Fourier transform continuous time, non-periodic

spectrum
- where?
- how big?
- how shifted?

output = maybe a constant sum. exp. input $e^{\pm j \, \text{time} \, \text{frequency}}$

$-$ time → freq.

$+$ freq → time

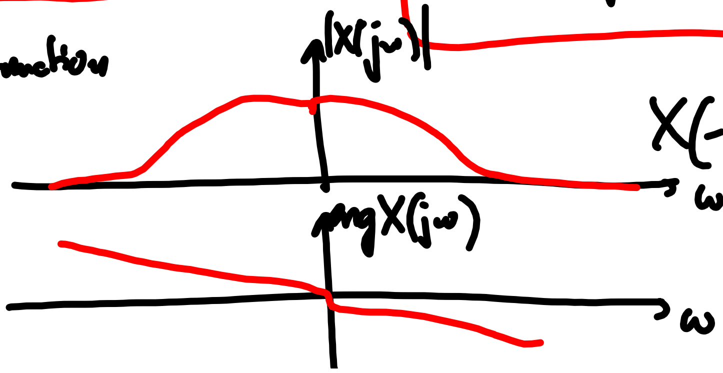
Analysis

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-jt\omega} dt$$

spectral function

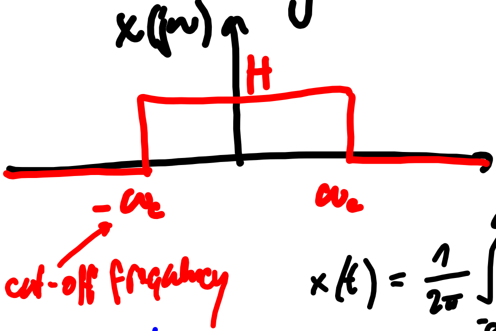
Synthesis (inverse F.T.)

$$x(t) = \frac{1}{2\pi} \int X(j\omega) e^{+j\omega t} d\omega$$

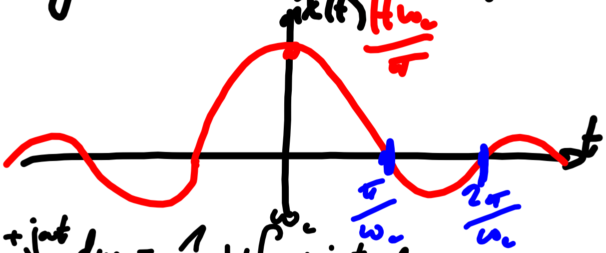


$$X(-j\omega) = X^*(j\omega)$$

Exercise: signal corresponding to a square spectrum



$\longrightarrow$



$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{+j\omega t}\,d\omega=\frac{1}{2\pi}H\int_{-\omega_c}^{\omega_c}e^{+j\omega t}\,d\omega=$$

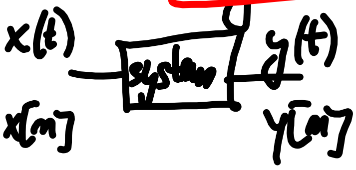
S.T.: $\int_{-b}^{b}e^{+jxy}\,dy=2b\,\mathrm{sinc}(bx)$

$$\frac{H}{\cancel{2}\pi}\cancel{2}\omega_c\,\mathrm{sinc}(\omega_c t)=\frac{H\omega_c}{\pi}\sin(\omega_c t)$$

