DISCRETE-TIME SIGNALS

Periodicity of a cosine

discrete time

$x[n] = C_1 \cos(\omega_1 n + \varphi_1)$

ω_1 normalized angular frequency

440Hz an $F_s = 8000$Hz

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n = np.arange(8000)
x = C1 * np.cos(2*np.pi*440/8000 * n)
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$\cancel{N_1 = \frac{2\pi}{\omega_1}}$

$\omega_1(N_1 + n) - \omega_1 n = k2\pi$

$\omega_1 N_1 + \cancel{\omega_1 n} - \cancel{\omega_1 n} = k2\pi$

$N_1 = k\frac{2\pi}{\omega_1}$ take k in order to have N_1 integer.

continuous time

$x(t) = C_1 \cos(\omega_1 t + \varphi_1)$

T_1 period? $f_1 = \frac{1}{T_1}$ $\omega_1 = 2\pi f_1$

$T_1 = \frac{2\pi}{\omega_1}$

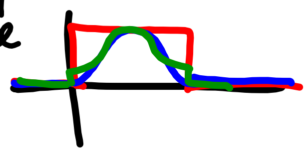
$\omega_1 = \frac{\pi}{8}$ $N_1 = \frac{2\pi}{\frac{\pi}{8}} = 16$

$\omega_1 = \frac{8\pi}{31}$ $N_1 = \frac{2\pi}{8\pi/31} = \frac{31}{4} = \cancel{7.75}$ $N_1 = k\frac{2\pi}{\frac{8\pi}{31}} = k\frac{31}{4} = 31$ with $k=4$

$\omega_1 = \frac{1}{6}$ $N_1 = k\frac{2\pi}{\frac{1}{6}} = k\,12\pi$ Not possible $\Rightarrow$ not periodic

Operations with discrete signals

Getting sequence of N samples: $x[n] = \begin{cases} \text{non-zero} & \text{for } n \in 0 \dots N-1 \\ 0 & \text{elsewhere} \end{cases}$



n	-3	-2	-1	0	1	2	3	4	5	6	7	8
$s[n]$	5	4	3	5	3	2	-1	-2	1	2	5	
$x[n]$	0	0	0	5	3	2	-1	0	0	0	0	0
$w[n]$	0	0	0	1	1	1	1	0	0	0	0	0

window

$x[n] = w[n] \cdot s[n] = R_4[n]\, s[n]$

rectangular length

Making a sequence periodic

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	...
$x[n]$				5	3	2	-1						
$y[n]$	3	2	-1	5	3	2	-1	5	3	2	-1		

$y[n] = x(\mathrm{mod}_N\, n)$

Periodization with a shift

$\boxed{3 \quad 2 \quad -1 \quad 5 \quad 3 \quad 2 \quad -1 \quad 5 \quad 3 \quad 2 \quad -1}$

$g[n] = x(\mathrm{mod}_N (n-m))$ m is the delay.

Circular shift

n	-3	-2	-1	0	1	2	3	4	5	6
$x[n]$				5	3	2	-1			
$y[n]$				2	-1	5	3			

$R_N[n]$
≈ the executioner :(

$y[n] = R_N[n]\, x(\mathrm{mod}_N(n-m))$ m is the delay...

CONVOLUTION

$x_1 = [1 \ 2 \ 3 \ 4]$ $N=4$ samples
$x_2 = [1 \ 1 \ -1 \ -1]$

STANDARD (LINEAR) CONVOLUTION

$$y[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$ 2N-1 samples

$(x^3+2x^2+3x+4)(x^3+x^2-x-1) =$ can be obtained by a convolution!

