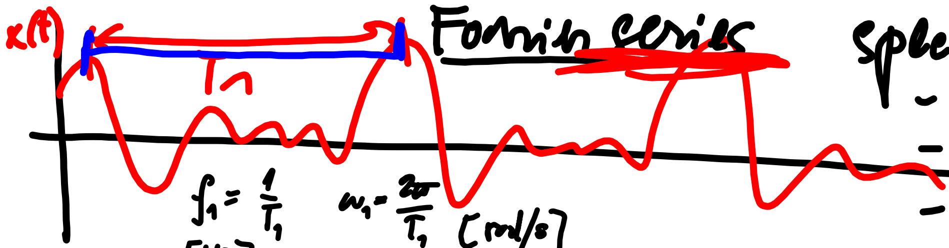
Fourier series



spectral analysis
- where?
- how much?
- how shifted?

$f_1 = \frac{1}{T_1}$ [Hz] $\omega_1 = \frac{2\pi}{T_1}$ [rad/s]

$$x(t) = c_{-k}e^{-jk\omega_1 t} \dots c_{-2}e^{-j2\omega_1 t} + c_{-1}e^{-j\omega_1 t} + c_0 + c_1 e^{j\omega_1 t} + c_2 e^{j2\omega_1 t} + \dots + c_k e^{jk\omega_1 t}$$

$C_1\cos(\omega_1 t + \varphi_1)$

$C_2\cos(2\omega_1 t + \varphi_2)$

$C_k\cos(k\omega_1 t + \varphi_k)$

"harmonics"

"harmonically related complex exps/cosines"

synthesis

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_1 t}}$$

analysis

$$\boxed{c_k = \frac{1}{T_1}\int_{T_1} x(t)\, e^{-jk\omega_1 t}\, dt}$$

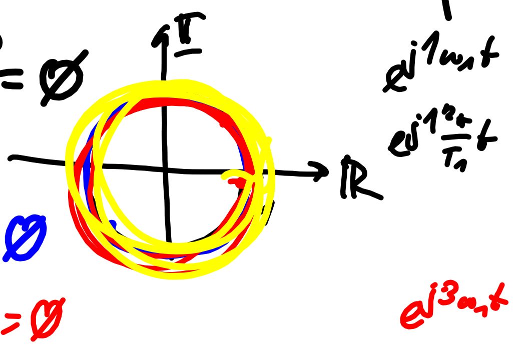
Projection onto bases
unknown signal: $x(t)$
analysis signal (basis): $e^{jk\omega_1 t}$

CHECK ORTHOGONALITY

$b_k(t) = e^{jk\omega_1 t}$ $b_l(t) = e^{jl\omega_1 t}$ $e^a \cdot e^b = e^{a+b}$

$$\int_{T_1} b_k(t)\cdot b_l^*(t)\,dt = \int_{T_1} e^{jk\omega_1 t}\cdot e^{-jl\omega_1 t}\,dt = \int_{T_1} e^{j\omega_1 \underbrace{(k-l)}_{\text{int}} t}\,dt = 0 \quad \Big| \; \omega_1 = \frac{2\pi}{T_1}$$

ORTHOGONAL YES! $k-l=1$ $\int = \emptyset$



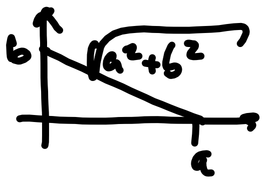
$k-l=-1$ $\int = \emptyset$

$k-l=3$ $\int = \emptyset$

NORMALITY

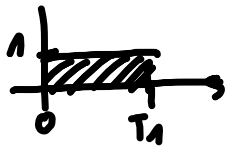
$\|x\| = 1$ $\qquad \|[0{,}1 \;\; 3 \;\; 5 \;\; 6]\| = \sqrt{0{,}1^2 + 3^2 + 5^2 + 6^2}$