$\omega_c t=\pi$

$t=\frac{\pi}{\omega_c}$

Systems

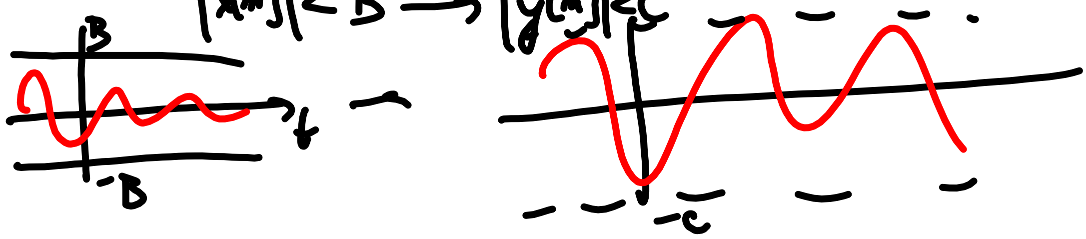


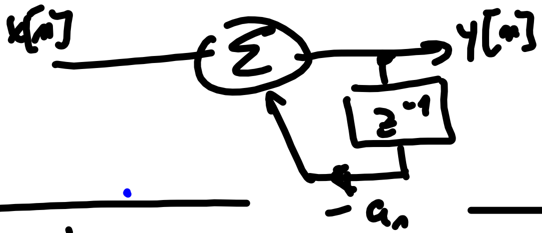
① Causality

$$y(t) = f\big(x(\tau \le t), y(\tau < t)\big)$$
$$y[n] = f\big(x(m \le n), y(m < n)\big)$$

② Stability bounded input → bounded output

$$|x(t)| < B \longrightarrow |y(t)| < C$$
$$|x[n]| < B \longrightarrow |y[n]| < C$$





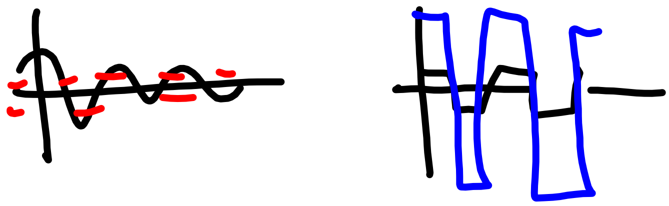
$|a_1| < 1$

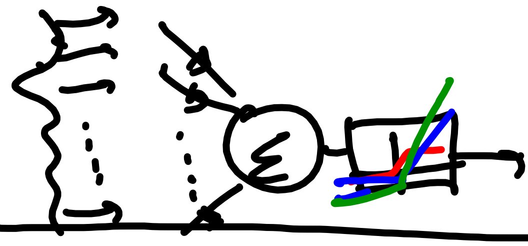
Time invariance

$$x(t) \longrightarrow y(t)$$
$$x(t-\tau) \longrightarrow y(t-\tau)$$

Linearity

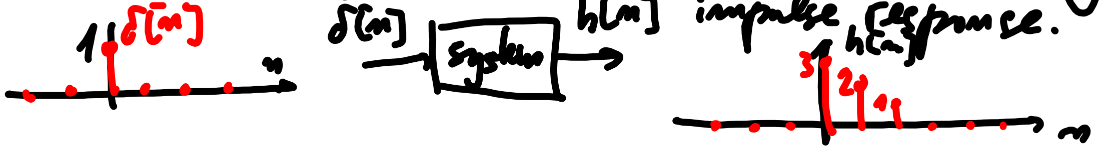
$$x_1(t) \longrightarrow y_1(t)$$
$$x_2(t) \longrightarrow y_2(t)$$
$$a x_1(t) + b x_2(t) \longrightarrow a y_1(t) + b y_2(t)$$

