DFT Discrete F.T.

N samples of signal $x[n]$ ⟶ N samples of spectrum $X[k]$

output = maybe constant · sum. op. · input · $e^{\pm j\,\text{freq.}\cdot\text{time}}$

DFT "analysis"

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

fundamental freq.

IDFT "synthesis"

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+jk\frac{2\pi}{N}n}$$

k: 0 … k … N-1

k	0 … N	
$f = k\frac{F_s}{N}$	F_s	standard frequency [Hz]
$f = \frac{k}{N}$	1	normalized fr [ ]
$\omega = k\frac{2\pi}{N}$	2π	normalized angular freq. [rad]
$\omega = k\frac{2\pi F_s}{N}$	$2\pi F_s$	standard angular freq. [rad/s]

properties of DFT:

$$\cancel{X[k] = X^*[-k]} \qquad X[k] = X^*[N-k]$$

circular shift of signal

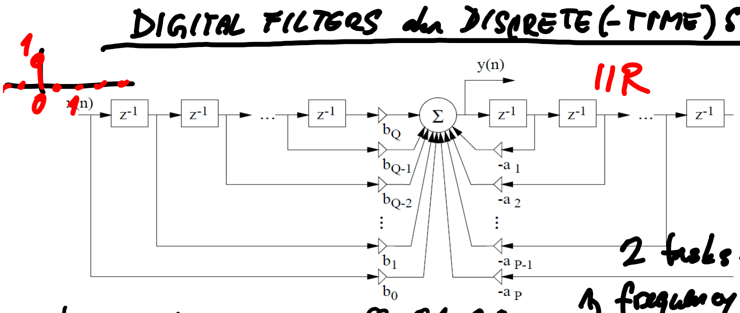
$$y[n] = x[\mathrm{mod}_N(n-m)] \qquad Y[k] = Y[k]\,e^{-jk\frac{2\pi}{N}m}$$

convolution of 2 signals

$$y[n] = x_1[n] \circledast x_2[n] \qquad Y[k] = X_1[k]\,X_2[k]$$

FFT = DFT for $N = 2^6$, usually 128, 256, 512, 1024, 2048

DIGITAL FILTERS aka DISCRETE(-TIME) SYSTEMS ...



m	0	1	2	3	.....	Q	Q+1	Q+2	..
h[m]	b_0	b_1	b_2	b_3		b_Q	0	0	

2 tasks:
1) frequency response.
2) stability

THEORY

Difference equation

$$y[n] = b_0x[n] + b_1x[n-1] + \dots + b_Qx[n-Q] - a_1y[n-1] \dots - a_Py[n-P]$$

$$y[n] = \sum_{k=0}^{Q} b_k x[n-k] - \sum_{k=1}^{P} a_k y[n-k]$$

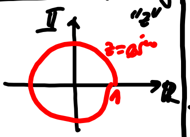
EXAMPLE

$$y[n] = x[n] - 0{,}5\,y[n-1]$$

FT (analog) $X(\omega)$

$X(j\omega) \approx$ Laplace $X(s)$

Z-transform $X(z) = \sum_{m=-\infty}^{\infty} x[m] z^{-m}$



$$Y(z) = X(z) - 0{,}5\,Y(z)z^{-1}$$

$$Y(z)[1 + 0{,}5z^{-1}] = X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1+0{,}5z^{-1}}$$

DTFT $\cancel{X(\omega)}$ $X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m}$

$x[n] \to X(z)$ $\quad a\,x[n] \to a\,X(z)$ $\quad x[n-k] \to X(z)z^{-k}$

$$Y(z) = b_0X(z) + b_1X(z)z^{-1} + \dots + b_QX(z)z^{-Q} - a_1Y(z)z^{-1} \dots - a_PY(z)z^{-P}$$

TRANSFER FUNCTION: $H(z) = \frac{Y(z)}{X(z)}$

$$H(e^{j\omega}) = \frac{1}{1+0{,}5e^{-j\omega}}$$

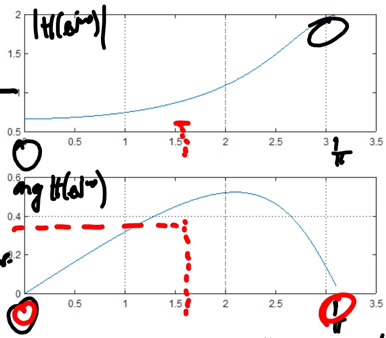
$$Y(z)[1 + a_1z^{-1} + \dots + a_Pz^{-P}] = X(z)[b_0 + b_1z^{-1} + \dots + b_Qz^{-Q}]$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + \dots + b_Qz^{-Q}}{1 + a_1z^{-1} + \dots + a_Pz^{-P}} = H(z) = \frac{\sum_{k=0}^{Q} b_k z^{-k}}{1 + \sum_{k=1}^{P} a_k z^{-k}}$$