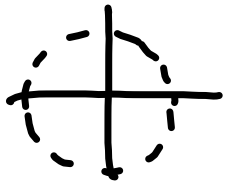
$b_k(t) = e^{jk\omega_1 t}$



$$\|b_k(t)\| = \int_{T_1} |b_k(t)|^2 dt = \int_{T_1} |\underbrace{e^{jk\omega_1 t}}_{1}|^2 dt =$$

$$= \int_{T_1} 1 \, dt = T_1$$





Exercise 1

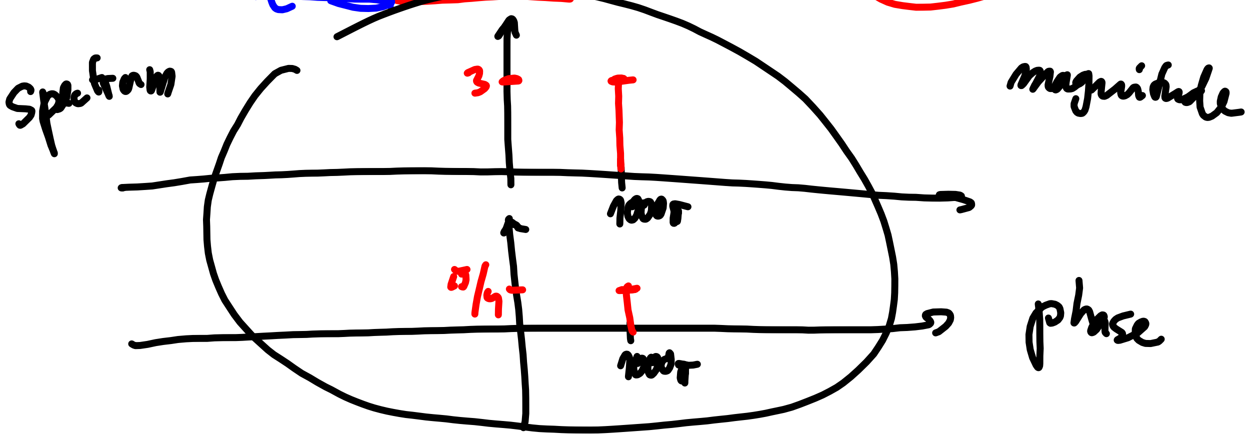
$$x(t) = 3e^{j(1000\pi t + \pi/4)} \Rightarrow \omega_1 = 1000\pi \text{ rad/s}$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_1 t} \quad k?$$

$$\rightarrow 3e^{j\pi/4} e^{j1000\pi t}$$

$$k = 1$$

$$c_1 = 3e^{j\pi/4}$$



Exercise 2

$$x(t) = 6\cos(1000\pi t + \pi/4) =$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_1 t}$$

$$\frac{6\left(e^{j(1000\pi t + \pi/4)} + e^{-j(1000\pi t + \pi/4)}\right)}{2} =$$