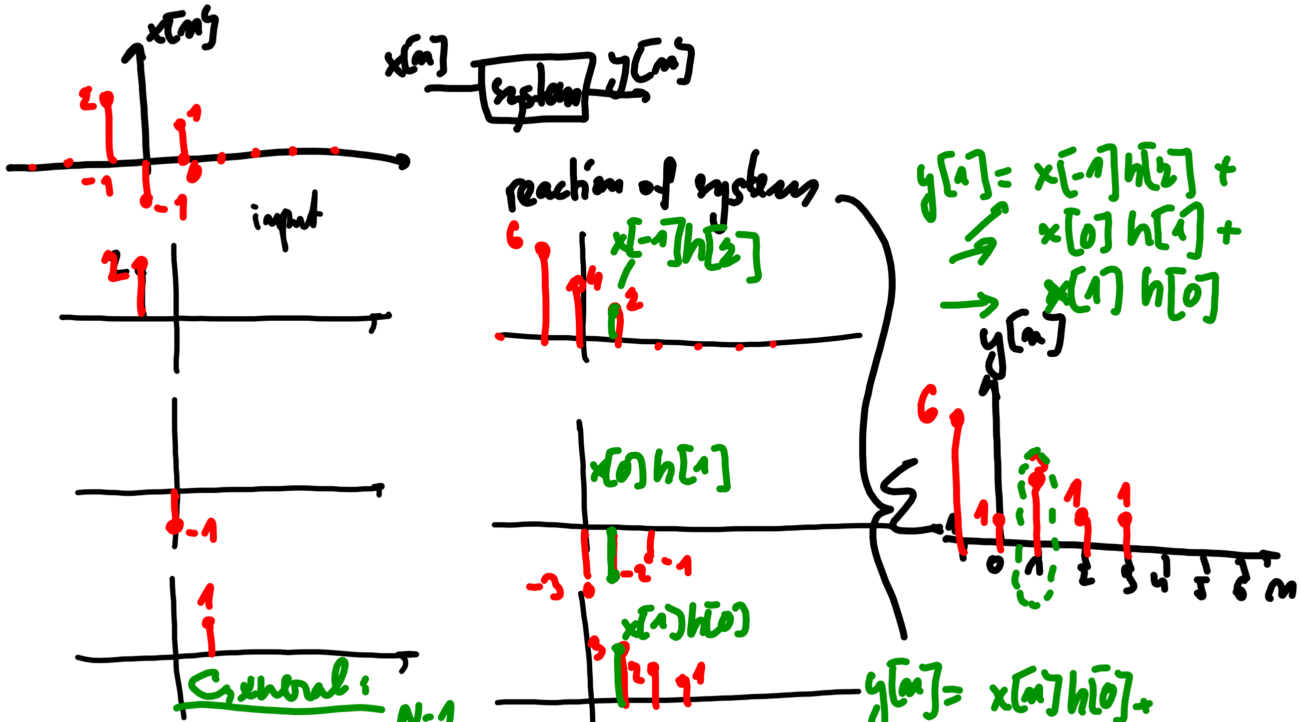




LTI - linear and time invariant.

Discrete-time system and its reaction to arbitrary input





$$y[1] = x[-1]h[2] + x[0]\,h[1] + x[1]\,h[0]$$

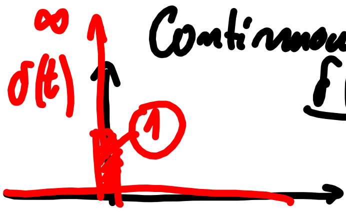
$$y[n] = x[n]h[0] + x[n-1]h[1] + x[n-2]h[2]$$

General:

$$y[n] = \sum_{k=0}^{N-1} x[n-k]\,h[k] = \sum_{k=n-N+1}^{n} x[k]\,h[n-k]$$

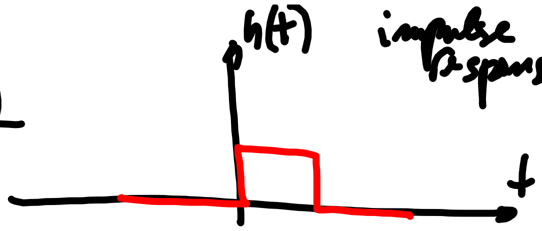
a convolution

$$y[n] = x[n] * h[n]$$



Continuous-time systems

$\delta(t) \rightarrow \boxed{\text{system}} \rightarrow h(t)$



impulse response

$x(t) \rightarrow \boxed{\text{system}} \rightarrow y(t)$

$$\boxed{y(t) = x(t) * h(t)}$$

convolution

discrete-time

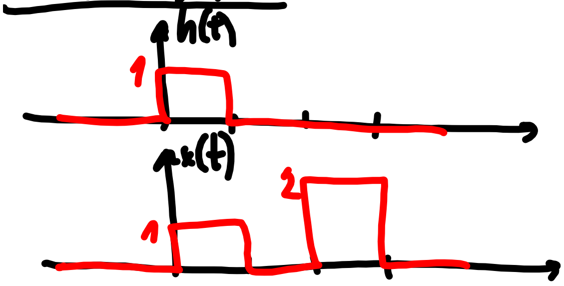
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

cont. time convolution integral

$$\boxed{y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau} =$$

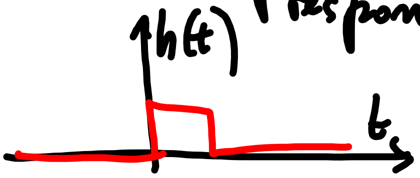
$$\boxed{= \int_{-\infty}^{\infty} h(t-\tau)\,x(\tau)\,d\tau}$$

Exercise:

