Random signals

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Signals at school and in the real world

Deterministic

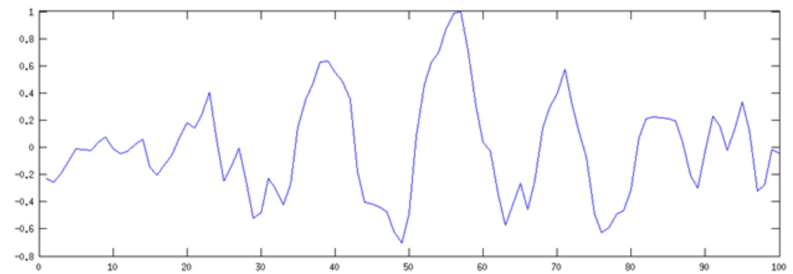
- Equation
- Plot
- Algorithm
- Piece of code

Can **compute**

Little information !

Random

- Don’t know for sure
- All different
- Primarily for „nature“ and „biological“ signals
- Can **estimate parameters**

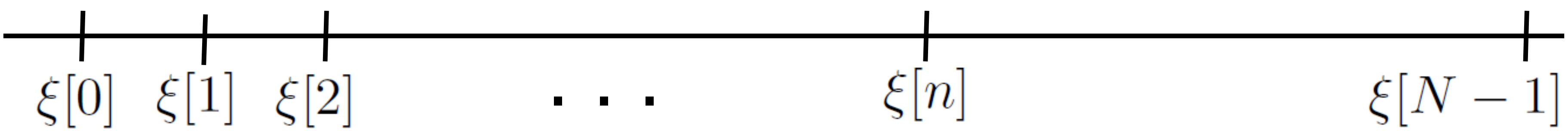


Examples

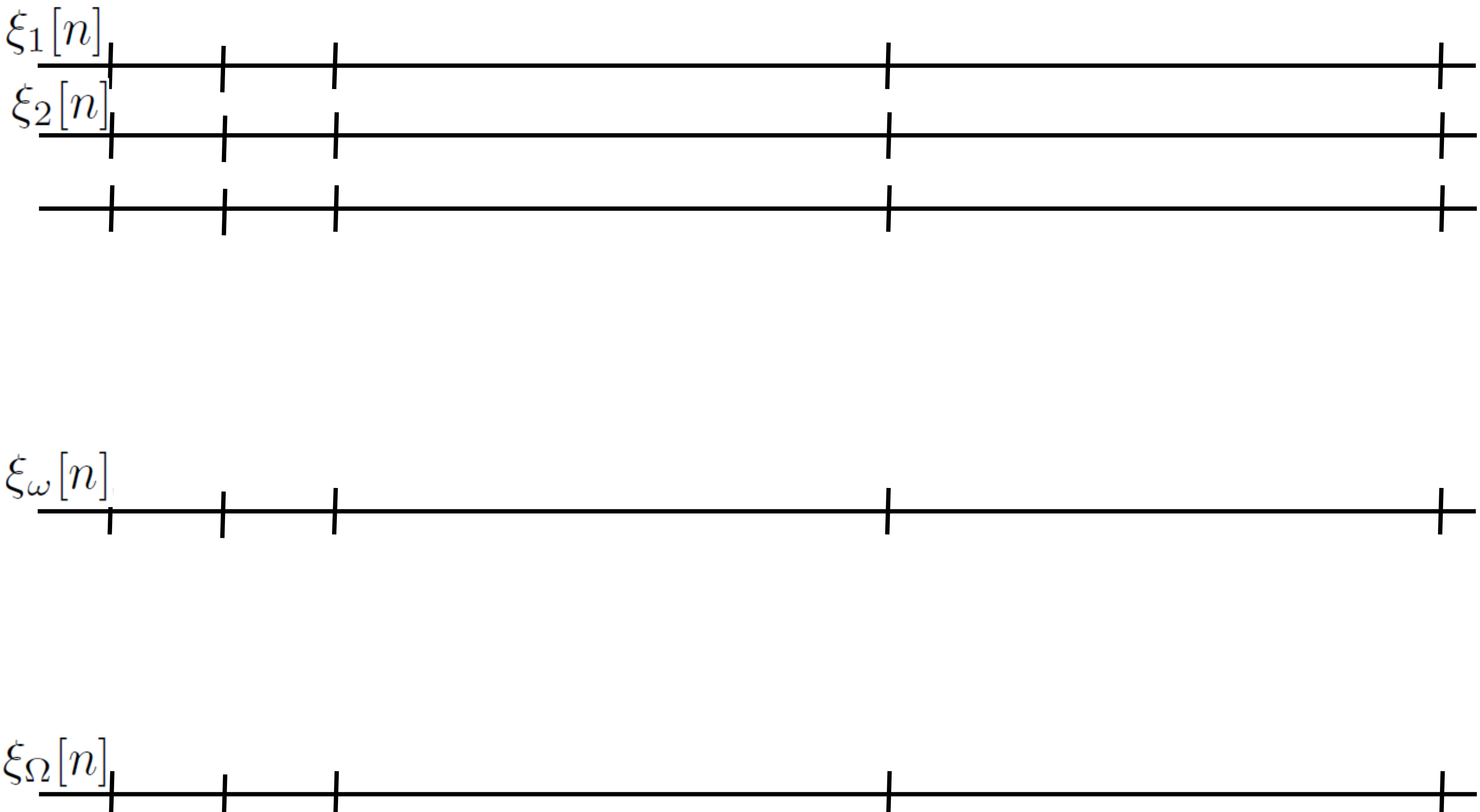
- Speech
- Music
- Video
- Currency exchange rates
- Technical signals (diagnostics)
- Measurements (of anything)
- … almost everything

Mathematically

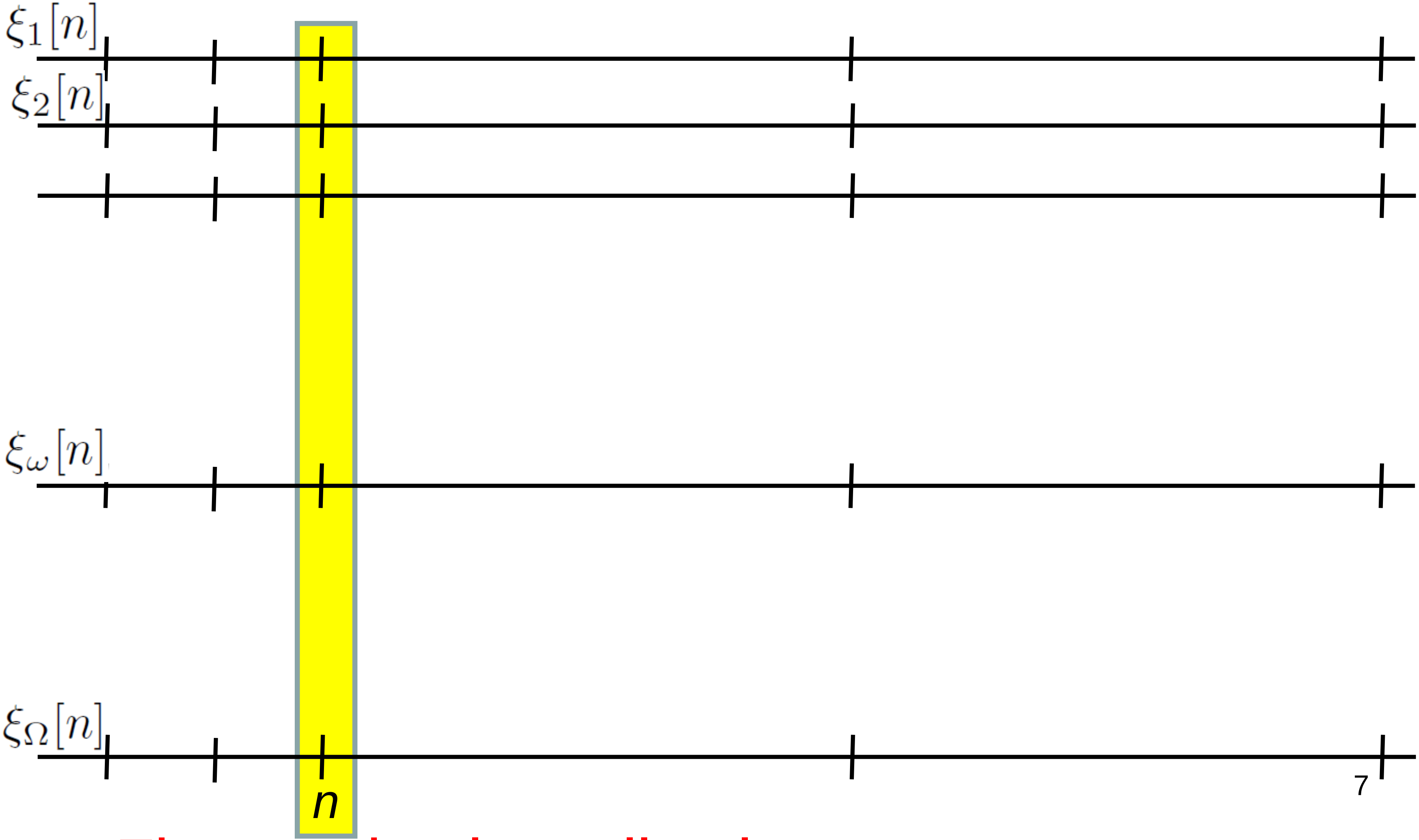
- Discrete-time only (samples)
- A system of random variables defined for each *n*
- For the moment, will look at them independently



Set of realizations



Ensemble estimates



- Fix n and select all values
- Estimate – the estimate will be valid only for this n

According to the range

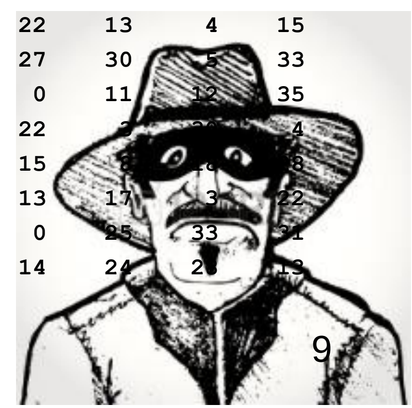
- Discrete range $\xi[n] \in [h_1, h_2, \ldots, h_H]$
  - Coin flipping
  - Dice
  - Roulette
  - Bits from a communication channel
- Real range $\xi[n] \in \mathcal{R}$
  - Strength of wind
  - Audio
  - CZK/EUR Exchange rate
  - etc

Discrete data

- 50 years of roulette *Ω=50x365* realizations
- *N=1000* games a day

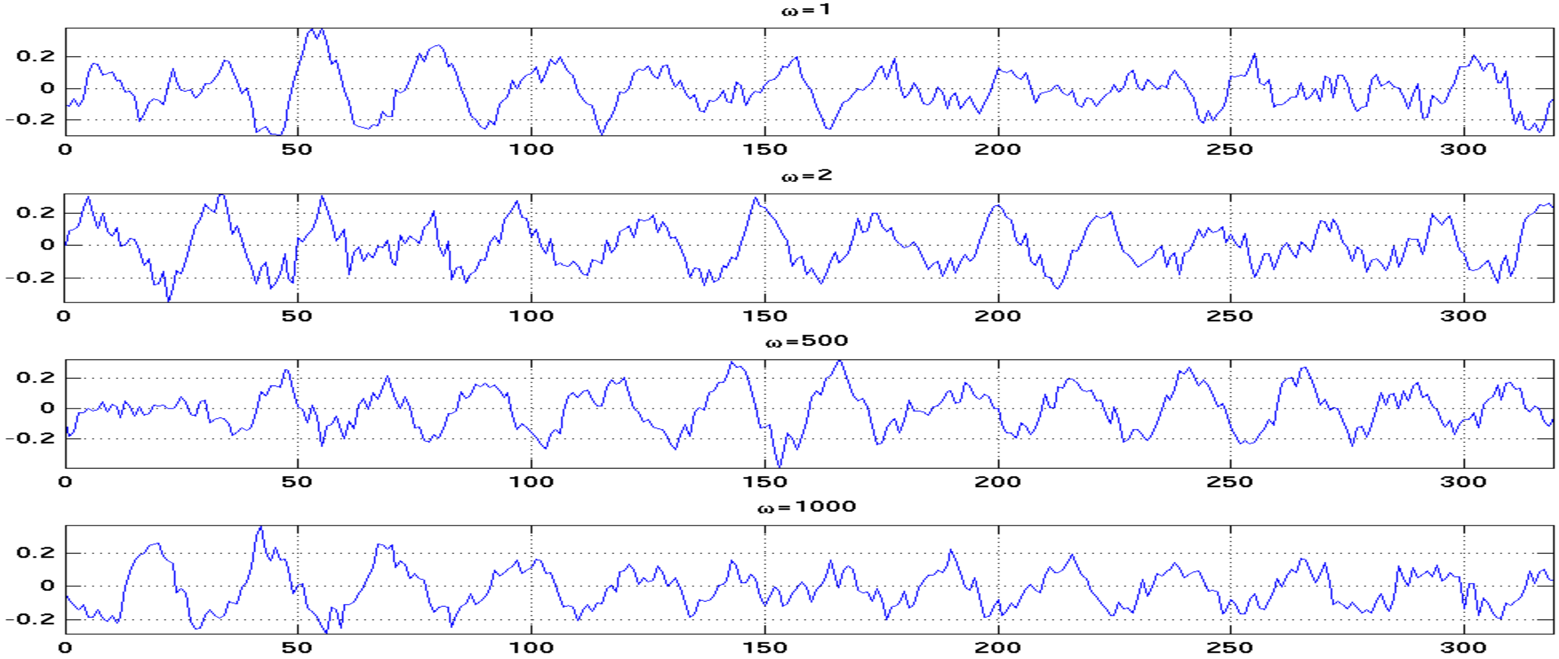
$$\xi[n] \in [h_1, h_2, \ldots h_H] = [0, 1, 2, \ldots, 36]$$

30	34	10	14	29	35	6	35	33	30	35	30	9	11	11	13	17	22	33	21
33	23	35	0	15	15	17	8	12	23	24	24	26	12	16	21	9	7	14	18
4	4	13	28	15	9	19	29	25	35	22	36	12	34	4	17	31	7	35	15
33	34	6	3	8	29	5	2	10	26	12	32	28	31	36	26	36	5	34	35
23	17	21	28	16	28	1	2	9	36	7	3	3	36	34	28	18	33	14	3
3	34	18	29	12	26	9	23	3	12	9	15	17	0	34	1	6	35	28	24
10	36	9	17	25	30	16	9	2	10	10	9	11	17	10	25	23	23	24	25
20	10	34	12	5	8	20	10	26	9	23	2	7	4	1	30	22	13	4	15
35	19	17	27	14	2	9	4	8	24	16	14	13	13	32	21	27	30	5	33
35	0	33	4	0	33	20	10	1	9	12	0	34	32	1	18	0	11	12	35
5	5	4	13	27	4	3	33	29	13	20	15	19	6	29	12	22	3	20	4
35	13	11	30	16	28	0	1	1	4	22	27	21	17	11	28	15	8	18	28
35	28	15	35	15	35	4	5	17	36	17	30	1	32	27	26	13	17	3	22
17	11	14	15	12	33	5	31	15	28	12	35	8	22	33	3	0	25	33	31
29	20	35	19	14	26	1	31	23	14	1	2	33	17	2	0	14	24	28	13



Continuous data

- $\Omega = 1068$ realizations of flowing water
- Each realization has 20ms, F_s=16kHz, so that N=320.



Describing random signal by **functions**

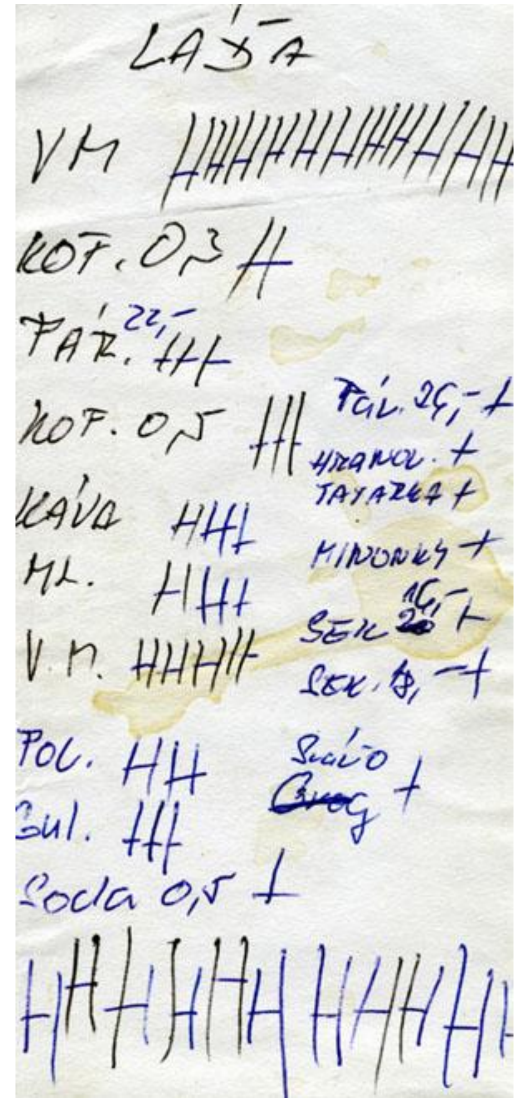
- CDF (cummulative distribution function)

$$F(x, n) = \mathcal{P}\{\xi[n] < x\}$$

- *x* is nothing random ! It is a value, for which we want to determine/measure CDF. For example „which percentage of population is shorter than 165cm?“ *x=165*

Estimation of probabilities of anything

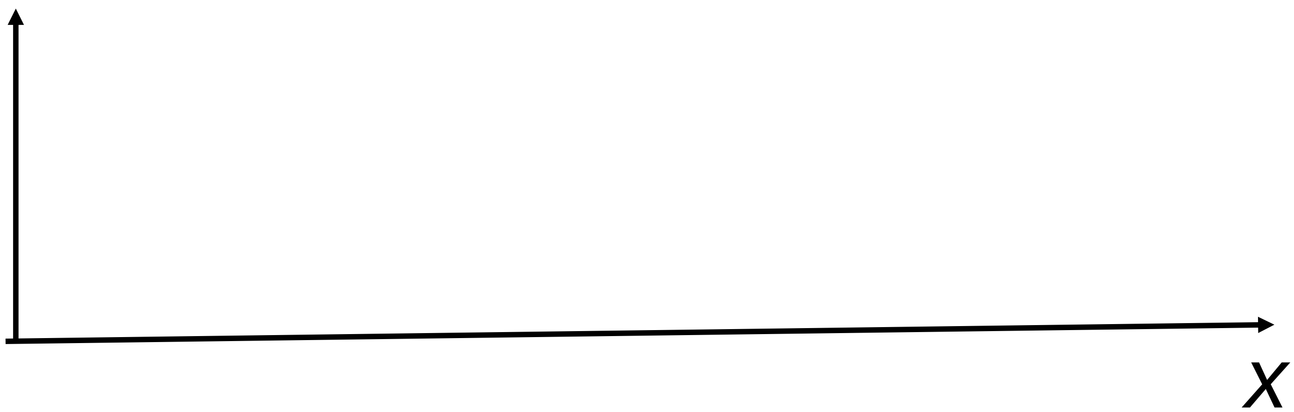
$$\text{probability} = \frac{\text{count}}{\text{total}}$$



Estimation of CDF from data

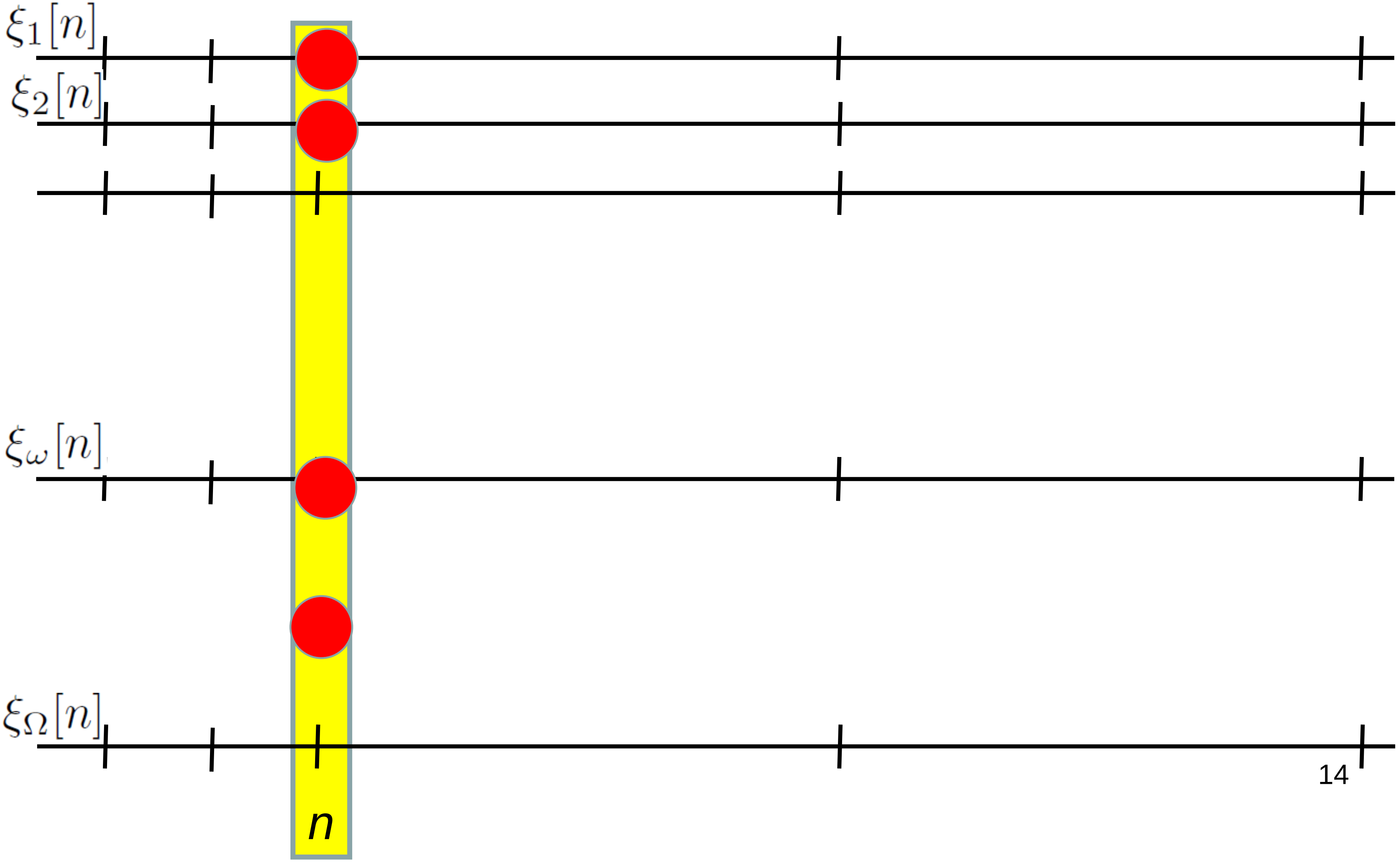
$$\hat{F}(x,n) = \frac{\mathrm{count}(\xi_\omega[n] < x)}{\Omega}$$

F(x,n)



How to divide *x* axis ?