$$\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$$

$$= 3e^{j\pi/4} e^{j1000\pi t} + 3e^{-j\pi/4} e^{-j1000\pi t}$$

$j(-1)1000\pi t$

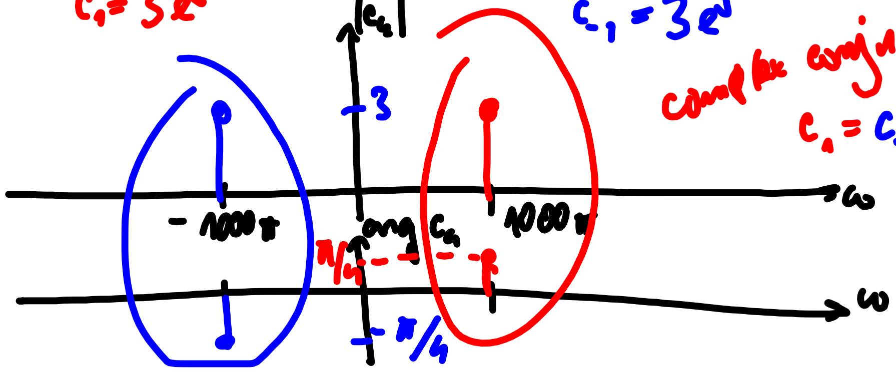
$\omega_1 = 1000\pi$ rad

$k = 1$ $k = -1$

$c_1 = 3e^{j\pi/4}$ $c_{-1} = 3e^{-j\pi/4}$

complex conjugate

$c_1 = c_{-1}^*$

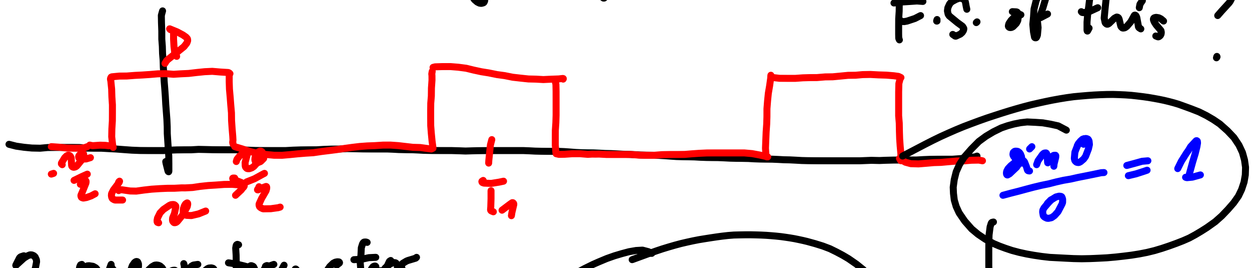


$|c_k| = |c_{-k}| = \frac{C_k}{2}$ - magnitude of the cosine

$\arg c_k = -\arg c_{-k} = \varphi_k$ - phase of the cosine.

Exercise 3 rectangular pulse-train

F.S. of this ?

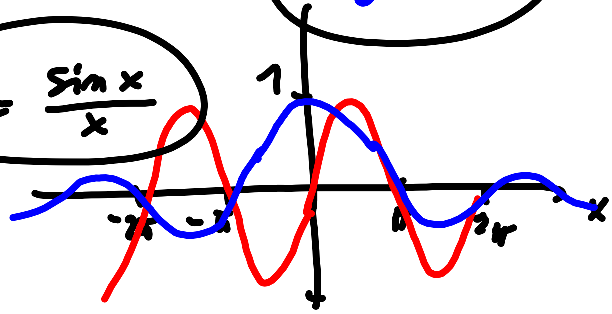


2 preparatory steps:

#1 cardinal sine $\mathrm{sinc}\, x = \frac{\sin x}{x}$

Matlab/Numpy ... $\frac{\sin \pi x}{\pi x}$

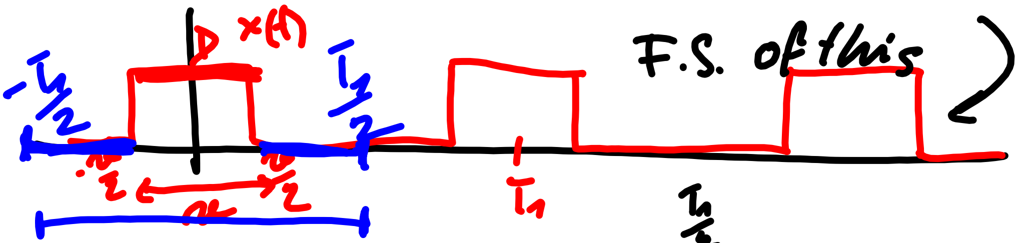
np.sinc(x/np.pi)