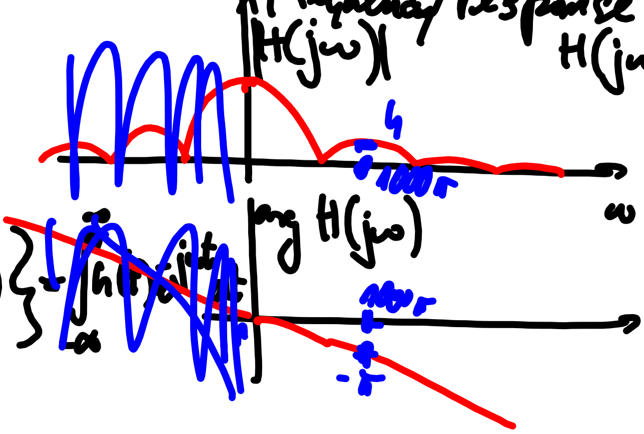


Frequency analysis of cont.-time systems

in time: impulse response



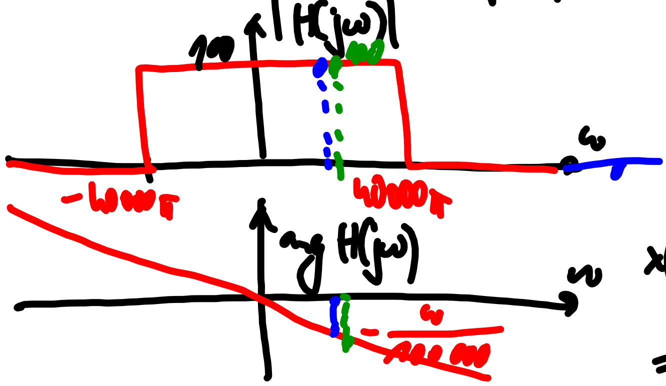
Frequency response $H(j\omega)$



$$H(j\omega) = F.T.\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$H(-j\omega) = H^*(j\omega)$$

Ideal Hi-Fi amplifier



input signal

$x(t) = 5\, e^{j20000\pi t}$ $\qquad e^{a+b} = e^a \cdot e^b$

$-\frac{20000\pi}{100000} = -0{,}2\pi$

$$x(t) = 500\, e^{j(20000\pi t - 0{,}2\pi)} = \underbrace{500\, e^{-j0{,}2\pi}}_{\text{const.}} \cdot \underbrace{e^{j20000\pi t}}_{\text{"living" comp. exponential.}}$$

input signal: $x(t) = 10 \cos\left(25000\pi t + \frac{\pi}{2}\right)$ $\qquad -\frac{25000\pi}{100000} = -0{,}25\pi$

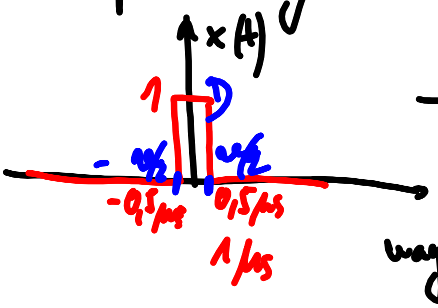
$$y(t) = 100 \cdot 10 \cos\left(25000\pi t + \frac{\pi}{2} - \frac{\pi}{4}\right) = 1000 \cos\left(25000\pi t + \pi/4\right)$$

complicated per. signal?

$x(t) = \sum c_k e^{jk\omega_0 t}$ — individual complex exps. (using F.S. to "dissect" the signal

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0)\, e^{jk\omega_0 t}$$

Non-periodic signal at the input



H(jω)