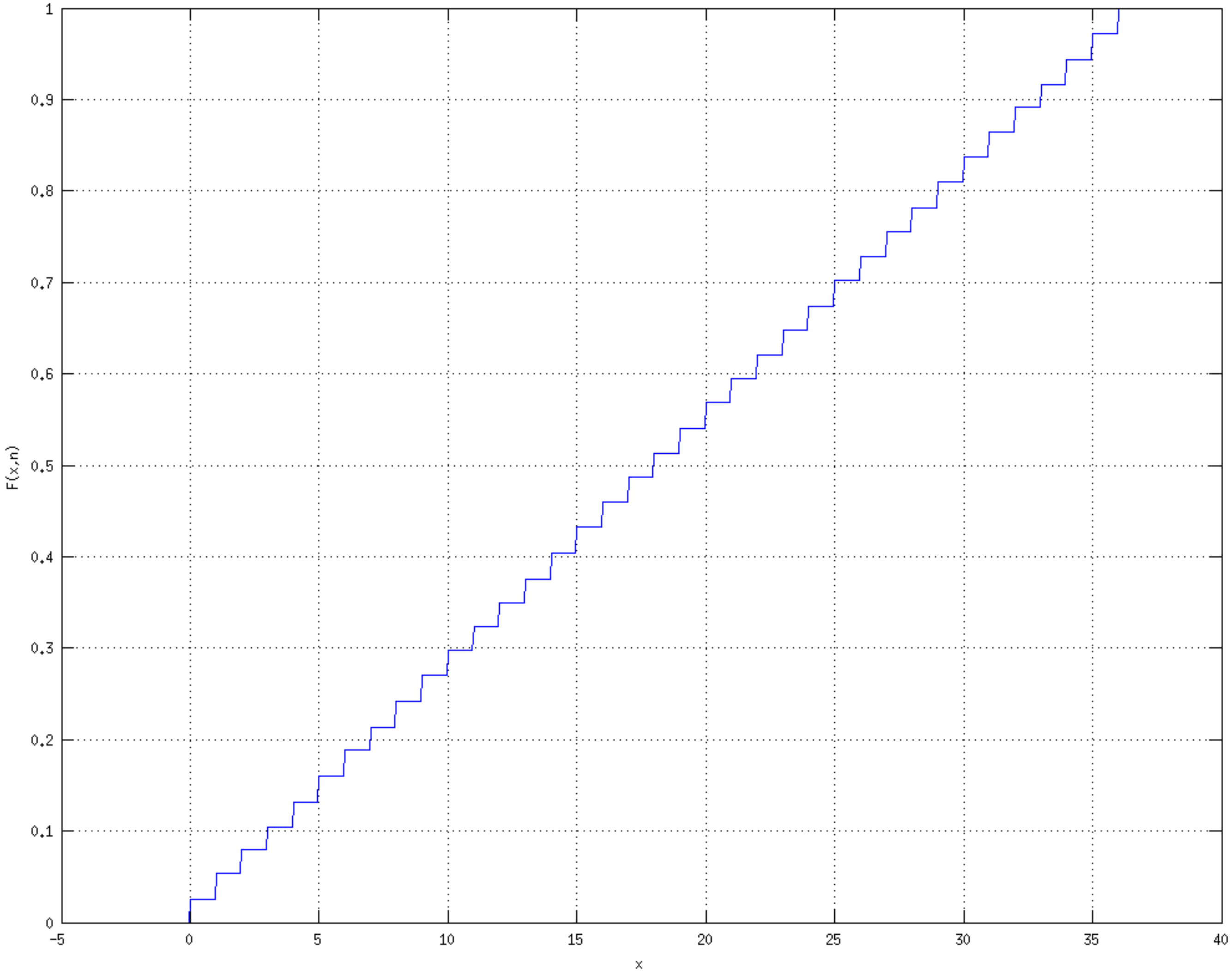
- Sufficiently fine
- But not useful in case the estimate is all the time the same.



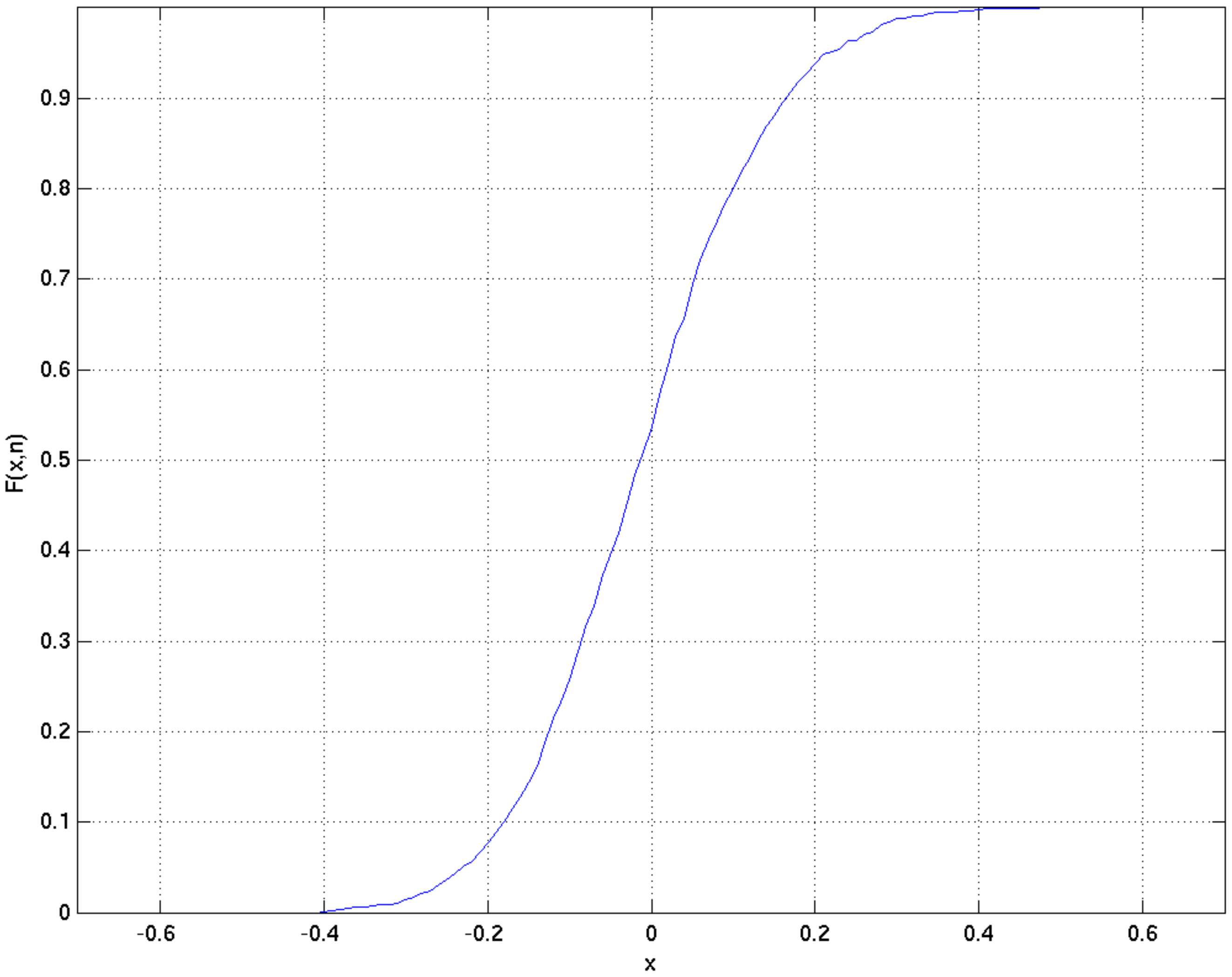
How many times was the value smaller than x=165 ?

P = 4 / 10, F(x,n) = 0.4

Estimation roulette



Estimation water



Probabilities of values

- Discrete range - OK $\mathcal{P}(X_i, n)$
- The mass of probabilities is

$$\sum_{\forall i} \mathcal{P}(X_i, n) = 1$$

- Estimation using the **counts**

$$\mathcal{P}(\hat{X}_i, n) = \frac{\mathrm{count}(X_i, n)}{\mathrm{total}[n]}$$

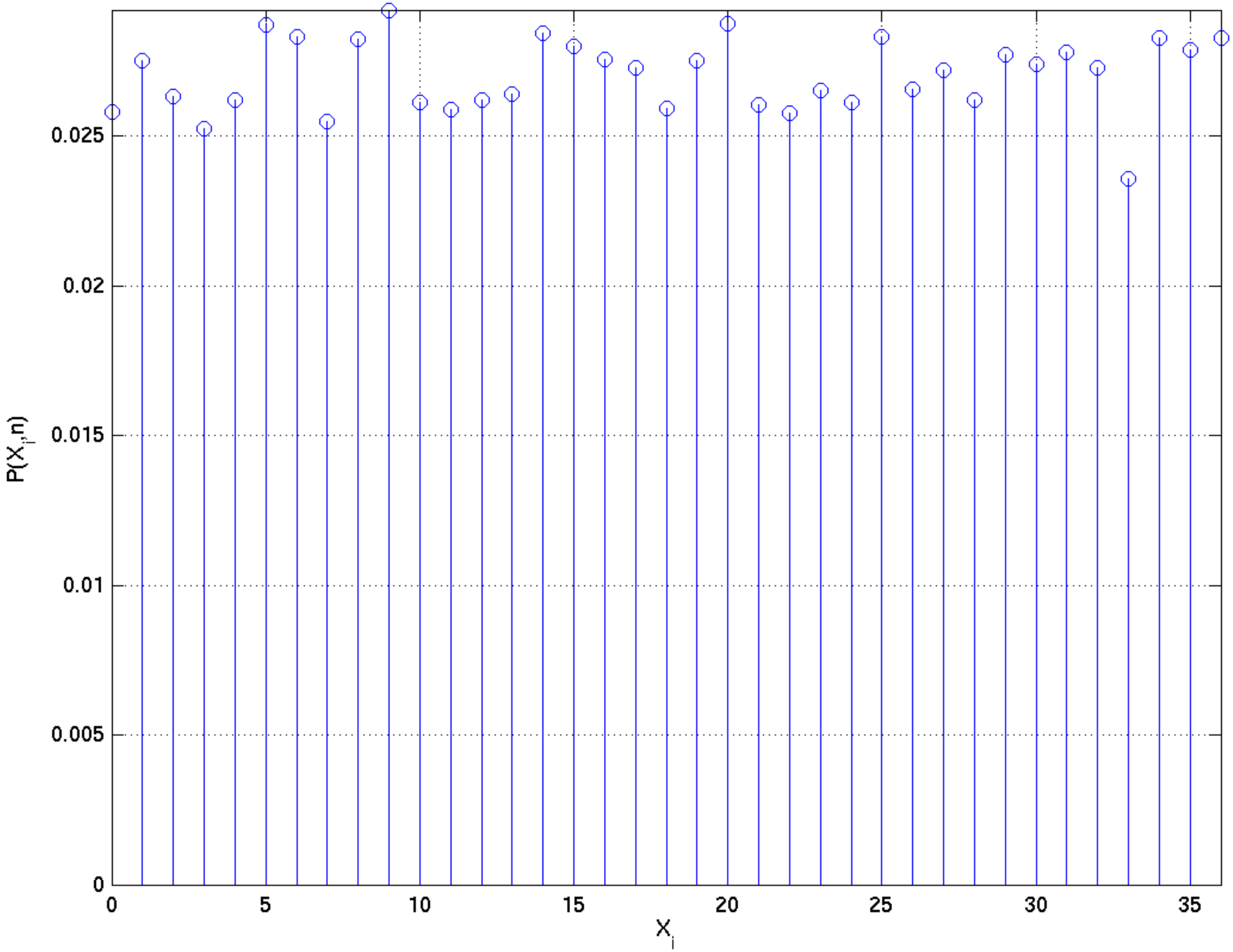
0

1

2

36

Result for roulette



$$\sum_{\forall i} \mathcal{P}(X_i, n) = 1$$

Continuous range

$$\mathcal{P}(x, n) = ???$$

- Nonsense or zero …

=> Needs **probability density**!

Real world examples

How many kms did the car run at time t ???

What is the mass of the ferment here, in coordinates x,y,z ???

Velocity

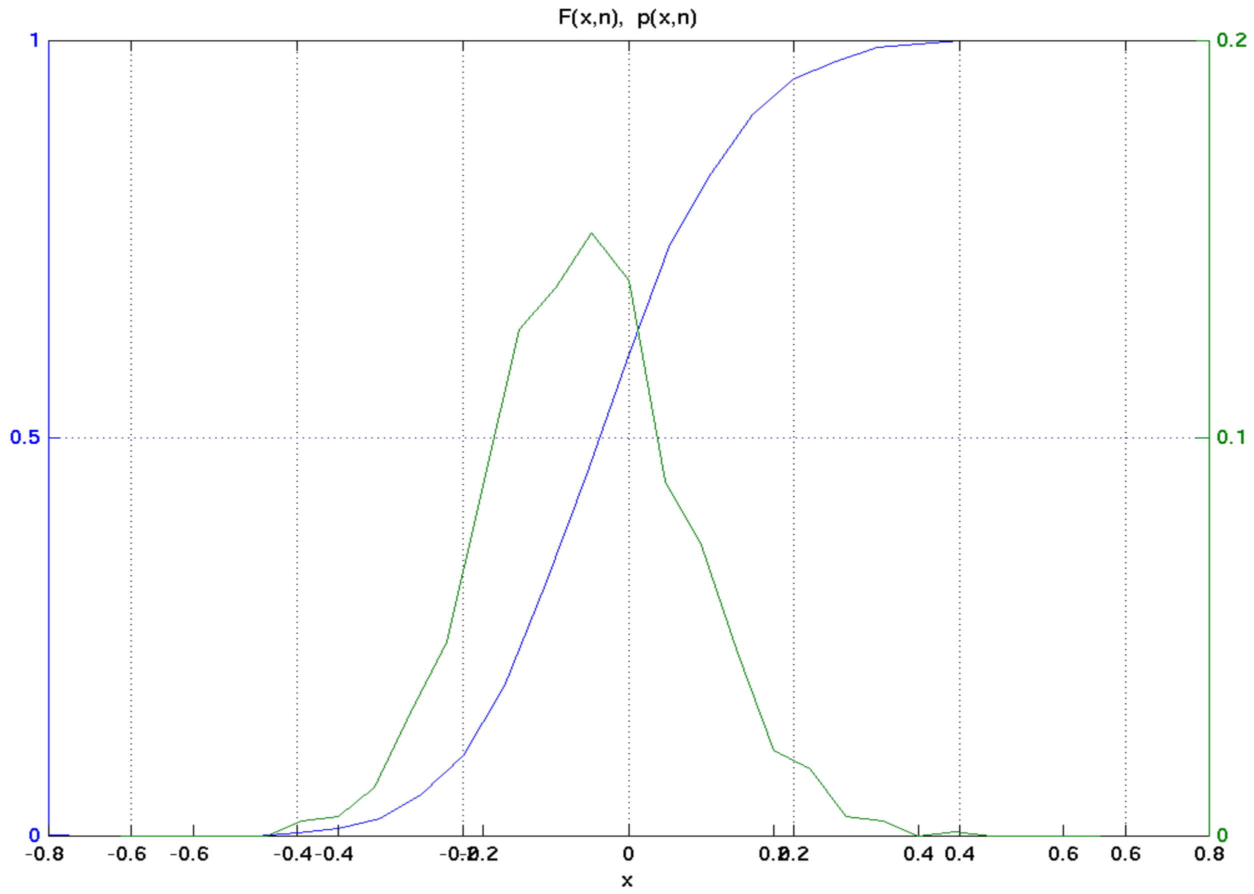
$$v(t) = \frac{dl(t)}{dt} = \frac{l(t_2) - l(t_1)}{t_2 - t_1} = \frac{\Delta l}{\Delta t}$$

Density

$$\rho(x, y, z) = \frac{dm}{dV} = \frac{m(x_1 \ldots x_2, y_1 \ldots y_2, z_1 \ldots z_2)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)} = \frac{\Delta m}{\Delta V}$$

Probability density function - PDF

$$p(x,n) = \frac{dF(x,n)}{dx}$$



Can we estimate it more easily?

$$v(t) = \frac{dl(t)}{dt} = \frac{l(t_2) - l(t_1)}{t_2 - t_1} = \frac{\Delta l}{\Delta t}$$

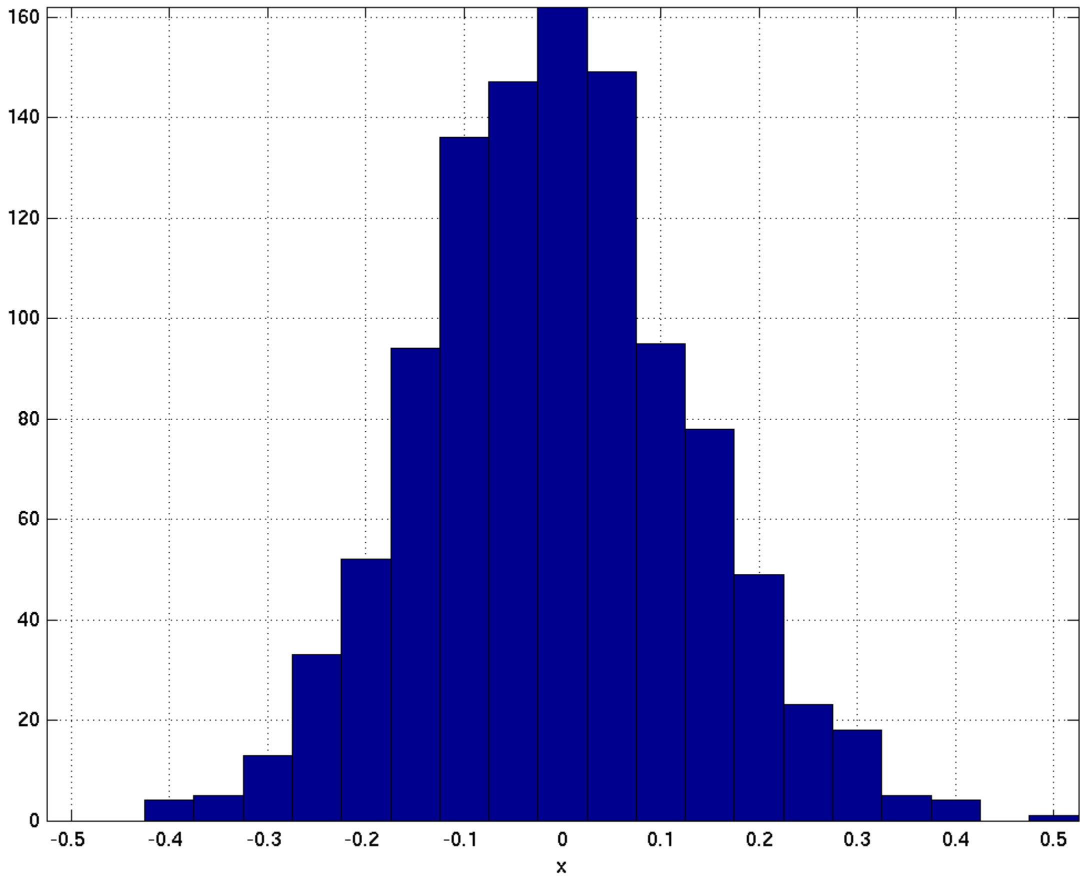
$$\rho(x, y, z) = \frac{dm}{dV} = \frac{m(x_1 \ldots x_2, y_1 \ldots y_2, z_1 \ldots z_2)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)} = \frac{\Delta m}{\Delta V}$$

$$p(x, n) = \frac{\text{probability}}{\text{normalization}}$$

Probabilities of **values** are nonsense, but we can use probabilities of intervals – **bins !**