#2 Euler's integral - Sebastian's tool (SP) $\sin x = \frac{e^{jx} - e^{-jx}}{2j}$

$$\int_{-b}^{b} e^{+jxy}\, dy = \left[\frac{e^{jxy}}{jx} \right]_{-b}^{b} = \frac{2\left(e^{jxb} - e^{-jxb}\right)}{2jx} = \frac{b\,2}{b\,x} \sin(bx) =$$

$$= \underline{2b\, \mathrm{sinc}\,(bx)}$$



F.S. of this

$\omega_1 = \frac{2\pi}{T_1}$

$$c_k = \frac{1}{T_1}\int_{T_1} x(t)\, e^{-jk\omega_1 t}\, dt = \frac{1}{T_1}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} x(t)\, e^{-jk\omega_1 t}\, dt = \frac{D}{T_1}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jk\omega_1 t}\, dt$$

$$\boxed{\int_{-b}^{b} e^{\pm jxy}\, dy = 2b\,\mathrm{sinc}\, bx}$$

$\boxed{\text{S.P}}$

$$= \frac{D}{T_1}\, \cancel{2}\frac{\tau}{\cancel{2}}\,\mathrm{sinc}\left(\frac{\tau}{2}k\omega_1\right) = \frac{D\tau}{T_1}\,\mathrm{sinc}\left(\frac{\tau}{2}k\omega_1\right)$$

by hand

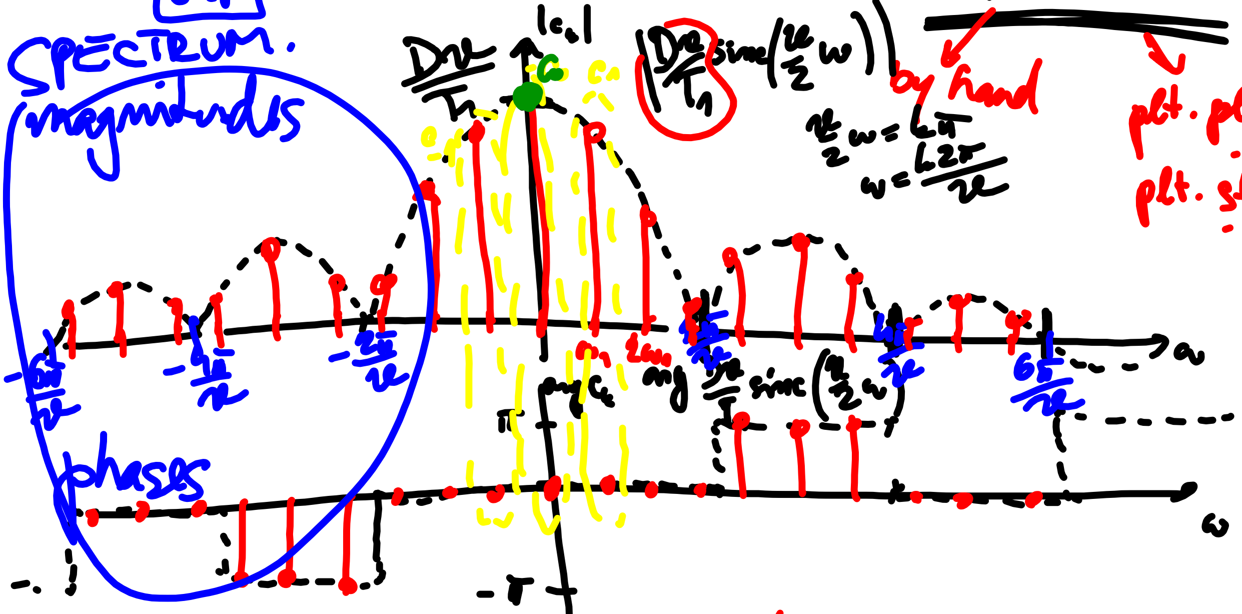
plt.plot ...

plt.stem ...

SPECTRUM.