Cyclic (periodic) convolution

$$y[n] = x_1[n] \tilde{*} x_2[n] = \sum_{k=0}^{N-1} x_1[k]\, x_2[\mathrm{mod}_N(n-k)]$$

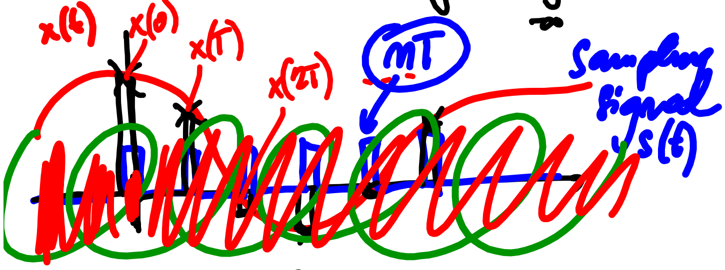
Circular convolution

$$y[n] = x_1[n] \, (N) \, x_2[n] = R_N[n] \sum_{k=0}^{N-1} x_1[k]\, x_2[\mathrm{mod}_N(n-k)]$$

SPECTRAL ANALYSIS OF DISCRETE SIGNALS

$x[n]$ → spectrum from this signal!

we have F.T. $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$ ← this works for cont. time signals!



$x_s(t) = x(t) \cdot s(t) = \sum_{m=-\infty}^{\infty} x(mT)\, \delta(t - mT)$

$$X(j\omega) = \int_{-\infty}^{\infty} \sum x(mT)\, \delta(t - mT)\, e^{-j\omega t}\, dt = \int x(mT)\, \delta(t - mT)\, e^{-j\omega mT}\, dt =$$

F_s sampling freq.
T sampling period
$T = \frac{1}{F_s}$

$$= \sum_{m=-\infty}^{\infty} x(mT)\, e^{-j\omega mT}$$

[rad/s]

$\omega m T = \frac{2\pi f m}{F_s}$ ← normalized frequency

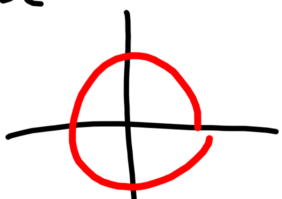
normalized angular frequency

$$\boxed{\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$$

[rad]

Discrete-time Fourier Transform DTFT

cont. time	discrete time
$\cancel{X(\omega)}$ $X(j\omega) \approx$ Laplace transform	$\cancel{X(\omega)}$ $\tilde{X}(e^{j\omega})$ z-transform



Example of DTFT

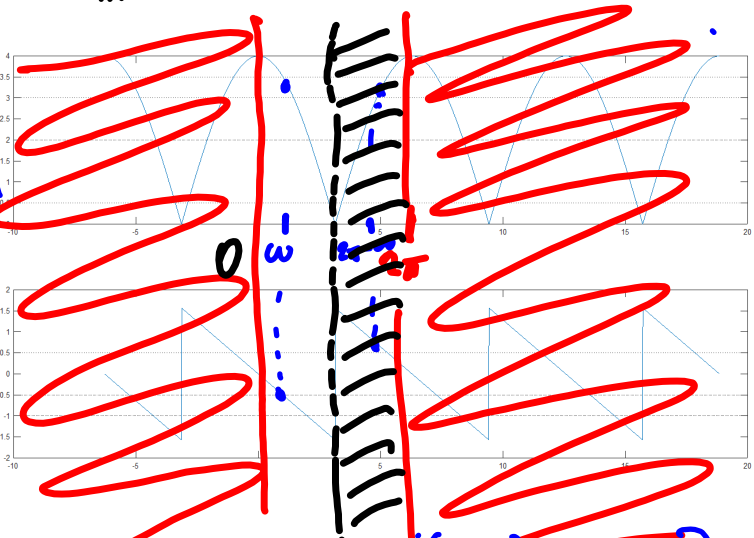
n	0	1	2	3	....
$x[n]$	2	2	0	0	0 ....

DTFT !

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = 2e^{-j\omega 0} + 2e^{-j\omega 1} = 2 + 2e^{-j\omega}$$

$|\tilde{X}(e^{j\omega})|$

ang $\tilde{X}(e^{j\omega})$



Periodicity ?

frequency [Hz]	F_s
angular freq. [rad/s]	$2\pi F_s$
normalized freq. [ ]	1
normalized angular freq [rad]	2π