Histogram

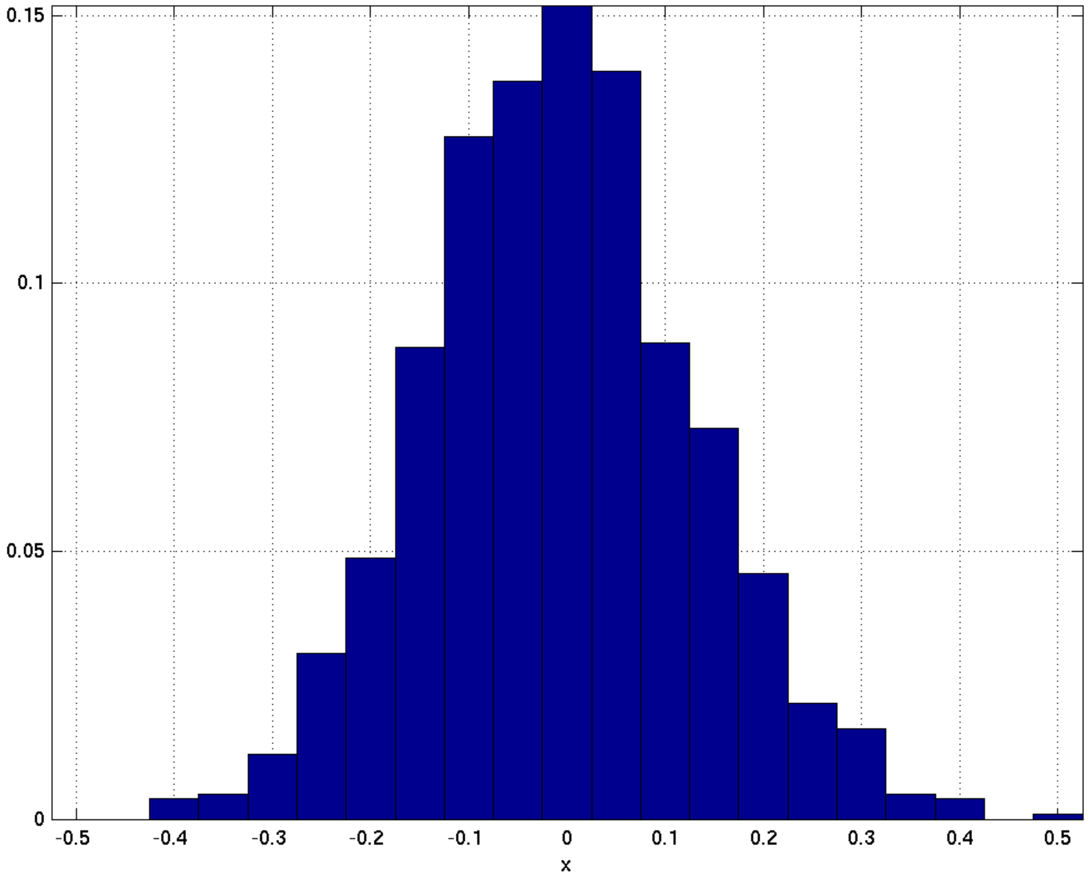
$$\text{histogram}(x \in interval, n) = \text{count}(x \in interval, n)$$



Bins !

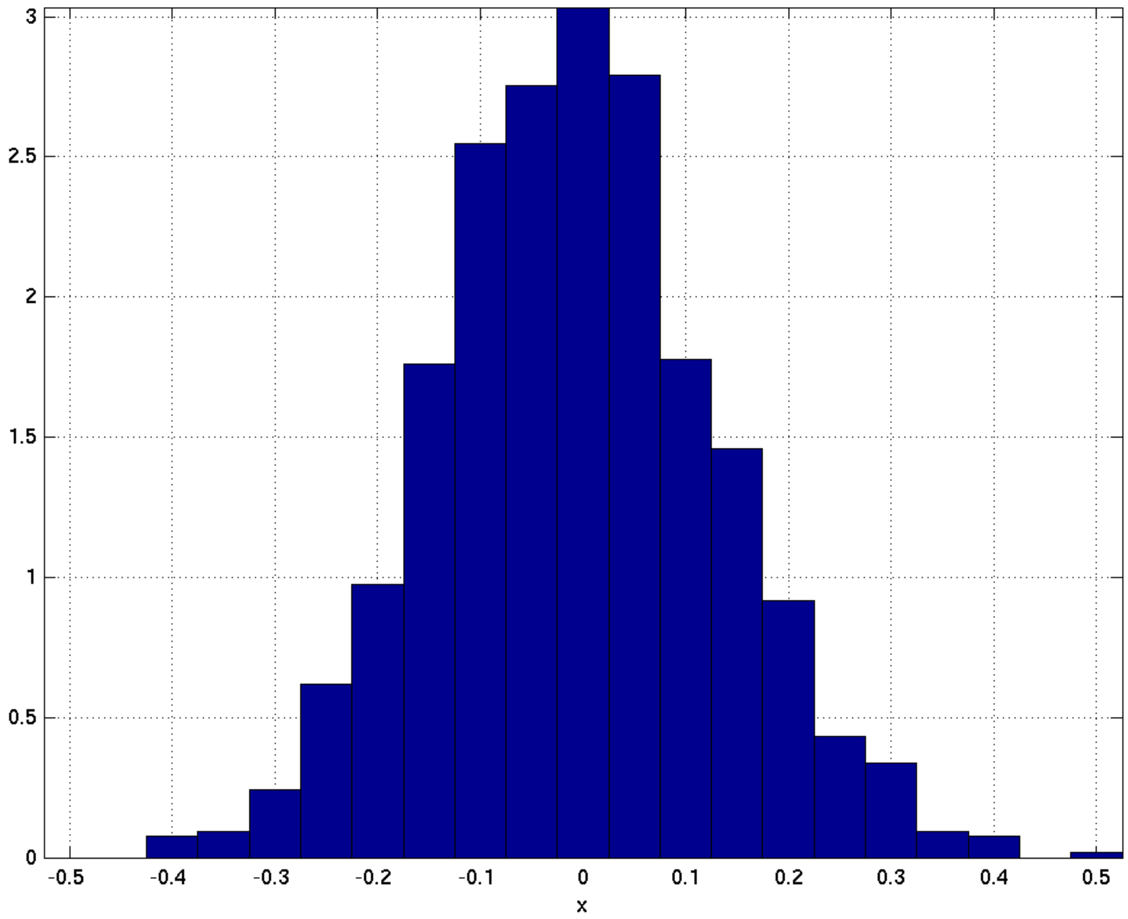
Probability

$$\mathcal{P}(x \in interval, n) = \frac{\text{count}(x \in interval, n)}{\Omega}$$



Probability density

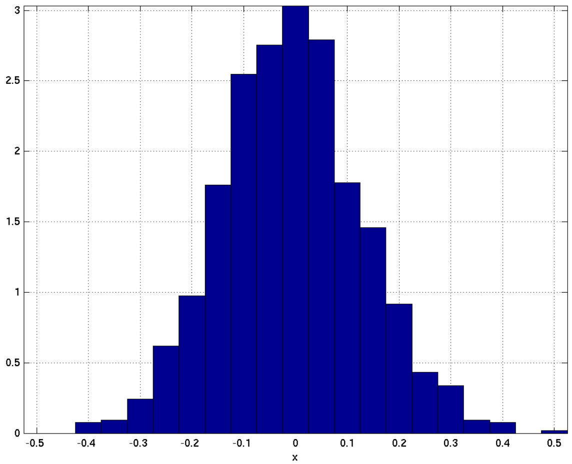
$$p(x \in interval, n) = \frac{\text{count}(x \in interval, n)}{\Omega|interval|}$$



How about the whole thing ?

$$\int_t v(t) =??$$

$$\iiint_V \rho(x,y,z) =??$$



$$\int_{x=-\infty}^{+\infty} p(x,n) = 1$$

Check this using the bins …

Joint probability or probability density function

- Any relations between samples in different times ?
- Are they independent or is there a link ?

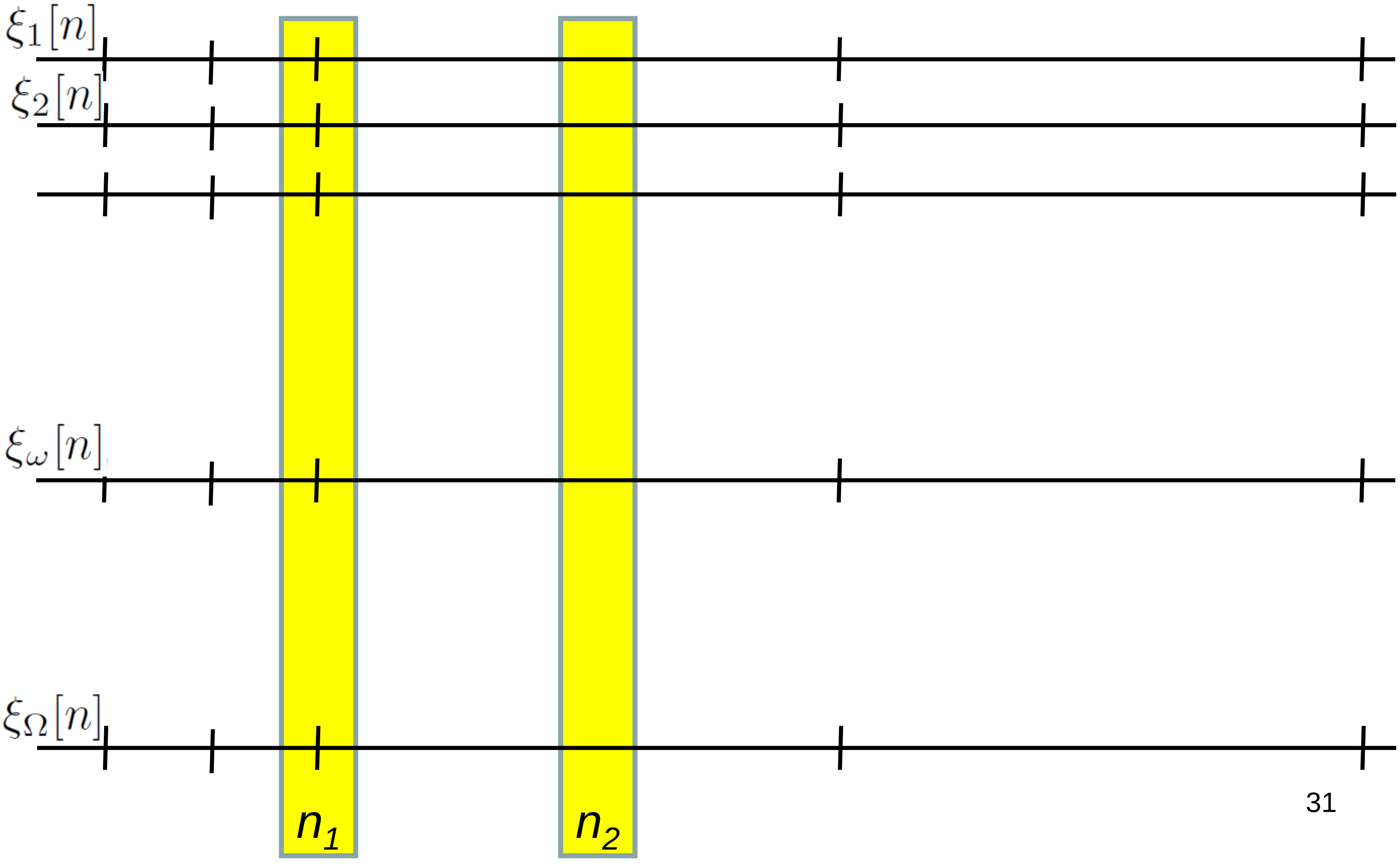
$$\mathcal{P}(X_i, X_j, n_1, n_2)$$

$$p(x_i, x_j, n_1, n_2)$$

Good for ?

- Looking for dependencies
- Spectral analysis

Two different times…

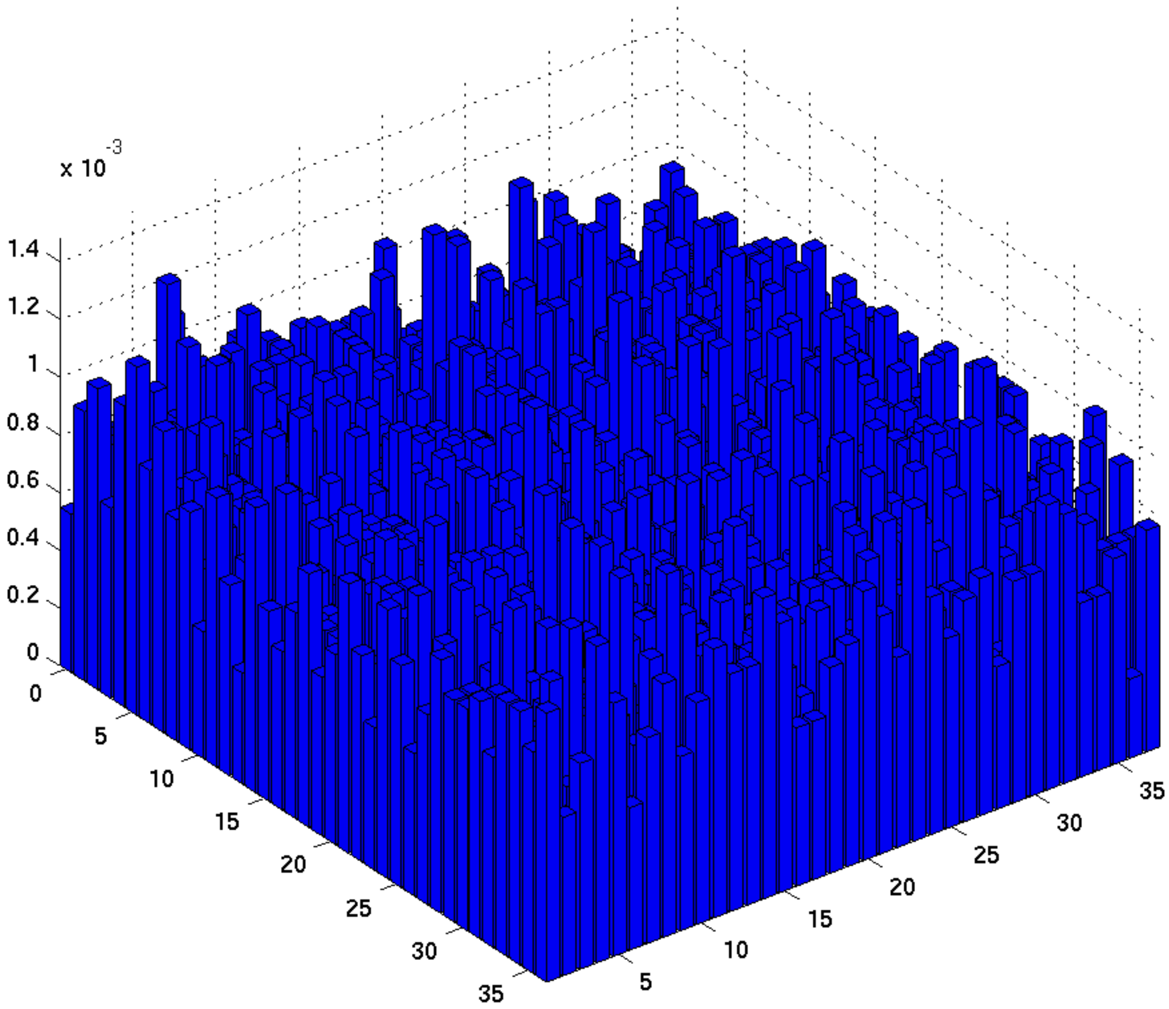


Estimations – again questions, now with “**and**”

Something at time n_1 **and** Something at time n_2

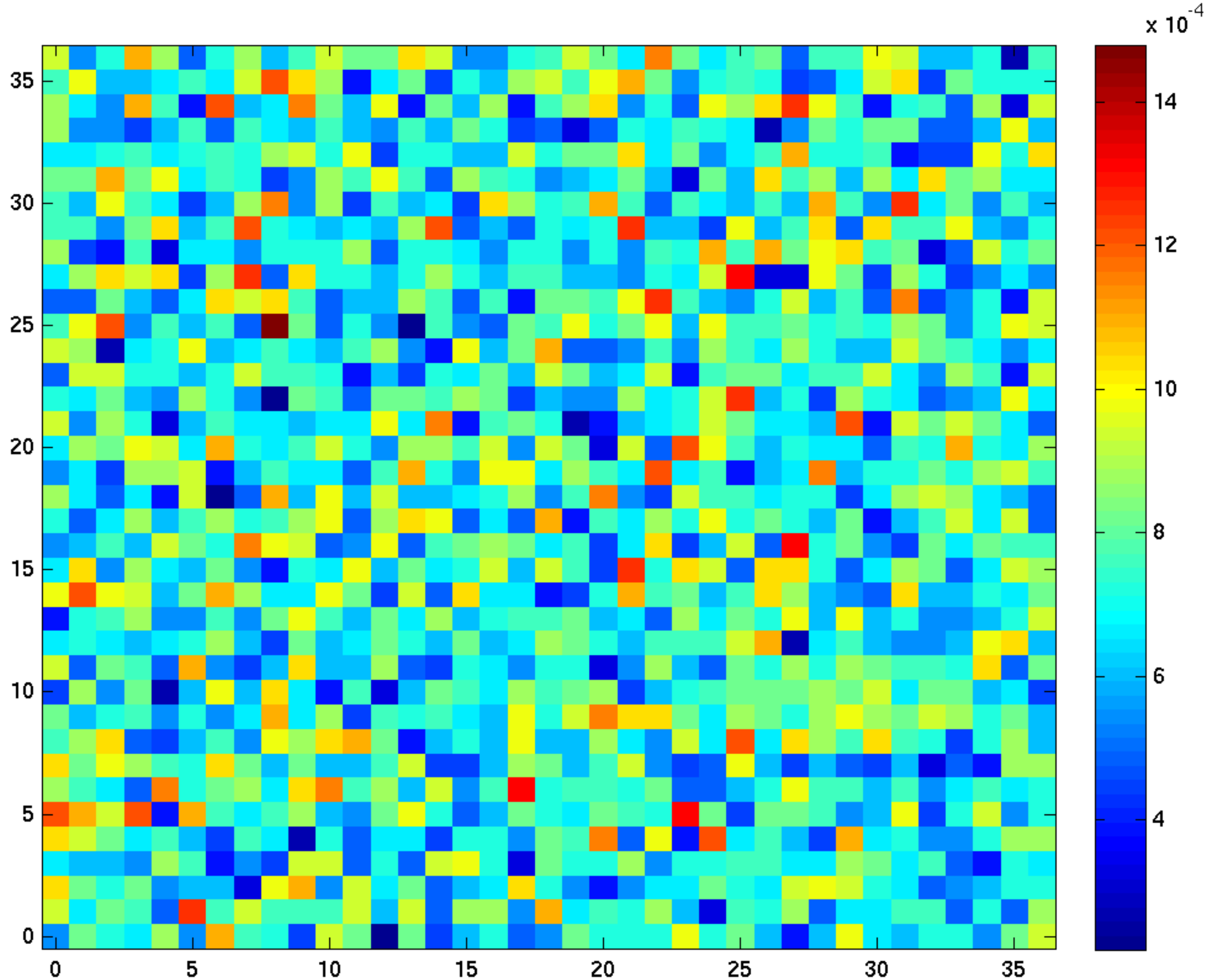
$$\text{joint probability} = \frac{\text{count that something happened } \textbf{simultaneously} \text{ in } n_1 \ \textbf{AND} \ n_2}{\text{total}}$$