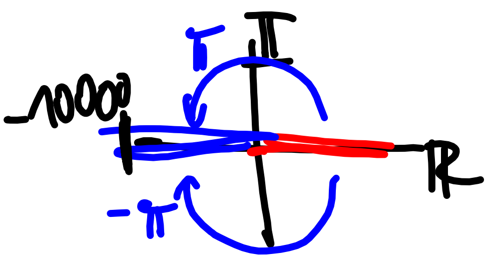
magnitudes

phases



$\frac{\tau}{2}\omega = k\pi$

$\omega = \frac{k2\pi}{\tau}$

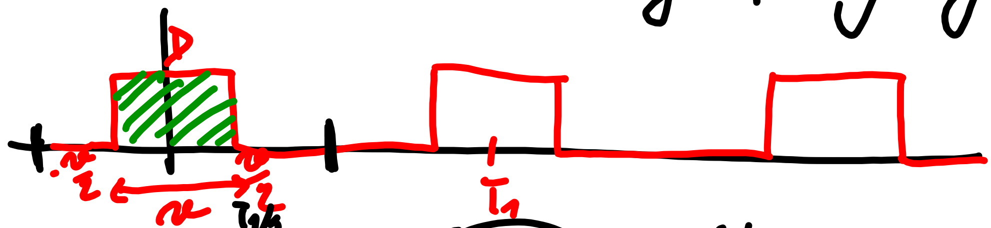


$1000\, e^{j\pi}$

$e^{-j\pi}$

$k\omega_1$ - multiples of basic angular freq.

Check of D.C. value average of my signal

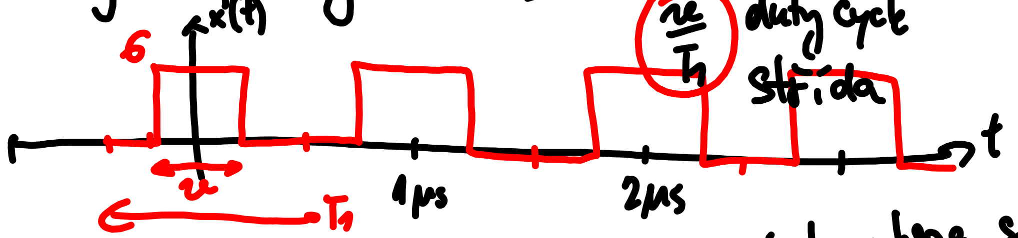


$$\bar{x} = \frac{1}{T_1}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} x(t)\,dt = \frac{D\tau}{T_1}$$

OK, same as c_0

Analysis of signal $D=6$ $f_0 = 1\,\text{MHz}$ $\tau = \frac{T_1}{2}$

$\frac{\tau}{T_1}$ duty cycle střída

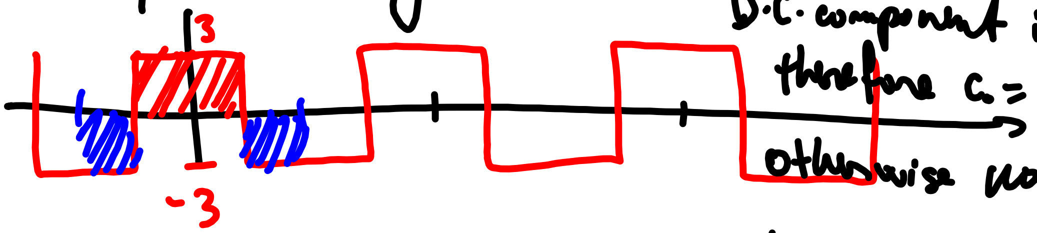


$\frac{2\pi}{\tau} = \frac{2\pi}{0{,}5\cdot 10^{-6}} = 4\pi\cdot 10^{6} = 4M\pi$ - points where sinc touches 0