$\int_{-b}^{b} e^{jxy} = 2b\,\mathrm{sinc}(bx)$

way 1: $h(t) = \mathrm{IFT}(H(j\omega))$ $y(t) = x(t) * h(t)$

way 2: ① spectrum; $X(j\omega) = \mathrm{F.T.}(x(t))$

② $Y(j\omega) = X(j\omega)\cdot H(j\omega)$

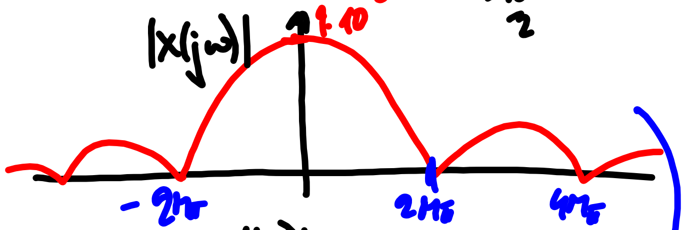
③ $y(t) = \mathrm{IF.T.}(Y(j\omega))$

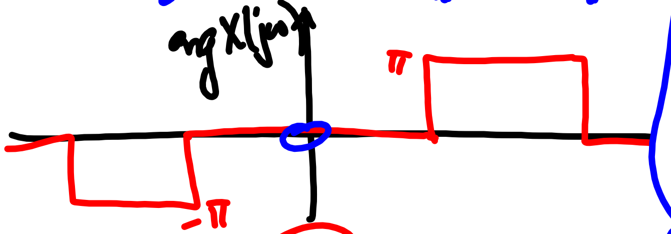
① $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = D\int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt = D\, 2\frac{\tau}{2}\,\mathrm{sinc}\left(\frac{\tau}{2}\omega\right) = D\tau\,\mathrm{sinc}\left(\frac{\tau}{2}\omega\right)$

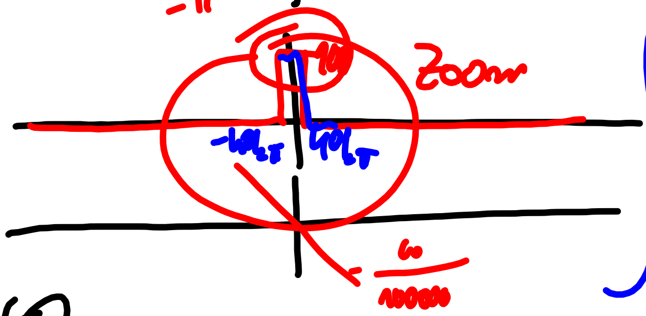
$= 1\cdot10^{-6}\,\mathrm{sinc}(0{,}5\cdot10^{-6}\,\omega)$

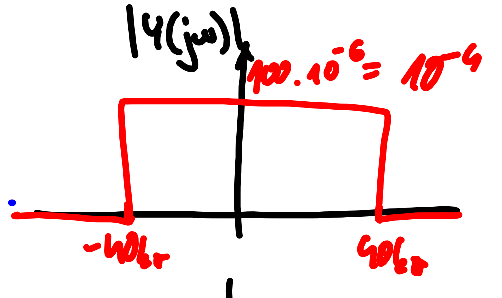
$0{,}5\cdot10^{-6}\,\omega = \pi$

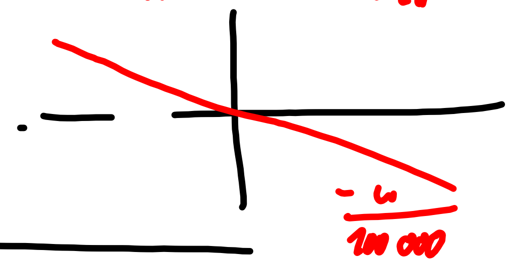
$\omega = 2\pi\cdot10^{6}$



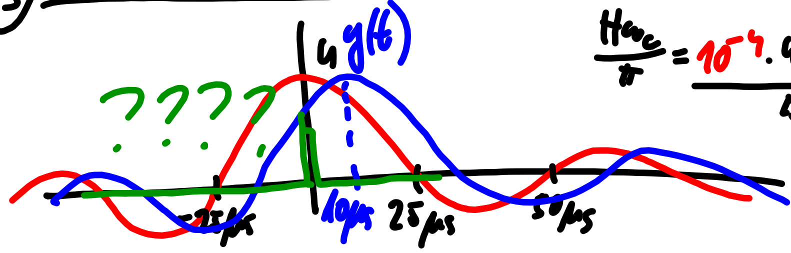








③



$\frac{H_{max}}{\pi} = \frac{10^{-4}\cdot 40\cdot10^{3}\pi}{\pi} = 4$

$\frac{\pi}{\omega_0} = \frac{\pi}{40\cdot10^{3}\pi} = \frac{1}{4}\cdot10^{-4} = 25\,\mu s$

$f_m(t) \longrightarrow FN(j\omega)$

$f_m(t-\tau) \longrightarrow FN(j\omega)\cdot e^{-j\omega\tau}$ — slope of decline is $-\tau$

$\tau = \frac{1}{100\,000} = 10\,\mu s$