Joint counts: n_1=10, n_2=11



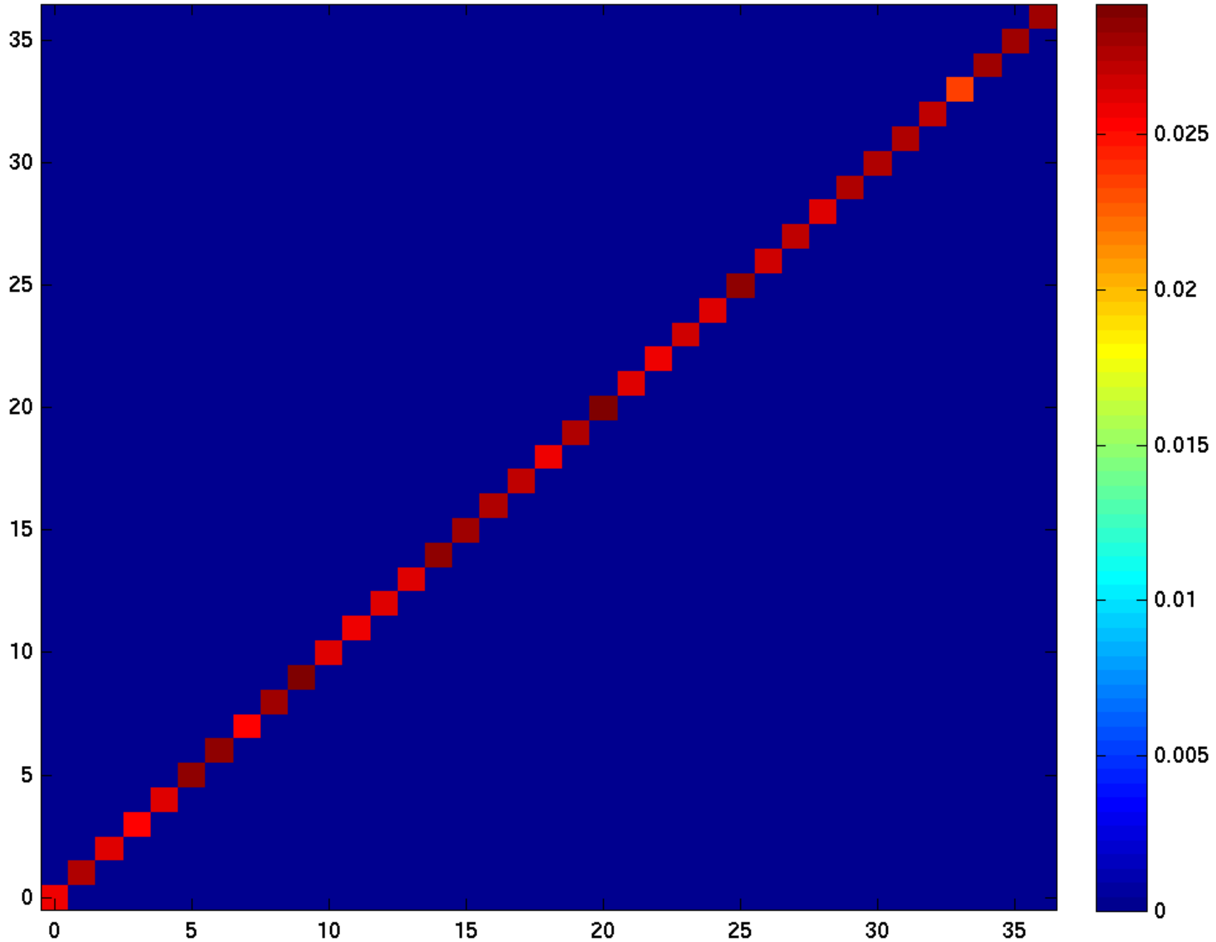
Joint probabilities: n_1=10, n_2=11

$$\hat{\mathcal{P}}(X_i, X_j, n_1, n_2) = \frac{\text{count}(\xi[n_1] = X_i \ \ \textbf{AND} \ \ \xi[n_2] = X_2)}{\Omega}$$



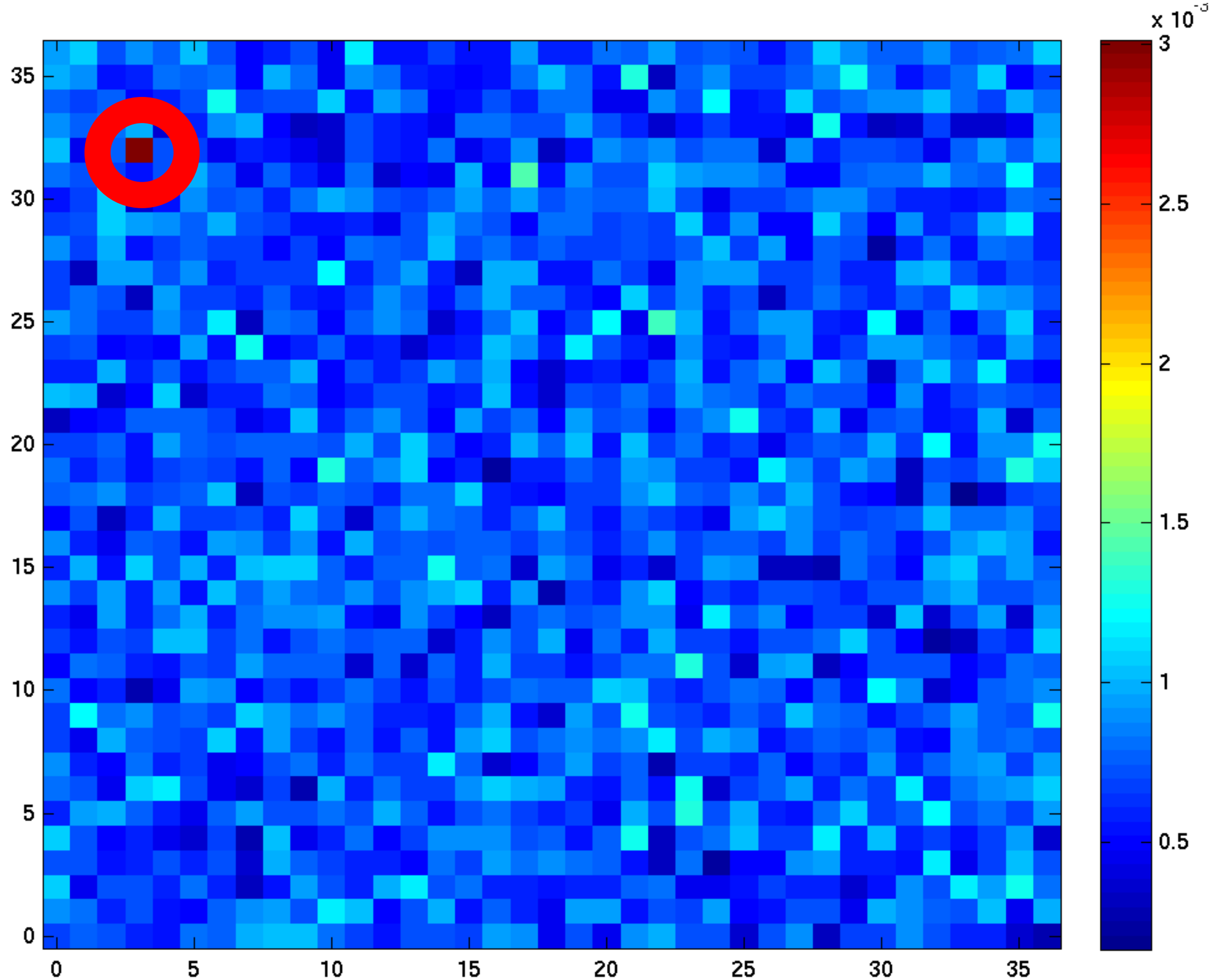
Joint probabilities: n_1=10, n_2=10

$$\hat{\mathcal{P}}(X_i, X_j, n_1, n_2) = \frac{\text{count}(\xi[n_1] = X_i \ \ \textbf{AND} \ \ \xi[n_2] = X_2)}{\Omega}$$



Joint probabilities: n_1=10, n_2=13

$$\hat{\mathcal{P}}(X_i, X_j, n_1, n_2) = \frac{\text{count}(\xi[n_1] = X_i \ \ \textbf{AND} \ \ \xi[n_2] = X_2)}{\Omega}$$



Continuous range

- Probabilities will not work…

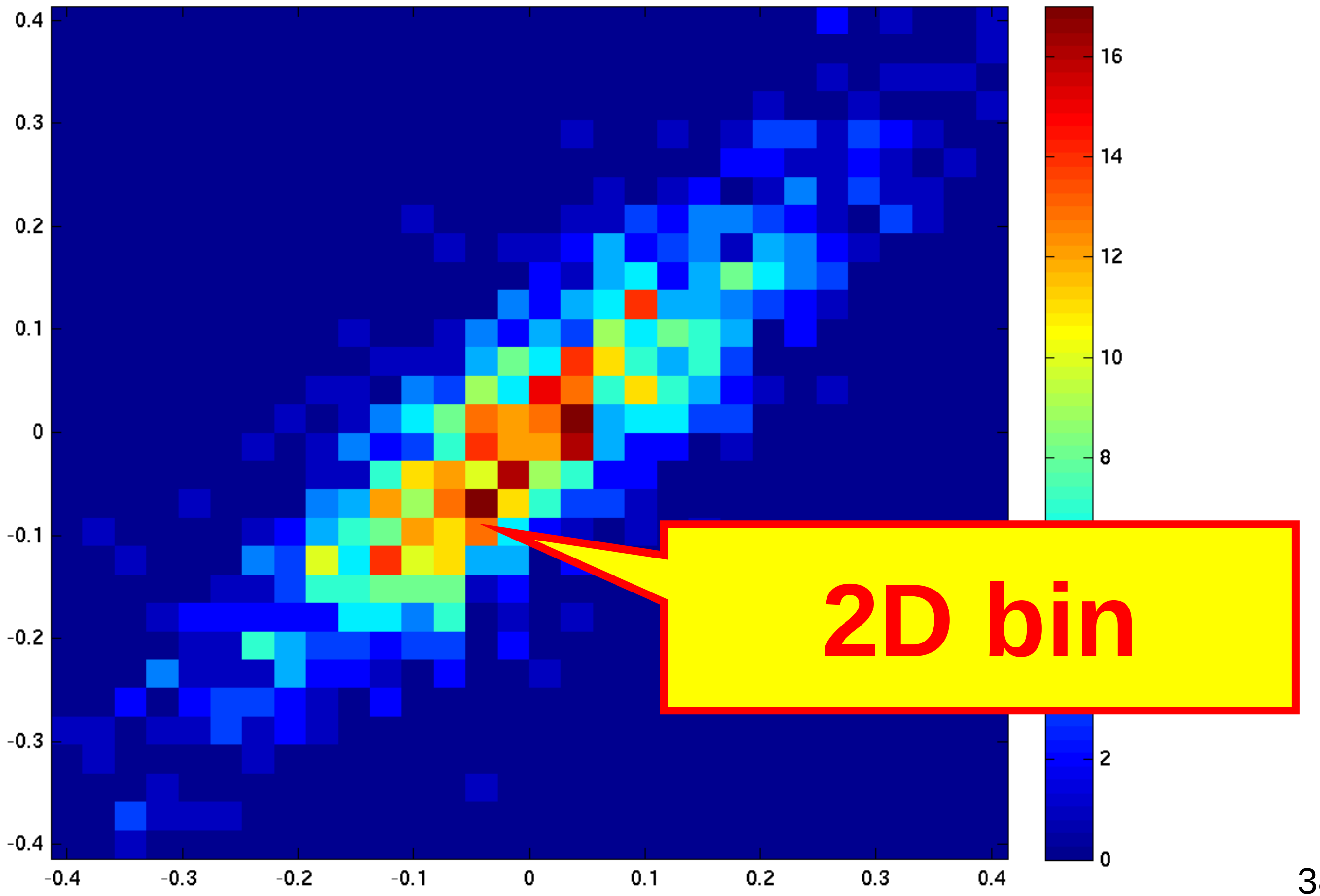
Histogram

=> Probabilities of 2D bins

=> Probability densities in 2D bins

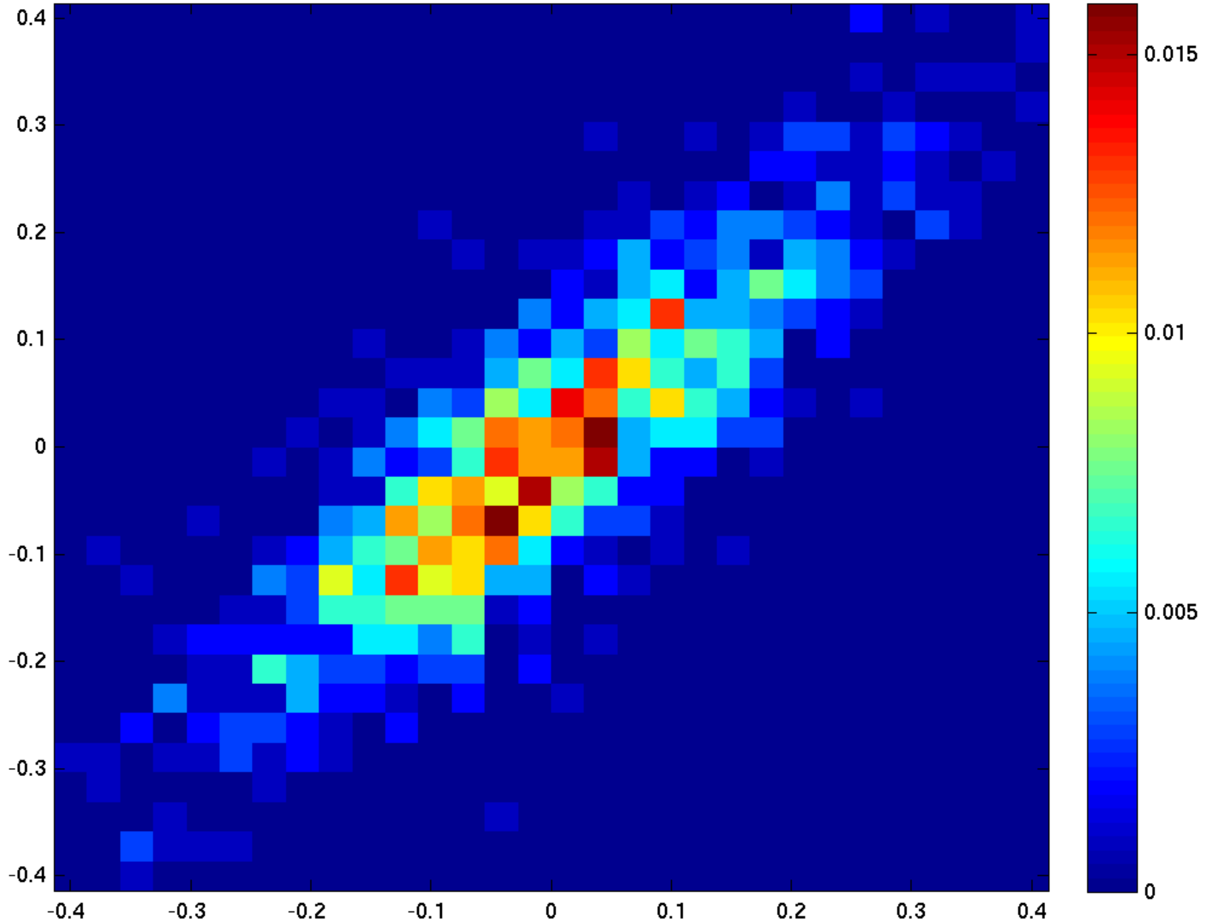
Joint histogram – counts, n_1=10, n_2=11

$$\text{histogram}(x_1 \in interval_1, x_2 \in interval_2, n_1, n_2) = \text{count}(x_1 \in interval_1, n_1 \;\; \textbf{AND} \;\; x_2 \in interval_2, n_2)$$



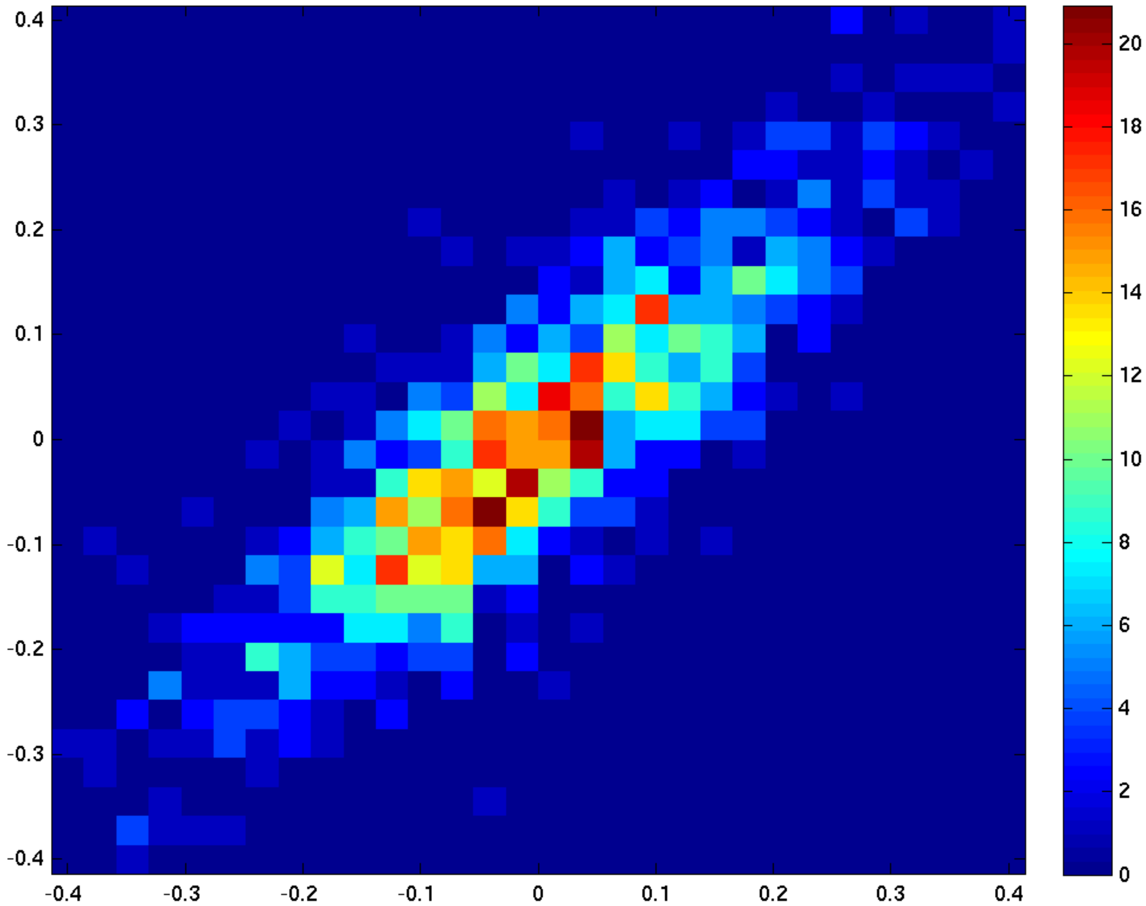
Joint probabilities of bins, n_1=10, n_2=11

$$\mathcal{P}(x_1 \in interval_1, x_2 \in interval_2, n_1, n_2) = \frac{\text{count}(x_1 \in interval_1, n_1 \quad \textbf{AND} \quad x_2 \in interval_2, n_2)}{\Omega}$$