$\omega_1 = 2\pi\cdot 1\cdot 10^{6} = 2\pi\cdot 10^{6} = 2M\pi$

$\frac{D\tau}{T_1} = \frac{6\cdot 0{,}5\,\cancel{\mu s}}{1\,\cancel{\mu s}} = 3$

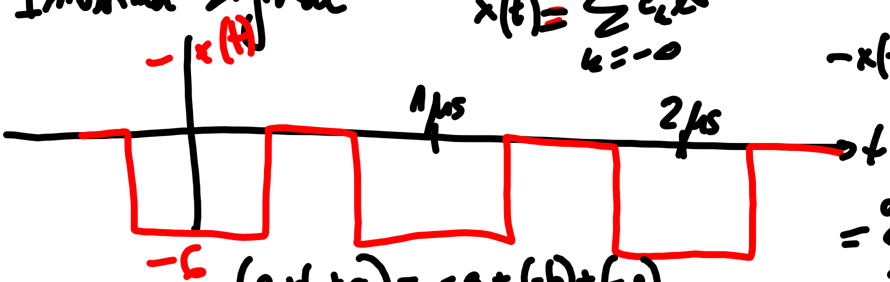
Vertically shifted signal



D.C. component is 0, therefore $c_0 = 0$, otherwise no change!

Inverted signal

$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_1 t}$



$-x(t) = -\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_1 t} =$

$= \sum_{k=-\infty}^{\infty} (-c_k) e^{jk\omega_1 t}$

$-(a+b+c) = -a+(-b)+(-c)$

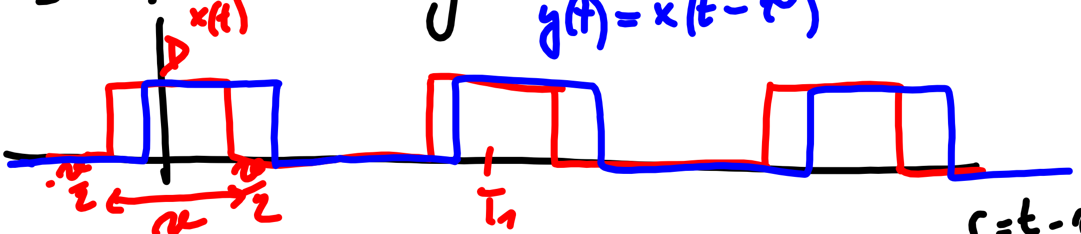
Flip sign of all coefficients → don't touch magnitudes!
→ in angles $0 \longrightarrow \pi$ or $-\pi$ (positive to negative)
π or $-\pi \longrightarrow 0$ (negative to positive)

Synthesis of the orig signal

$$x(t) = \underbrace{\cancel{c_{-2}e^{-j2\omega_1 t}}}_{0} + c_{-1}e^{-j\omega_1 t} + c_0 + c_1 e^{j\omega_1 t} + \underbrace{\cancel{c_2 e^{j2\omega_1 t}}}_{0}$$

F.S. of shifted signal

$$y(t) = x(t-\tau)$$



$r = t - \tau$

$t = r + \tau$

$$c_k = \frac{1}{T_1}\int_{T_1} x(t)\, e^{-jk\omega_1 t}\, dt \qquad c_k' = \frac{1}{T_1}\int_0^{T_1} x(\underbrace{t-\tau}_{r})\, e^{-jk\omega_1 t}\, dt =$$

$$\frac{1}{T_1}\int_{T_1} x(r)\, e^{-jk\omega_1 (r+\tau)}\, dr = \frac{1}{T_1}\int_{T_1} x(r)\cdot e^{-jk\omega_1 r}\, e^{-jk\omega_1 \tau}\, dr = e^{-jk\omega_1 \tau}\, \frac{1}{T_1}\int x(r)\, e^{-jk\omega_1 r}\, dr$$

c_k

$$\boxed{c_k' = \boxed{c_k}\, e^{-jk\omega_1\tau}}$$ F.S. coefficients of the shifted signal.