matlab freqz(B,A)
Python mp.freqz

Frequency response

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{b_0 + b_1e^{-j\omega} + \dots + b_Qe^{-jQ\omega}}{1 + a_1e^{-j\omega} + \dots + a_Pe^{-jP\omega}} = \frac{\sum_{k=0}^{Q} b_k e^{-jk\omega}}{1 + \sum_{k=1}^{P} a_k e^{-jk\omega}}$$



Factorization of transfer function

$$H(z) = \frac{b_0z^{-Q}\left(z^Q + \frac{b_1}{b_0}z^{Q-1} + \dots + \frac{b_Q}{b_0}\right)}{z^{-P}\left(z^P + a_1z^{P-1} + \dots + a_P\right)}$$

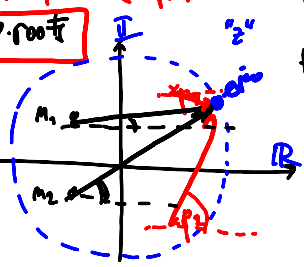
$ax^2 + bx + c = 0$ $\quad x_{1/2} = $ 2 real or 2 complex-conj. numbers

$a(x-x_1)(x-x_2)$

$$H(z) = \frac{b_0z^{-Q}(z-n_1)(z-n_2)\dots(z-n_Q)}{z^{-P}(z-p_1)(z-p_2)\dots(z-p_P)}$$

n - roots of numerator "zero points"
p - roots of the denominator "poles"

$\boxed{\text{np.roots}}$



$$H(e^{j\omega}) = \frac{b_0e^{-j\omega Q}(e^{j\omega}-n_1)(e^{j\omega}-n_2)\dots}{e^{-j\omega P}(e^{j\omega}-p_1)(e^{j\omega}-p_2)\dots}$$

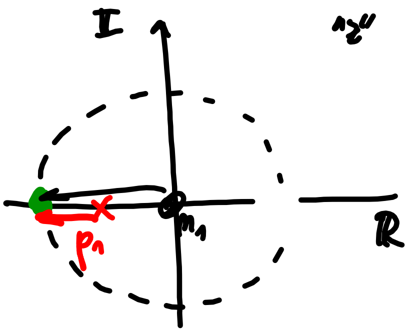
$$\left|\frac{a\,b}{c\,d}\right| = \frac{|a||b|}{|c||d|}$$

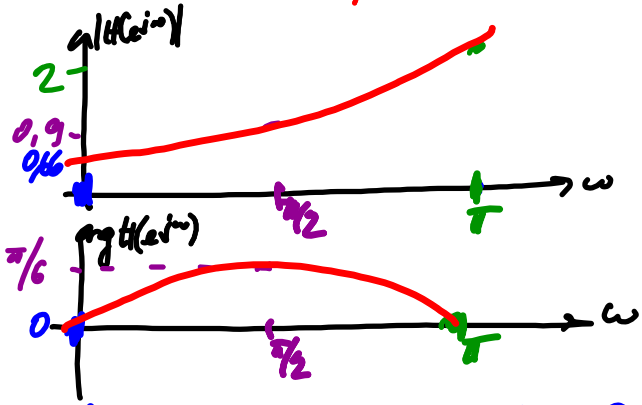
$$|H(e^{j\omega})| = b_0 \frac{\text{product of lengths of black vectors}}{\text{product of lengths of red vectors.}}$$

ang $H(e^{j\omega}) = \omega(P-Q) +$ sum of angles of black vectors
$-$ sum of angles of red vectors.

$$H(z) = \frac{1}{1+0{,}5z^{-1}} = \frac{1}{z^{-1}(z+0{,}5)} = \frac{z-0}{z-(-0{,}5)}$$

$$H(e^{j\omega}) = \frac{e^{j\omega}-0}{e^{j\omega}-(-0{,}5)}$$





$$|H| = \frac{1}{1{,}5} = 0{,}66 \qquad \arg H = 0 - 0 = 0$$

$$|H| = \frac{1}{1{,}11} = 0{,}9 \qquad \arg H = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

$$|H| = \frac{1}{0{,}5} = 2 \qquad \arg H = \pi - \pi = 0$$

Stability

$|p_k| < 1$

CZSa