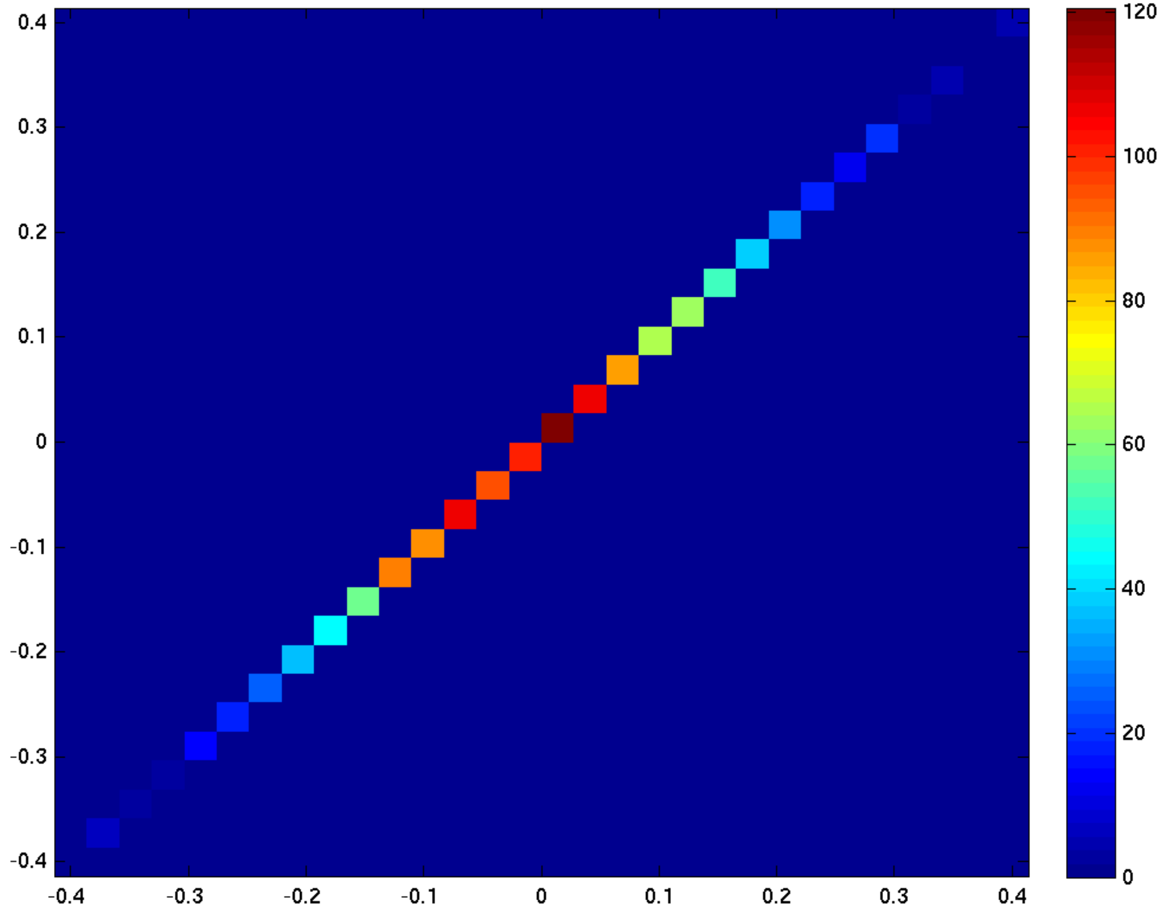
Joint probability density function, n_1=10, n_2=11

$$p(x_1 \in interval_1, x_2 \in interval_2, n_1, n_2) = \frac{\text{count}(x_1 \in interval_1, n_1 \ \ \textbf{AND} \ \ x_2 \in interval_2, n_2)}{\Omega |interval_1||interval_2|}$$



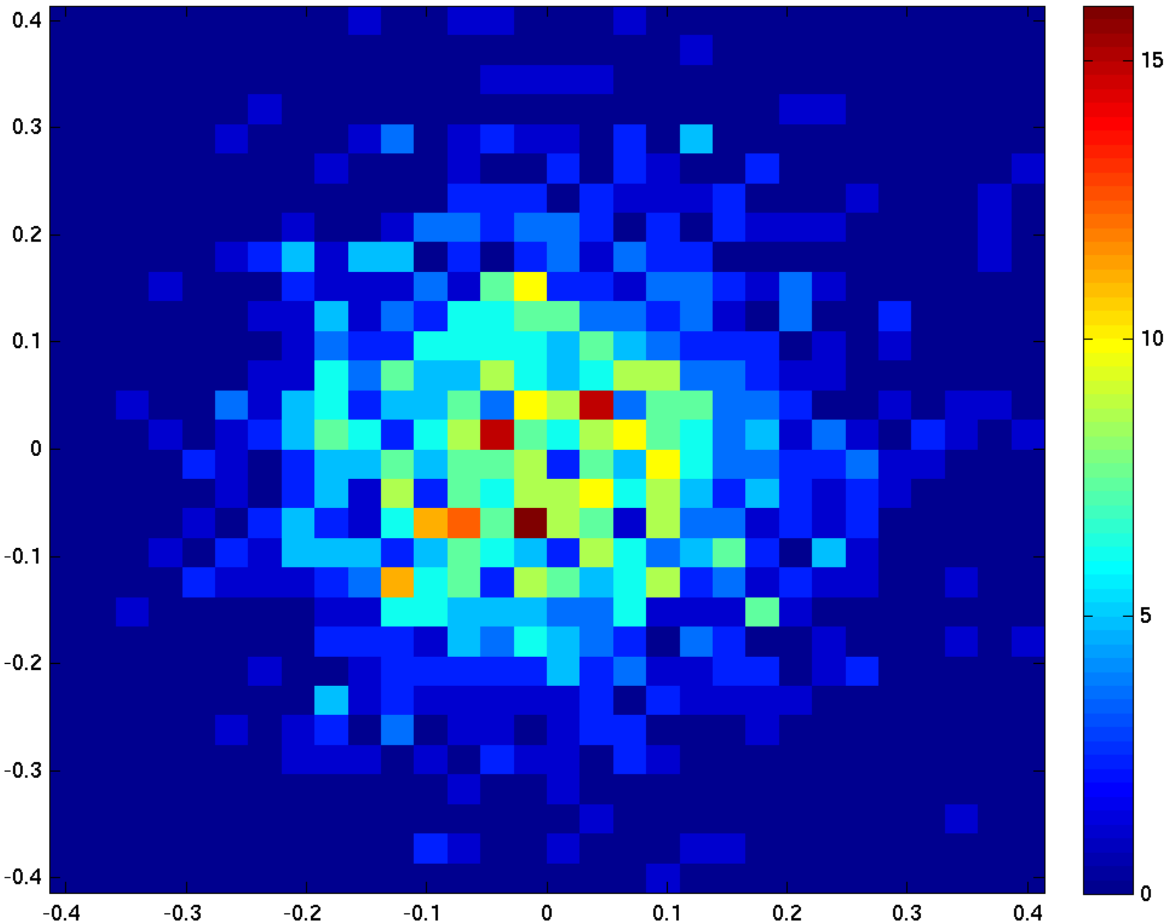
Joint probability density function, n_1=10, n_2=10

$$p(x_1 \in interval_1, x_2 \in interval_2, n_1, n_2) = \frac{\text{count}(x_1 \in interval_1, n_1 \;\; \textbf{AND} \;\; x_2 \in interval_2, n_2)}{\Omega |interval_1||interval_2|}$$



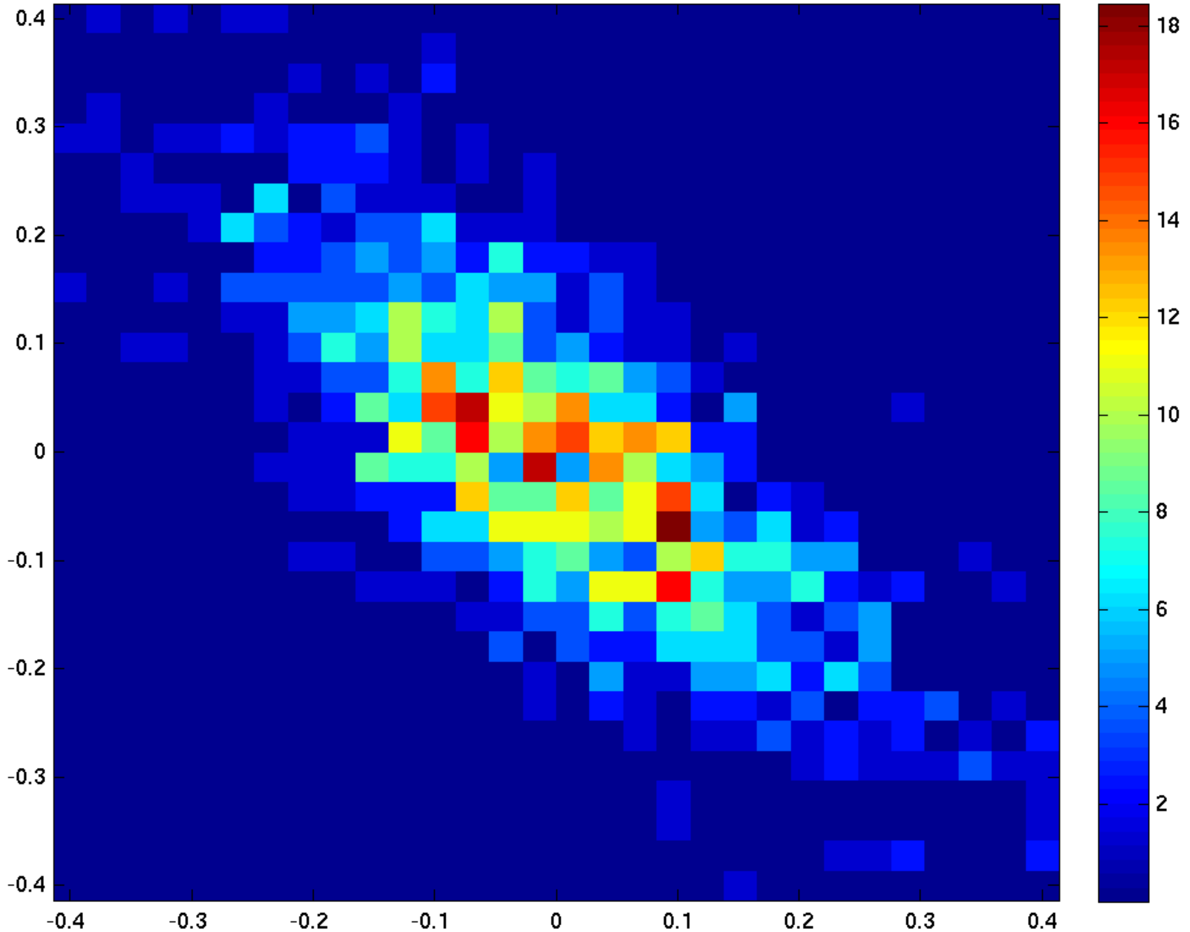
Joint probability density function, n_1=10, n_2=16

$$p(x_1 \in interval_1, x_2 \in interval_2, n_1, n_2) = \frac{\text{count}(x_1 \in interval_1, n_1 \;\; \textbf{AND} \;\; x_2 \in interval_2, n_2)}{\Omega |interval_1||interval_2|}$$



Joint probability density function, n_1=10, n_2=23

$$p(x_1 \in interval_1, x_2 \in interval_2, n_1, n_2) = \frac{\text{count}(x_1 \in interval_1, n_1 \ \ \textbf{AND} \ \ x_2 \in interval_2, n_2)}{\Omega|interval_1||interval_2|}$$



Moments

- Single numbers characterizing the random signal.
- Still at time *n*
- Expectation of something

Expectation = sum $_{\text{all possible values of } x}$
probability of *x*
times the thing that we're expecting

Sometimes a sum, sometimes an integral.

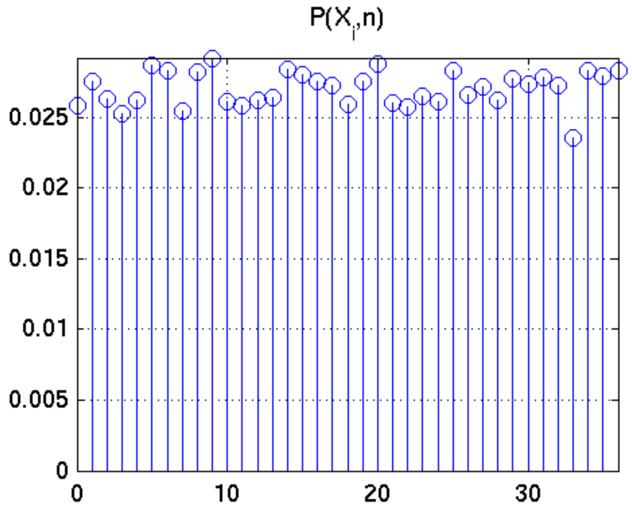
Mean value

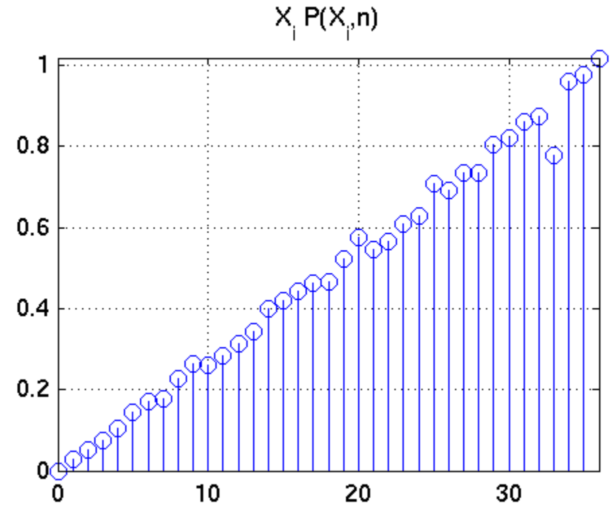
- Expectation of the value

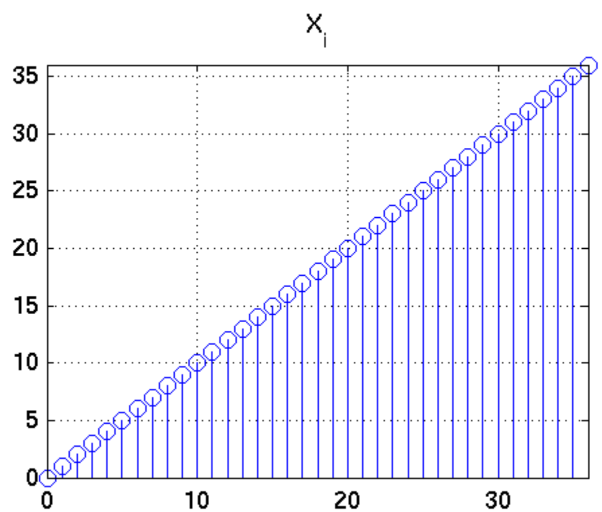
$$a[n] = E\{\xi[n]\}$$

Mean value – discrete range

$$a[n] = \sum_{\forall X_i} \mathcal{P}(X_i, n) X_i$$



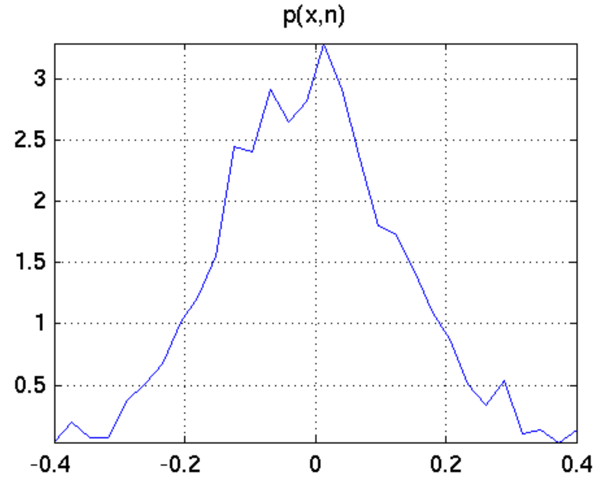


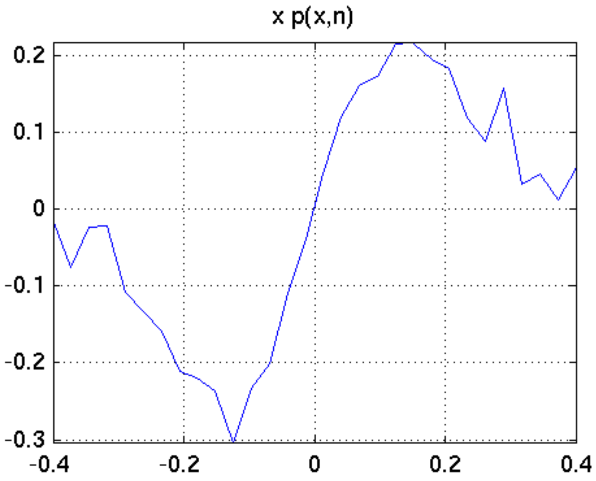


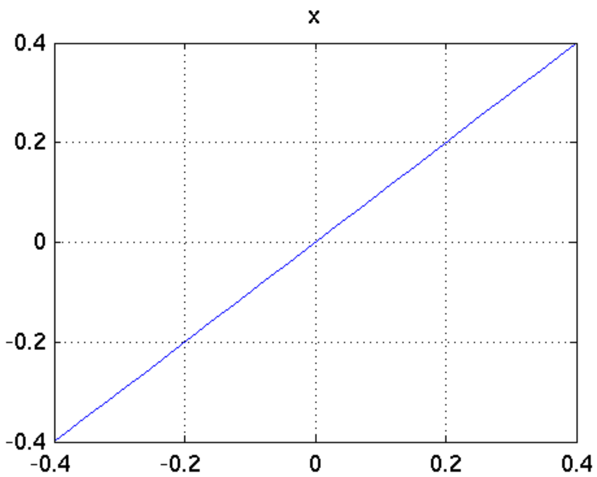
a[10] = 18.0422

Mean value – continuous range

$$a[n] = \int_x p(x,n)x dx$$







a[10] = -0.0073

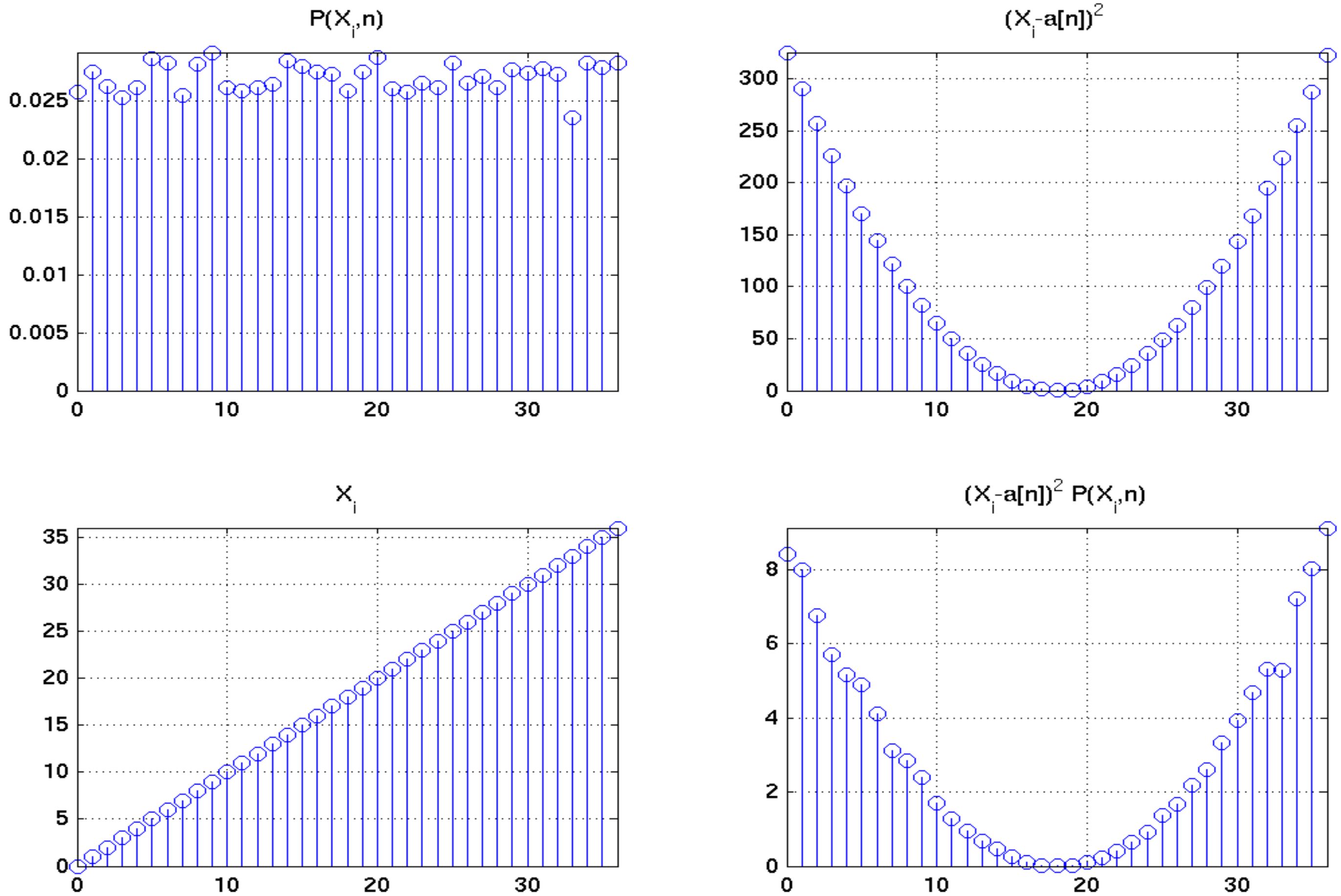
Variance (dispersion)