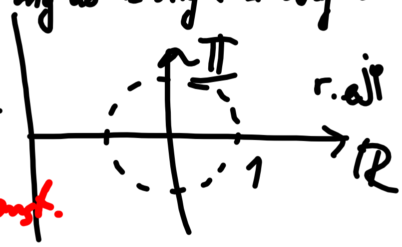
mult. of 2 complex numbers.

ab $\quad |ab| = |a|\,|b|$

ang ab = ang a + ang b

$$|c_k'| = |c_k|\,|\underbrace{e^{-jk\omega_1\tau}}_{1}| = |c_k|$$

$$\text{ang } c_k' = \text{ang } c_k + \text{ang } e^{-jk\omega_1\tau} = \text{ang } c_k - k\omega_1\tau$$



→ magnitudes : no change

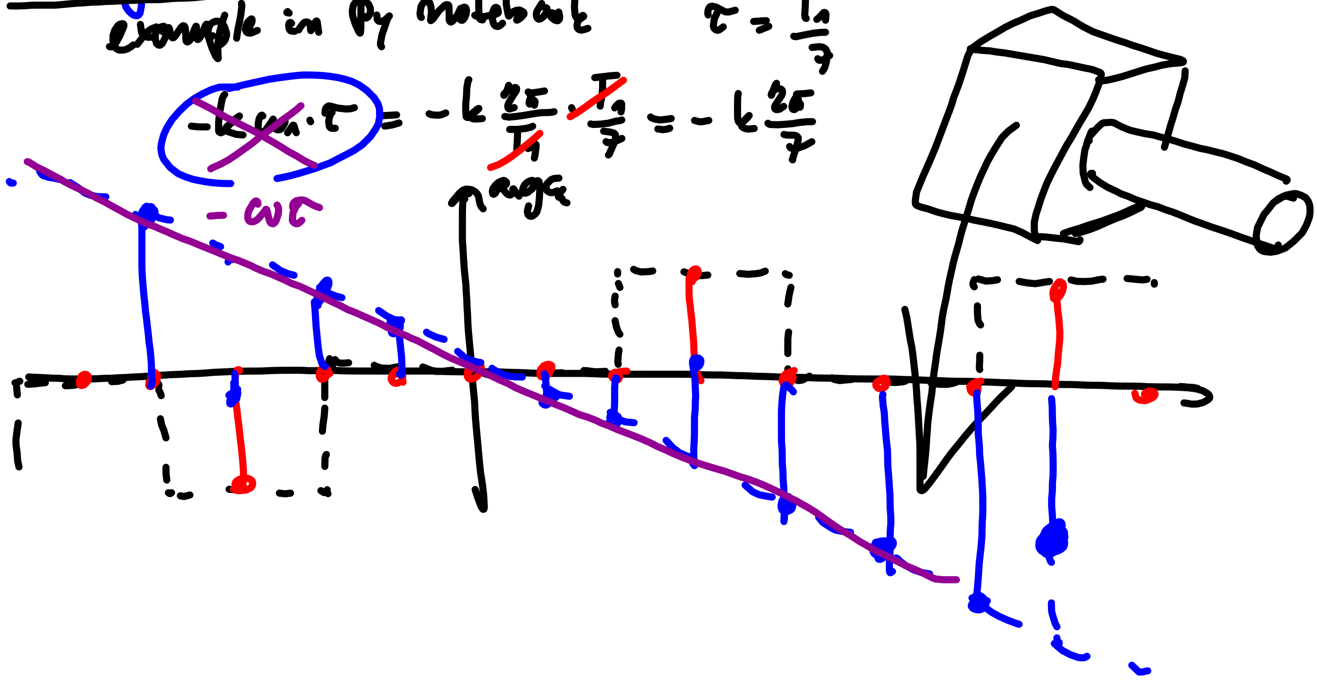
→ angles : subtraction of $k\omega_1\tau$

const

const.

example in Py notebook $\quad \tau = \frac{T_1}{7}$

$$-k\omega_1\cdot\tau = -k\frac{2\pi}{\cancel{T_1}}\cdot\frac{\cancel{T_1}}{7} = -k\frac{2\pi}{7}$$



NO LECTURE NEXT WEEK !

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