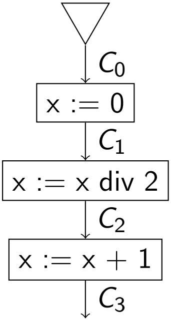

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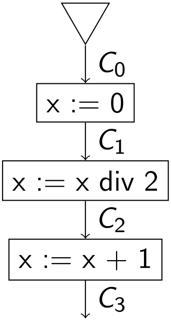

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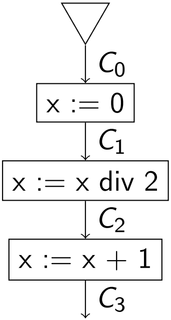

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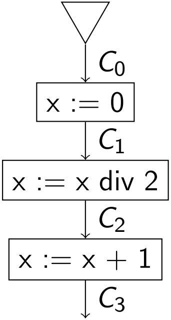

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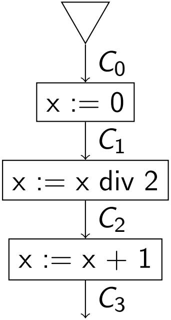

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- Notice that we obtain more information in C_3:
  - $\gamma_{\boldsymbol{A}}(\langle \top, \geq 0 \rangle) = \{0, 1, 2, \ldots\}$
  - $\gamma_{\boldsymbol{A}}(\langle odd, \geq 0 \rangle) = \{1, 3, 5, \ldots\}$
- This was a simple sequential example but such reductions can have a positive effect on widening and narrowing as well.

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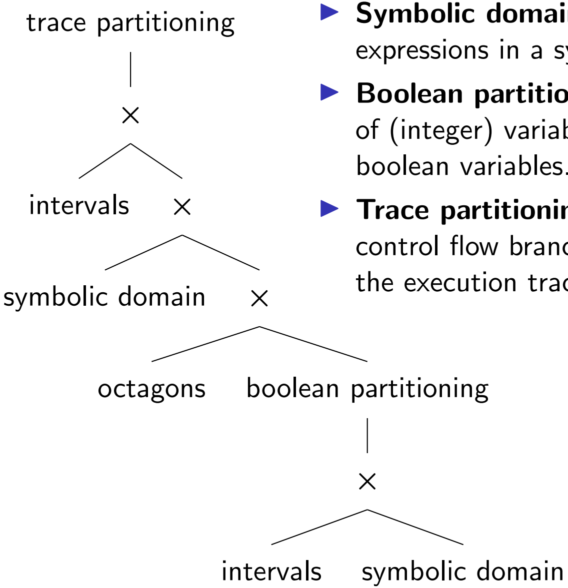
- Alternatively, reductions can be applied in a fixed order, e.g. Astrée [5].

Astrée example hierarchy [5]



- **Symbolic domain** propagates assigned expressions in a symbolic way [6].
- **Boolean partitioning** relates the values of (integer) variables to the values of boolean variables.
- **Trace partitioning** tracks history of control flow branches and values along the execution trace.

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[1] P. Cousot, R. Cousot, and L. Mauborgne, "The reduced product of abstract domains and the combination of decision procedures,", Mar. 2011, pp. 456–472, ISBN: 978-3-642-19804-5. DOI: 10.1007/978-3-642-19805-2_31.

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