- Expectation of zero-mean value squared
- Energy, power …

$$D[n] = E\{(\xi[n] - a[n])^2\}$$

Variance – discrete range

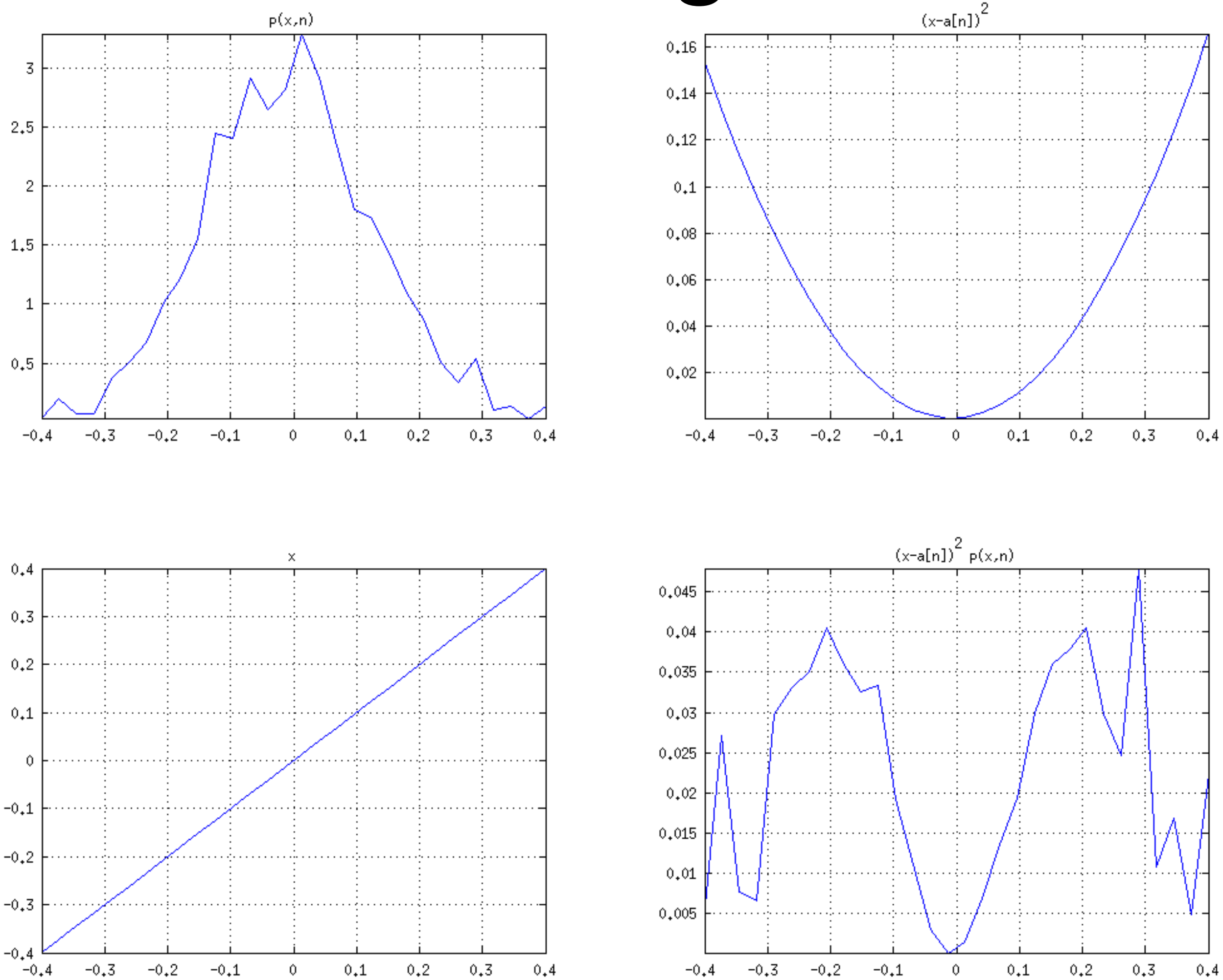
$$D[n] = \sum_{\forall X_i} \mathcal{P}(X_i, n)(X_i - a[n])^2$$



D[10] = 113.8563

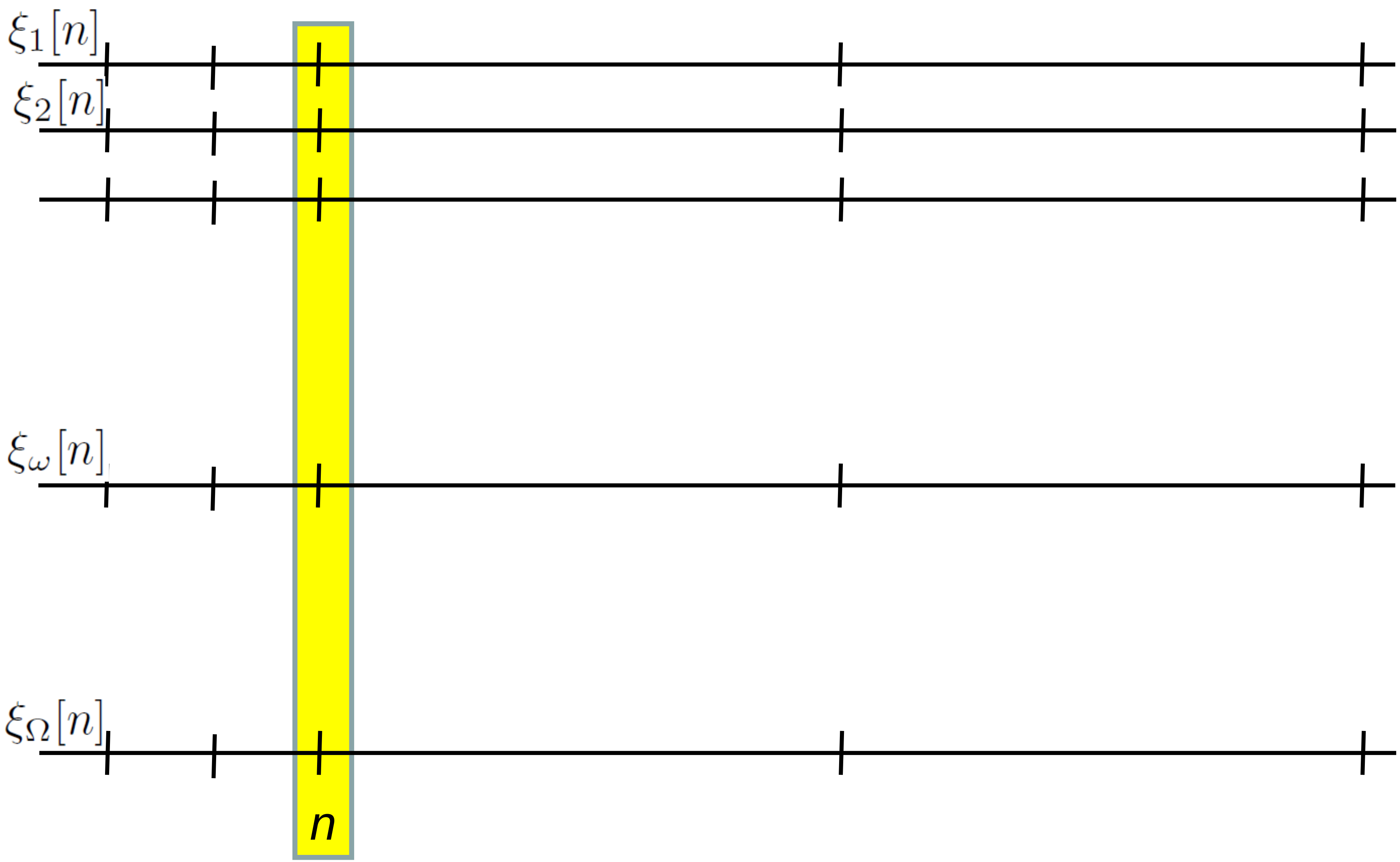
Variance – continuous range

$$D[n] = \int_x p(x,n)(x - a[n])^2 dx$$



D[10] =0.0183

Ensemble estimates



You know this from elementary school …

- Discrete range (roulette)
- $n_1 = 10$

$$\hat{a}[n] = \frac{1}{\Omega}\sum_{\omega=1}^{\Omega}\xi_\omega[n]$$

$\hat{a}[10] = 18.0422$

$$\hat{D}[n] = \frac{1}{\Omega}\sum_{\omega=1}^{\Omega}(\xi_\omega[n] - \hat{a}[n])^2$$

$\hat{D}[10] = 113.8563$

You know this from elementary school …

- Continuous range (water)
- $n_1 = 10$

$$\hat{a}[n] = \frac{1}{\Omega} \sum_{\omega=1}^{\Omega} \xi_\omega[n]$$

$\hat{a}[10] = -0.0069$

$$\hat{D}[n] = \frac{1}{\Omega} \sum_{\omega=1}^{\Omega} (\xi_\omega[n] - \hat{a}[n])^2$$

$\hat{D}[10] = 0.0183$

… The equations are the same ☺

Correlation coefficient

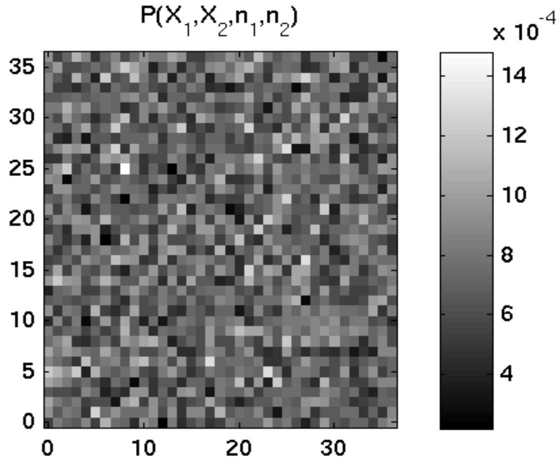
- Expectation of product of values from two different times

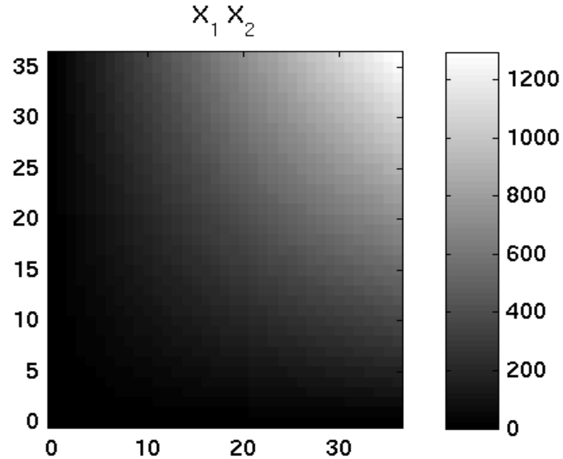
$$R[n_1, n_2] = E\{\xi[n_1]\xi[n_2]\}$$

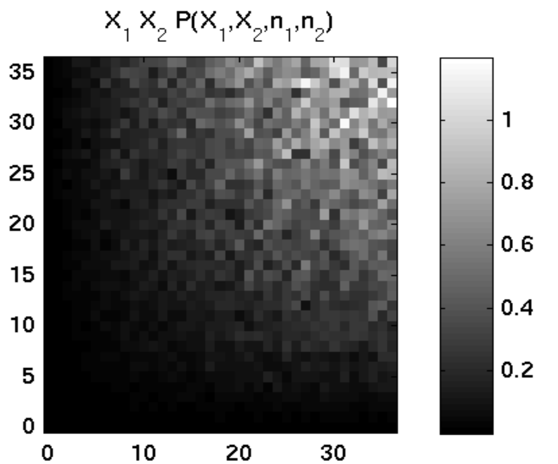
- What does it mean when $R[n_1, n_2]$ is
  - Big ?
  - Small or zero ?
  - Big negative ?

Discrete range, n_1=10, n_1=11

$$R[n_1, n_2] = \sum_{\forall X_1} \sum_{\forall X_2} \mathcal{P}(X_1, X_2, n_1, n_2) X_1 X_2$$



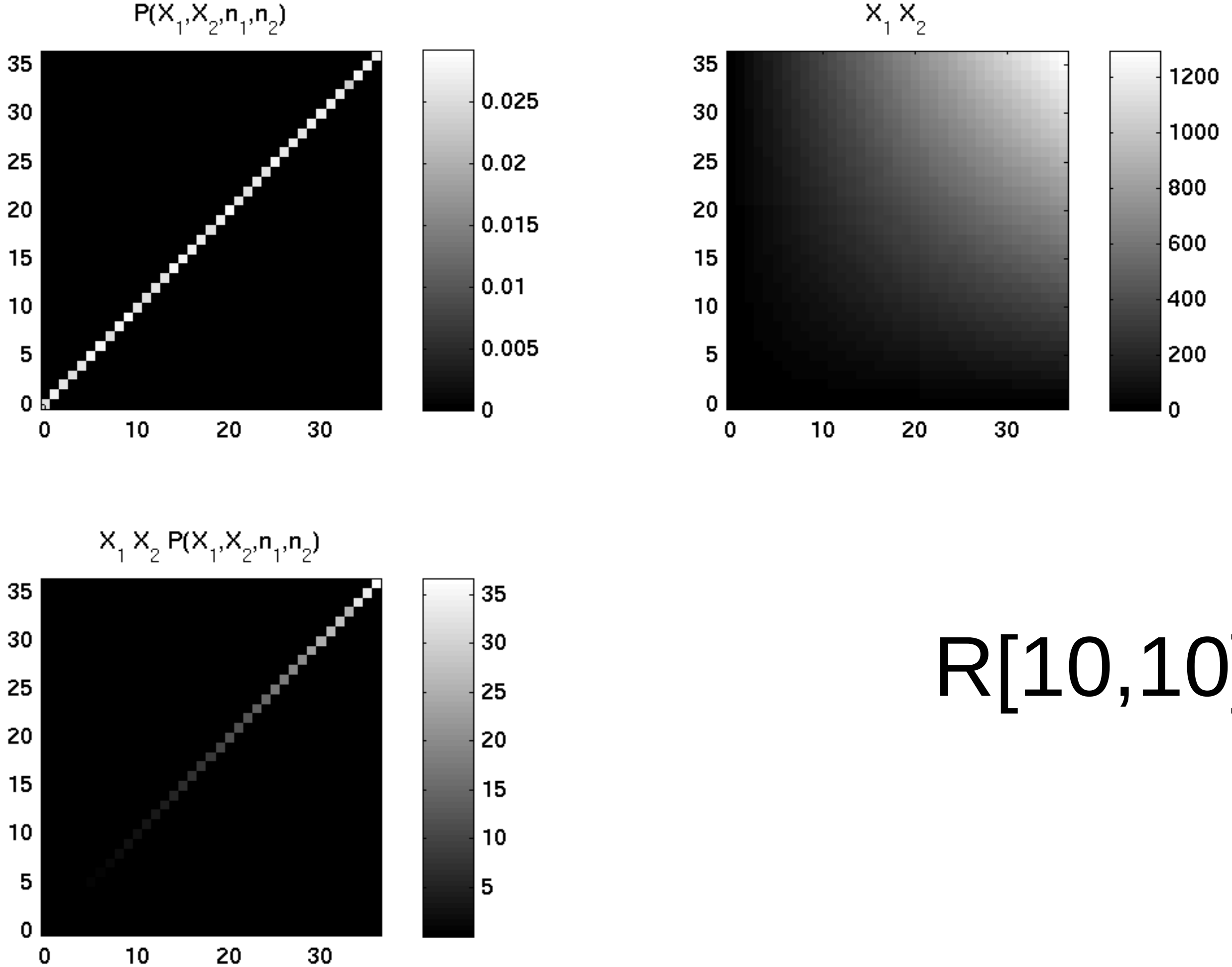




R[10,11] = 324.2020

Discrete range, n_1=10, n_2=10

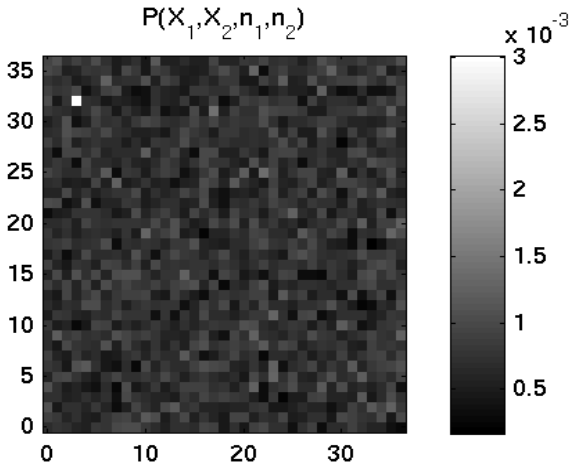
$$R[n_1, n_2] = \sum_{\forall X_1} \sum_{\forall X_2} \mathcal{P}(X_1, X_2, n_1, n_2) X_1 X_2$$

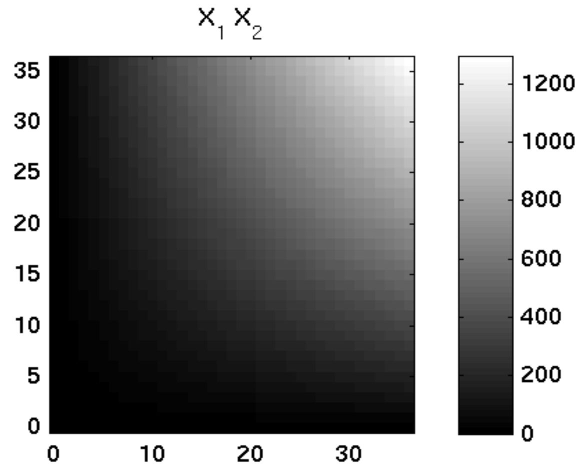


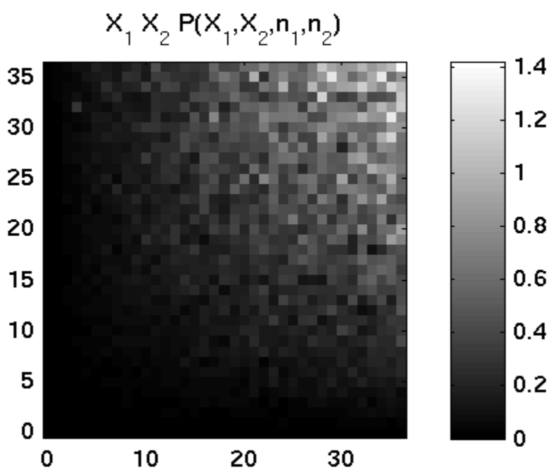
R[10,10] = 439.3770

Discrete range, n_1=10, n_2=13

$$R[n_1, n_2] = \sum_{\forall X_1} \sum_{\forall X_2} \mathcal{P}(X_1, X_2, n_1, n_2) X_1 X_2$$