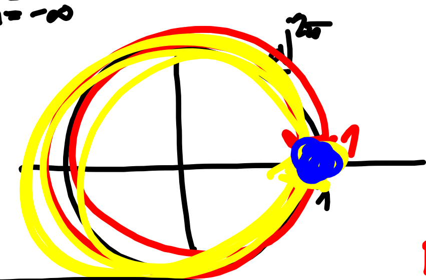
$$\left.\begin{aligned} |\tilde{X}(e^{j\omega})| &= |\tilde{X}(e^{j(2\pi-\omega)})| \\ \text{ang}\,\tilde{X}(e^{j\omega}) &= -\,\text{ang}\,\tilde{X}(e^{j(2\pi-\omega)}) \end{aligned}\right\} \quad \tilde{X}(e^{j\omega}) = \tilde{X}^*(e^{j(2\pi-\omega)})$$

Proof that DTFT is periodic with 2π

$e^{a+b} = e^a \cdot e^b$

$$\tilde{X}(e^{j(\omega+k2\pi)}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega+k2\pi)n} = \sum x[n] e^{-j\omega n} e^{-j2\pi k n} = 1$$

k - integer, n - integer } kn - integer



$$= \boxed{\sum x[n] e^{-j\omega n}} = X(e^{j\omega})$$

■ proven !

IDFT Inverse DFT

3.12. 14:30

Output = maybe a constant; Sum op.; $e^{\pm j}$ time freq; input

$$x[n] = \frac{1}{2\pi} \int_{0\,(-\pi)}^{2\pi\,(\pi)} \tilde{X}(e^{j\omega}) e^{+j\omega n} d\omega$$

$+$ freq → time

$-$ time → freq.

Freq. analysis of periodic discrete signals

n	-2	-1	0	1	2	3	4	5	6	....
$\tilde{x}[n]$	0	0	2	2	0	0	2	2	0	0 ...

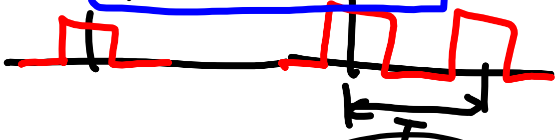
1 period has N samples

time	spectrum
sampling	periodization
periodization	discretization (sampling!)

BOTH!

cont. time periodic signal ... c_k

non-periodic signal $X(j\omega)$

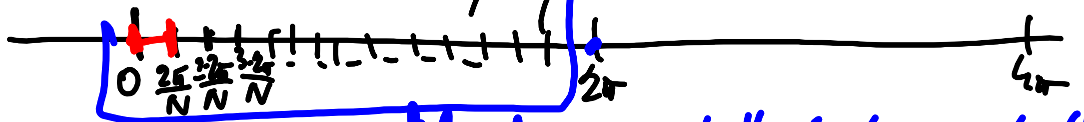


$\omega_1 = \frac{2\pi}{T}$

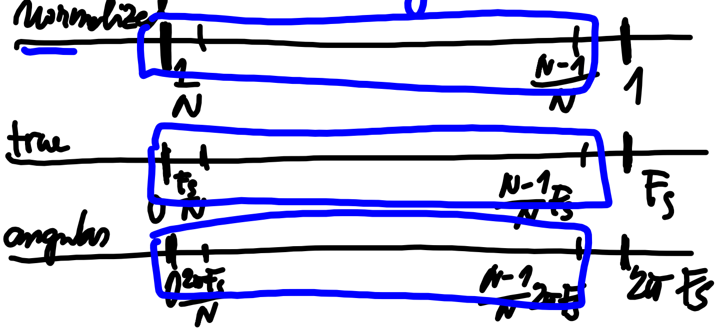
coefficients sitting at multiples of fundamental freq.

$\omega_1 = \frac{2\pi}{N}$

normalized angular frequency



N meaningful values



Let's just sample the DTFT!

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\boxed{\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n}}$$

DFS - Discrete Fourier Series

Periodicity of DFS: proof:

$$\tilde{X}[k+m_0N] = \sum \tilde{x}[n] e^{-j(k+m_0N)\frac{2\pi}{N}n} = \sum \tilde{x}[n] e^{-jk\frac{2\pi}{N}n} \cdot \underbrace{e^{-jm_0\cancel{N}\frac{2\pi}{\cancel{N}}n}}_{1} =$$

$e^{a+b} = e^a e^b$

$e^{-jm_0 n 2\pi} = 1$

$$\sum \tilde{x}[n] e^{-jk\frac{2\pi}{N}n} = \tilde{X}[k]$$ ■ proven!