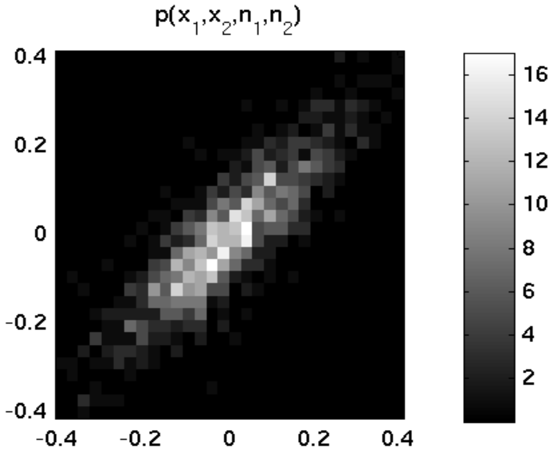


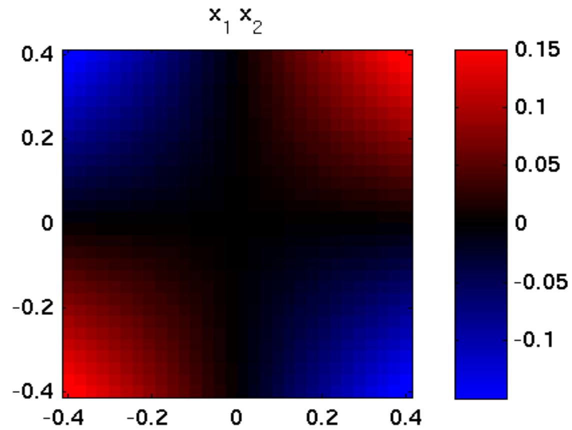


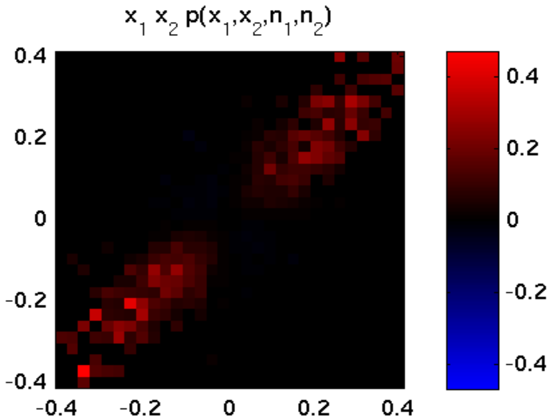
R[10,13] = 326.9284

Continuous range, n_1=10, n_2=11

$$R[n_1, n_2] = \int_{x_1} \int_{x_2} p(x_1, x_2, n_1, n_2) x_1 x_2 dx_1 dx_2$$



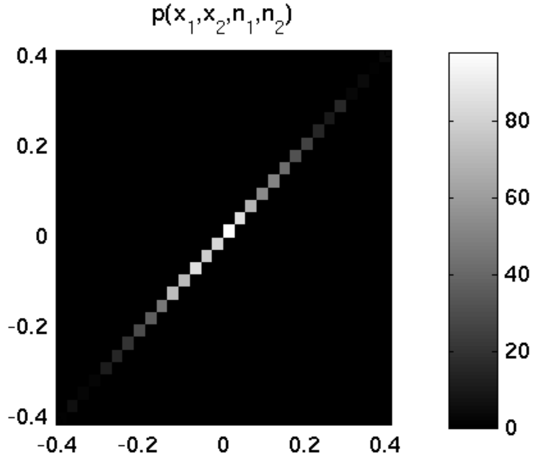


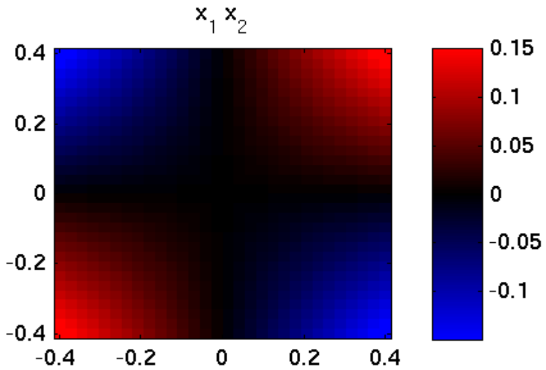


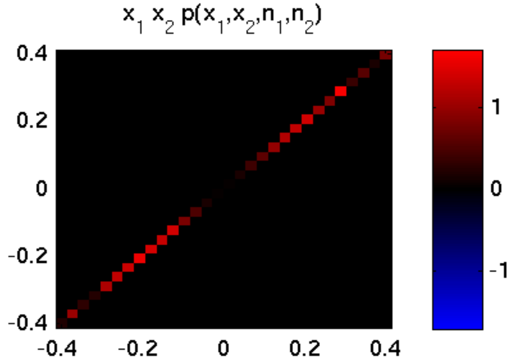
R[10,11] =0.0159

Continuous range, n_1=10, n_2=10

$$R[n_1, n_2] = \int_{x_1} \int_{x_2} p(x_1, x_2, n_1, n_2) x_1 x_2 dx_1 dx_2$$



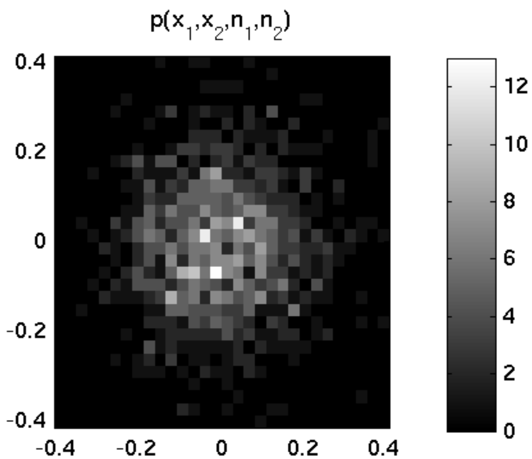


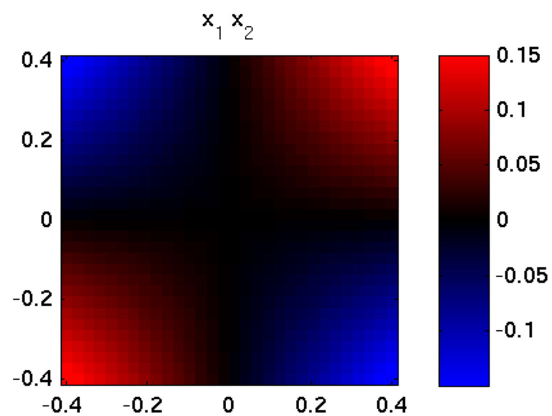


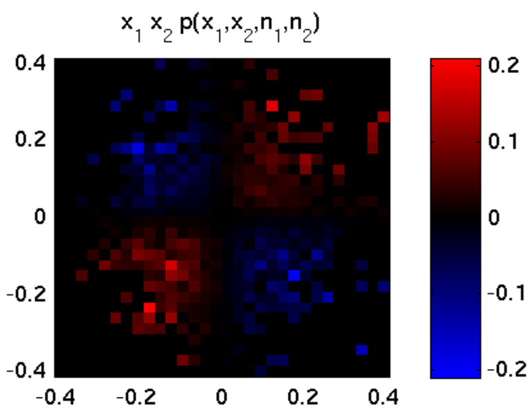
R[10,10] =0.0184

Continuous range, n_1=10, n_2=16

$$R[n_1, n_2] = \int_{x_1} \int_{x_2} p(x_1, x_2, n_1, n_2) x_1 x_2 dx_1 dx_2$$



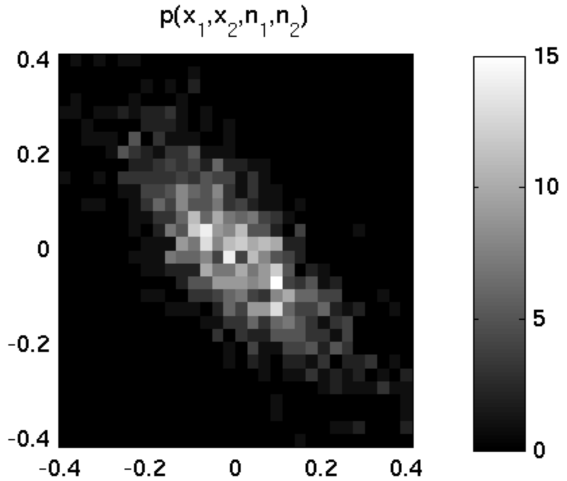


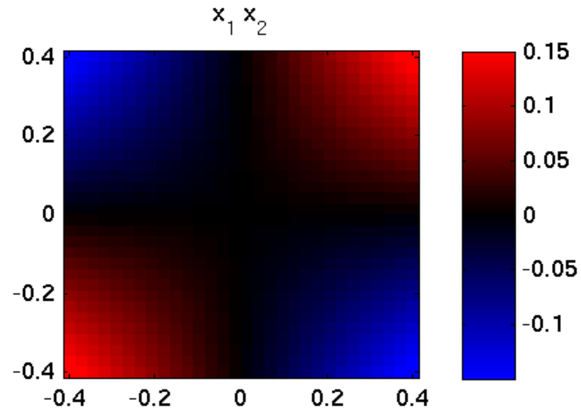


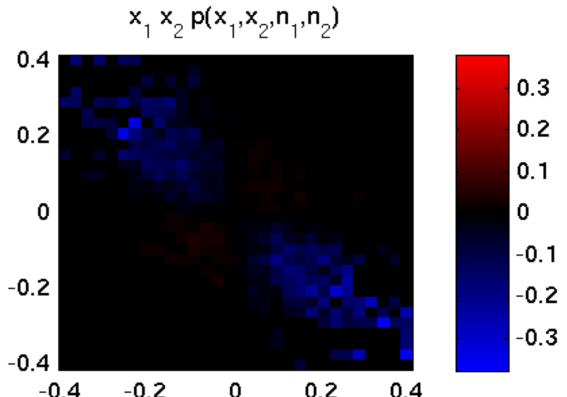
R[10,16] = 0.00038

Continuous range, $n_1=10, n_2=23$

$$R[n_1, n_2] = \int_{x_1} \int_{x_2} p(x_1, x_2, n_1, n_2) x_1 x_2 dx_1 dx_2$$

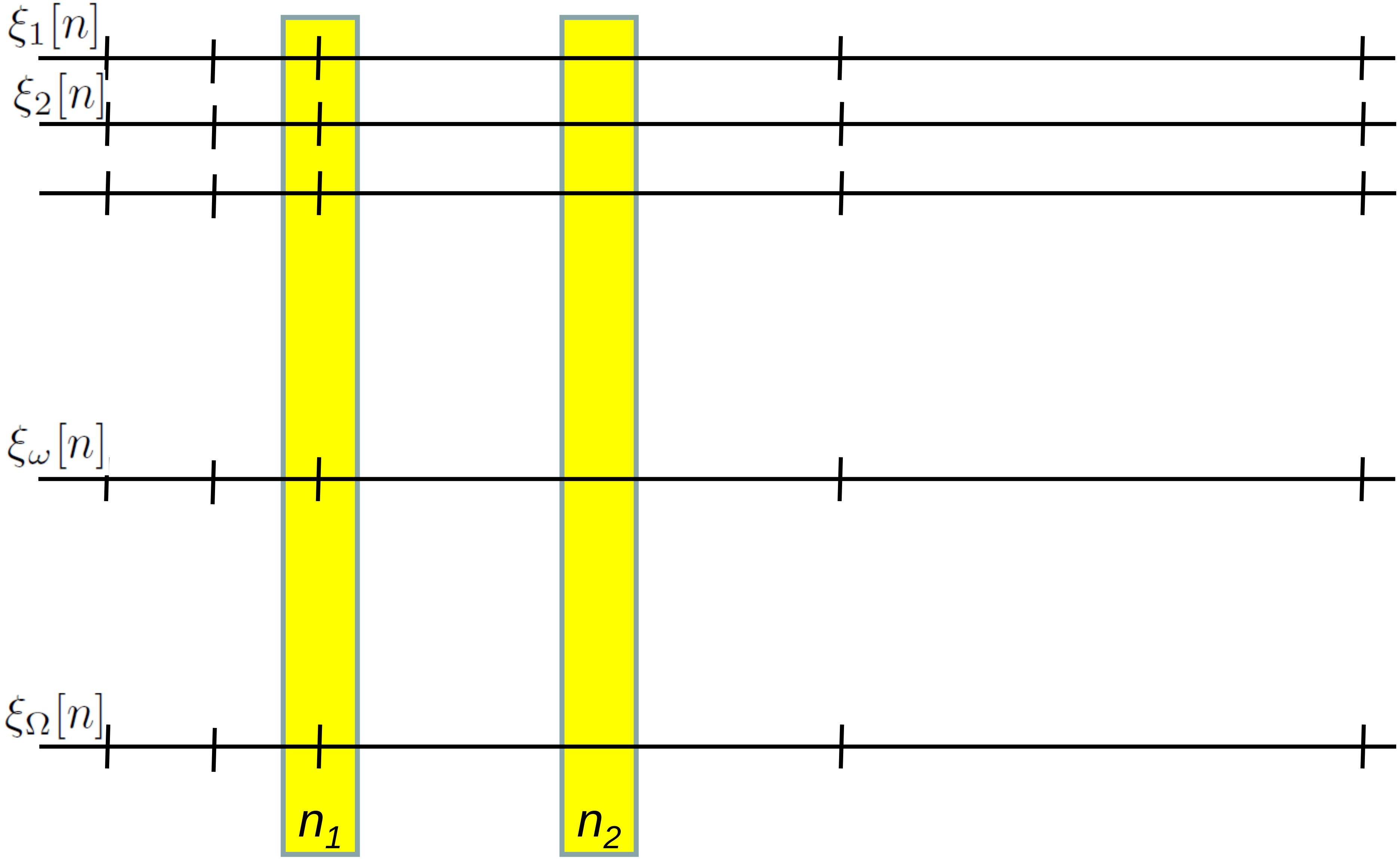






R[10,23] = -0.0139

Direct ensemble estimate



Discrete range

$$\hat{R}[n_1, n_2] = \frac{1}{\Omega} \sum_{\omega=1}^{\Omega} \xi_\omega[n_1] \xi_\omega[n_2]$$

R[10,10] = 439.3770

R[10,11] = 324.2020

R[10,13] = 326.9284

Continuous range

$$\hat{R}[n_1, n_2] = \frac{1}{\Omega} \sum_{\omega=1}^{\Omega} \xi_\omega[n_1] \xi_\omega[n_2]$$

R[10,10] = 0.0183

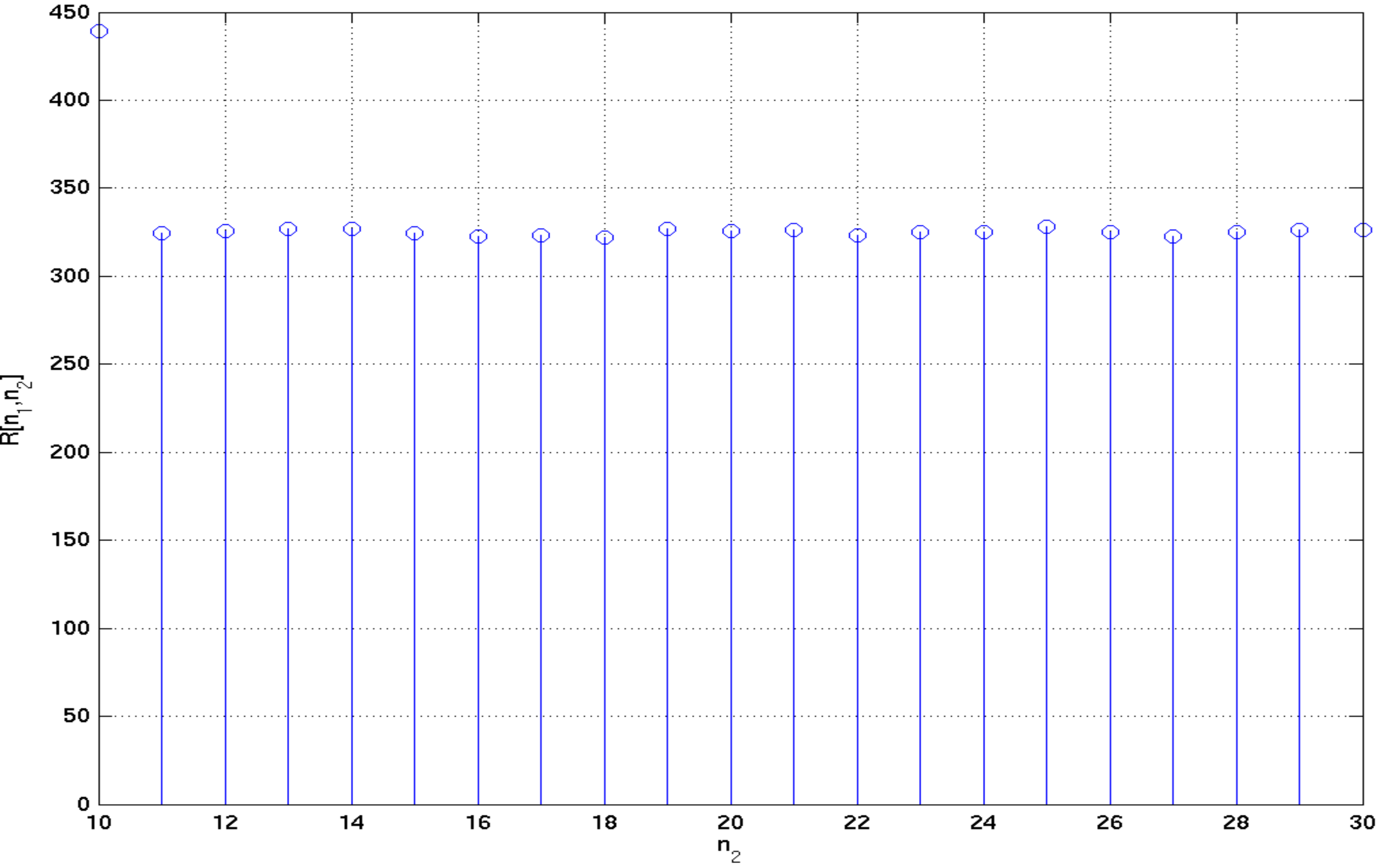
R[10,11] = 0.0160

R[10,16] = 3.8000e-04

R[10, 23] = -0.0140

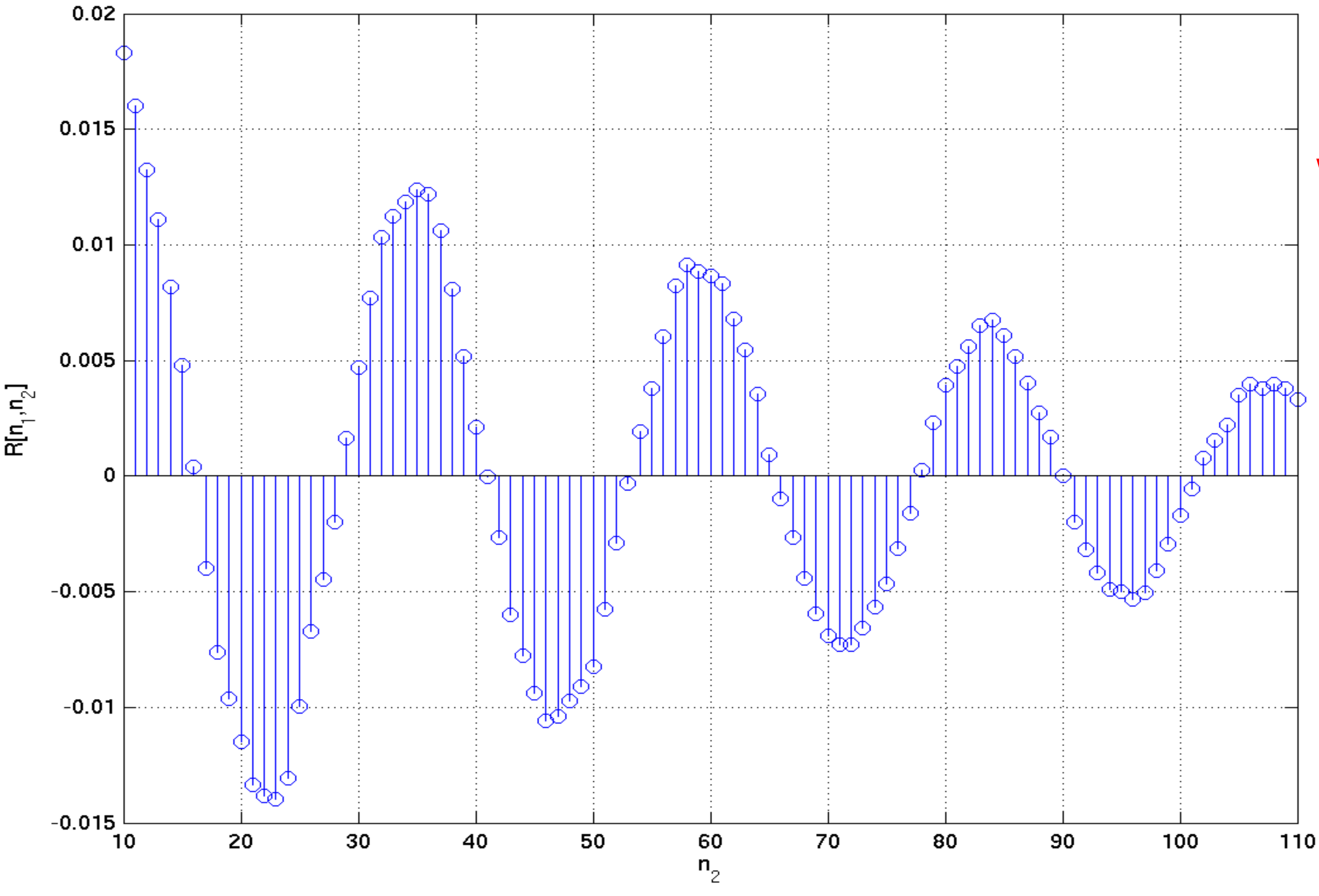
The same equations again ☺

Sequence of correlation coefficients – roulette



Useful ???

Sequence of correlation coefficients - water



Wow !!!

Stationarity

- The behavior of stationary random signal does not change over time (or at least we believe that it does not…)
- Values and functions independent on time *n*
- Correlation coefficients do not depende on n_1 and n_2, only on their difference $k=n_2-n_1$

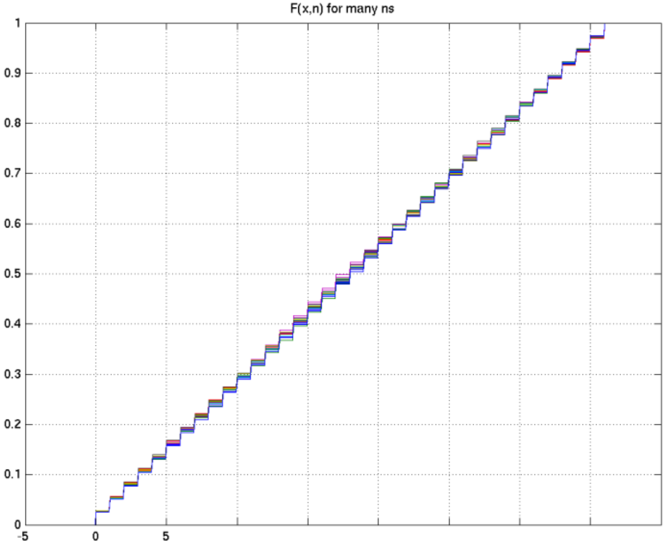
$$F(x,n) \rightarrow F(x) \quad p(x,n) \rightarrow p(x)$$

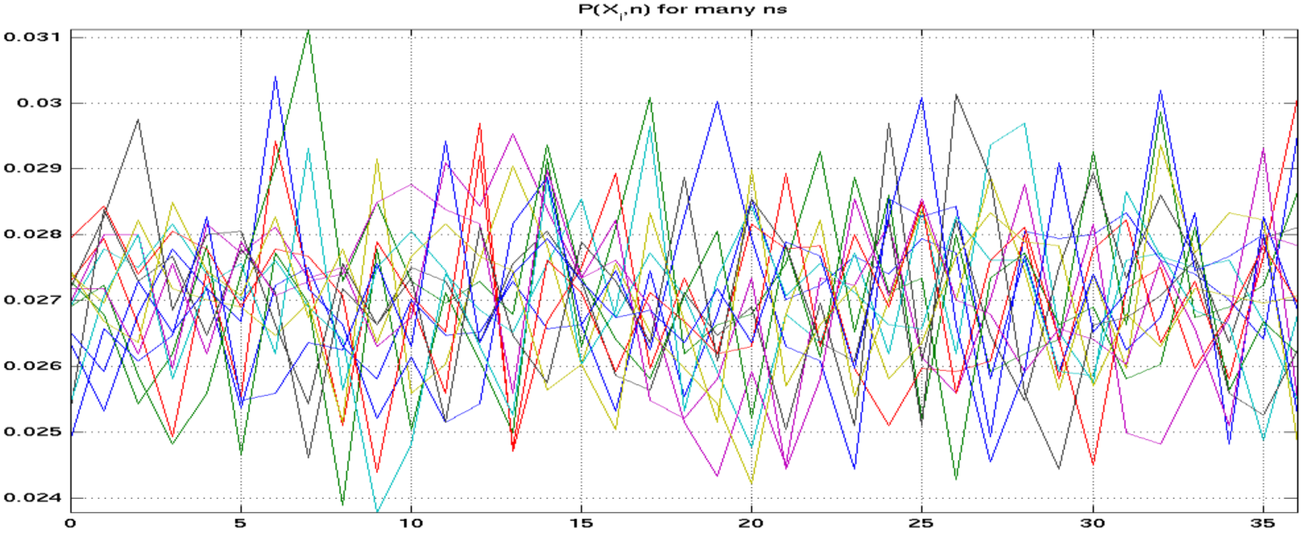
$$a[n] \rightarrow a \quad D[n] \rightarrow D \quad \sigma[n] \rightarrow \sigma$$

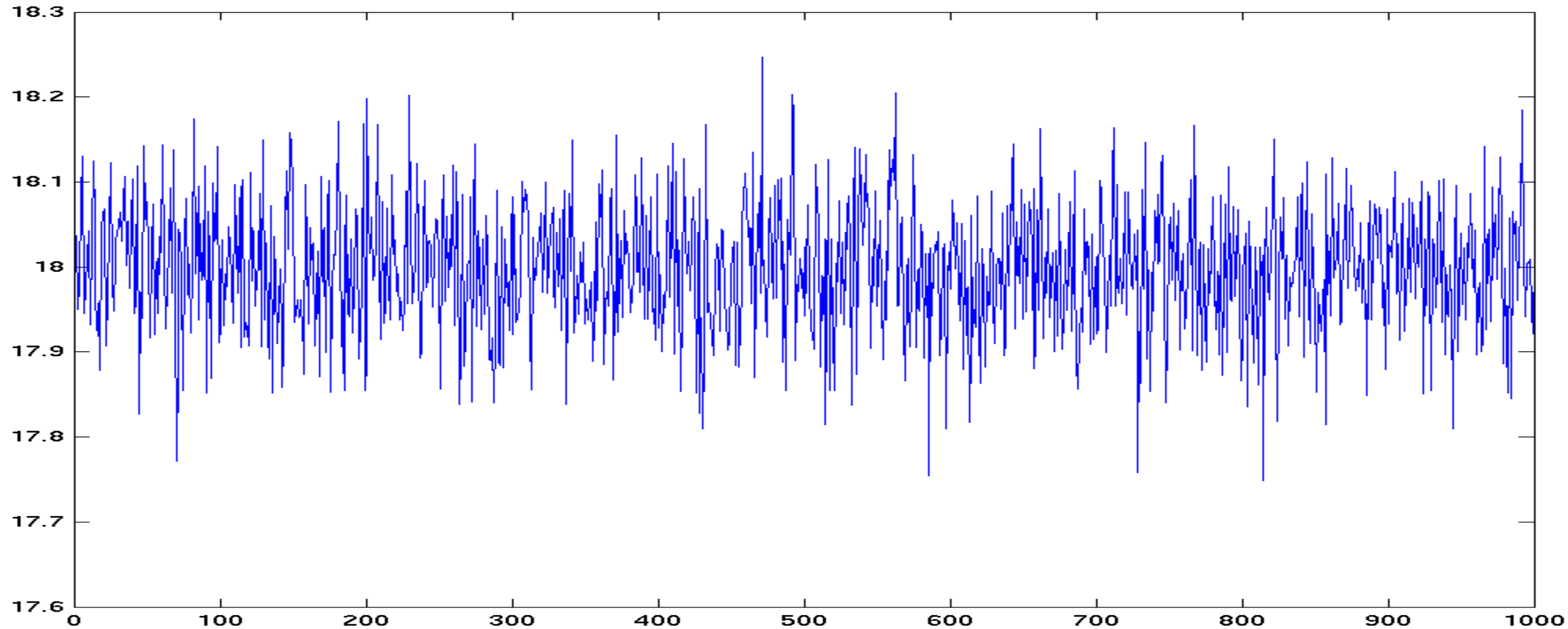
$$p(x_1, x_2, n_1, n_2) \rightarrow p(x_1, x_2, k)$$

$$R[n_1, n_2] \rightarrow R(k)$$

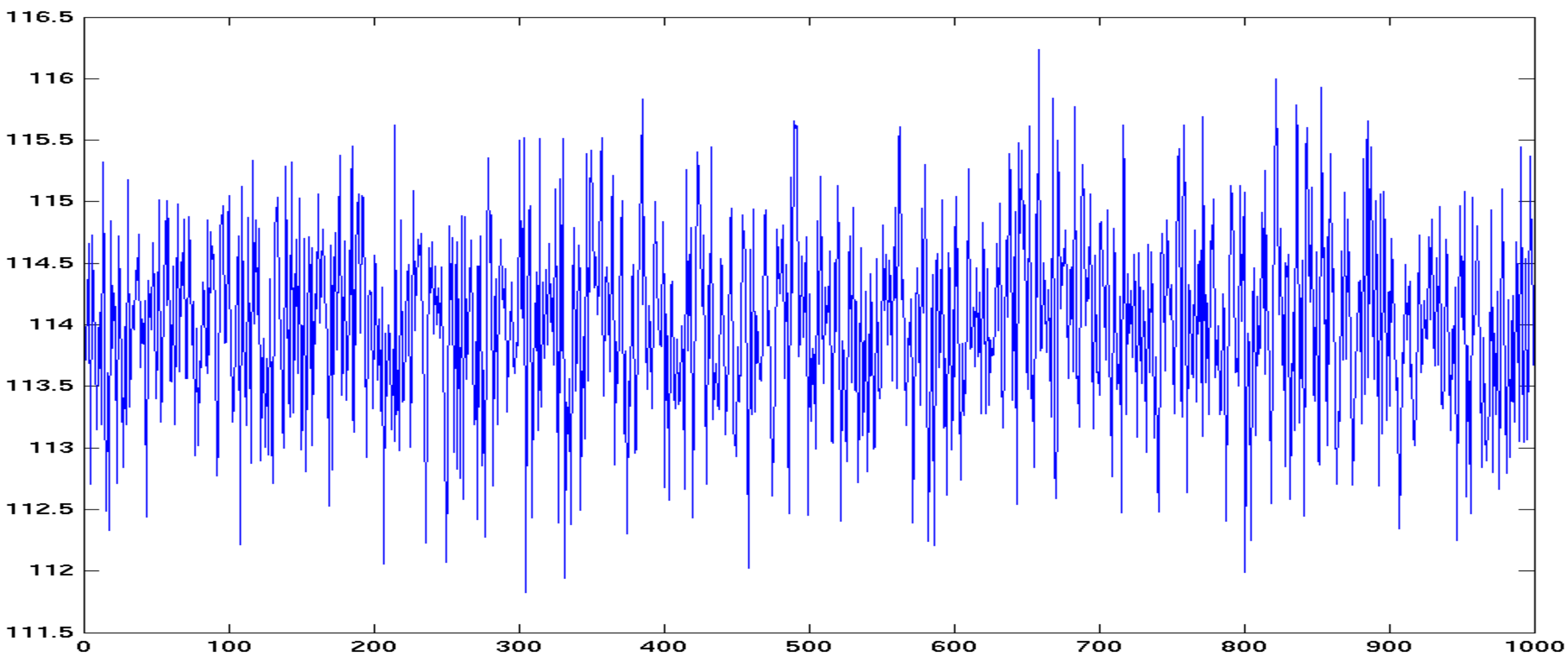
Is roulette stationary ?



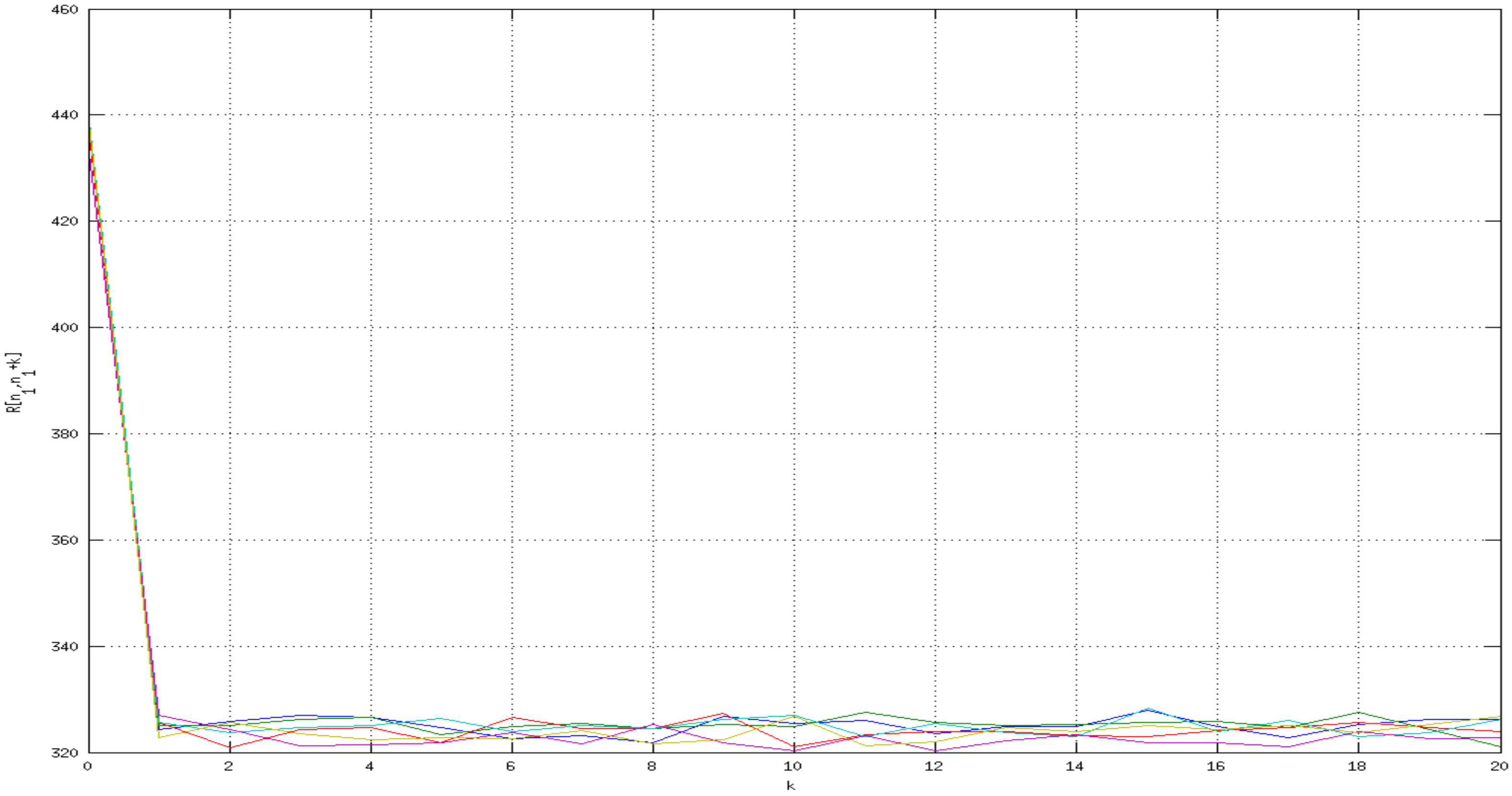




18.3
18.2
18.1
18
17.9
17.8
17.7
17.6
0
100
200
300
400
500
600
700
800
900
1000

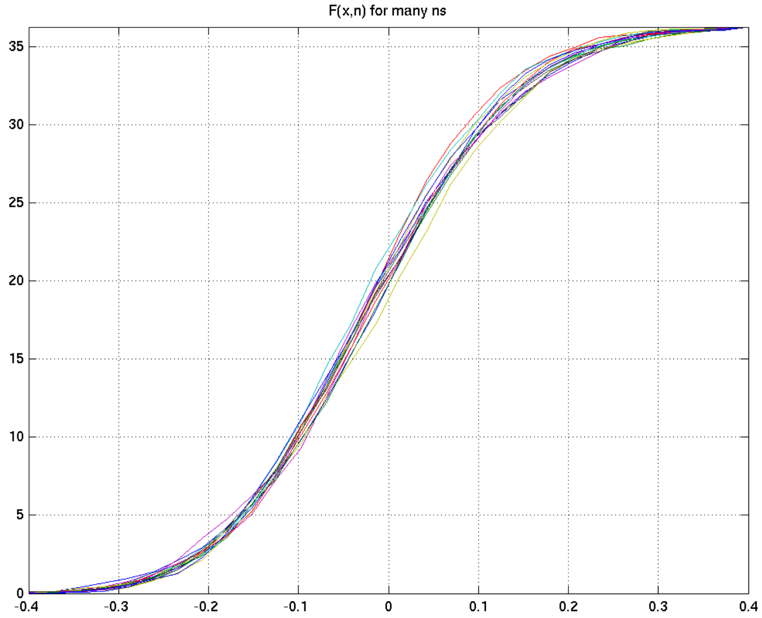


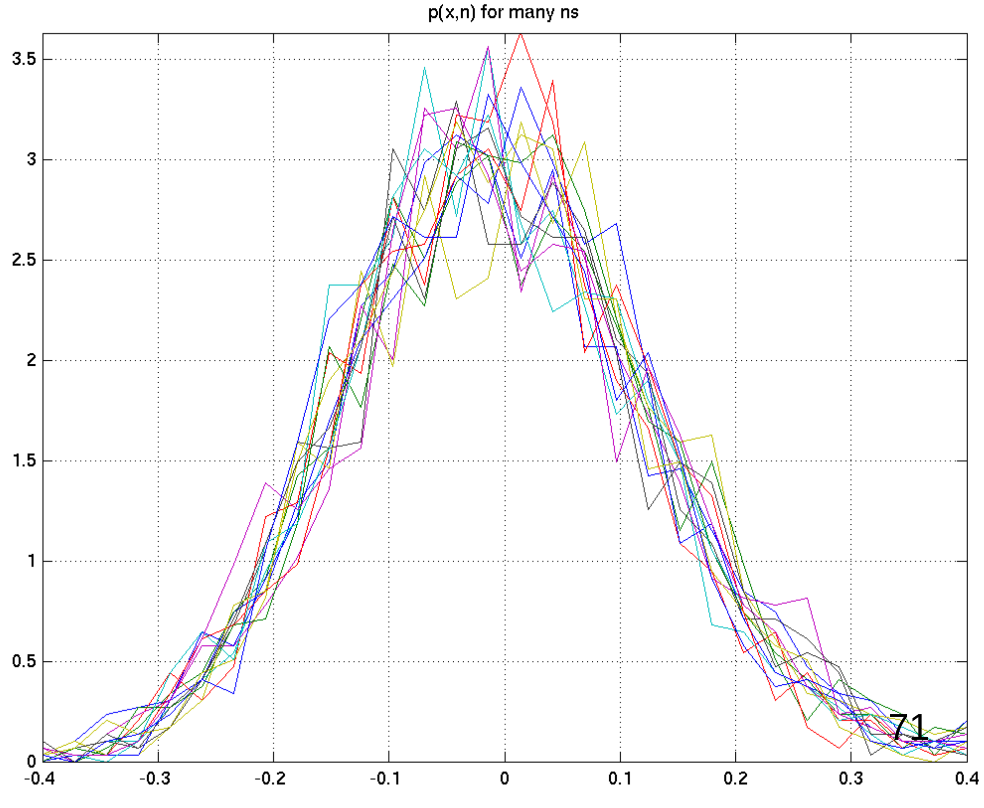
116.5
116
115.5
115
114.5
114
113.5
113
112.5
112
111.5
0
100
200
300
400
500
600
700
800
900
1000

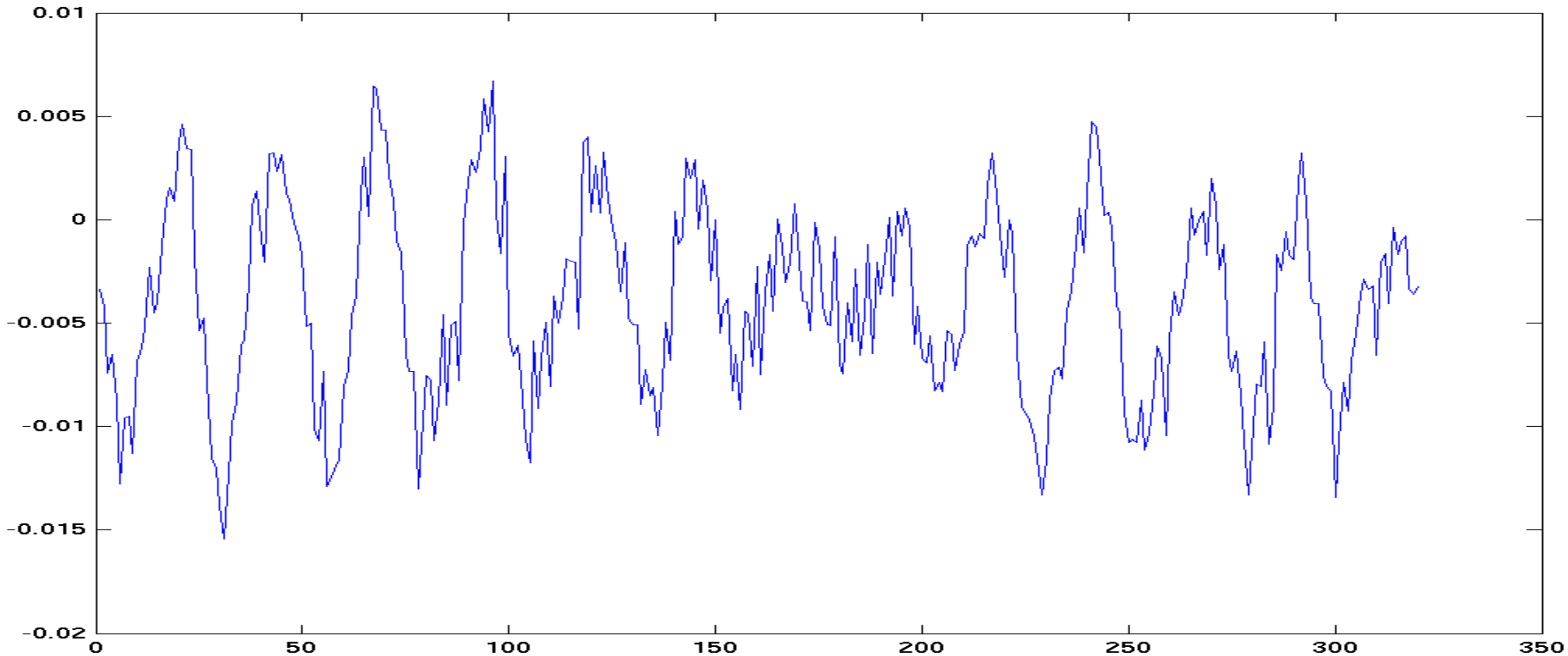


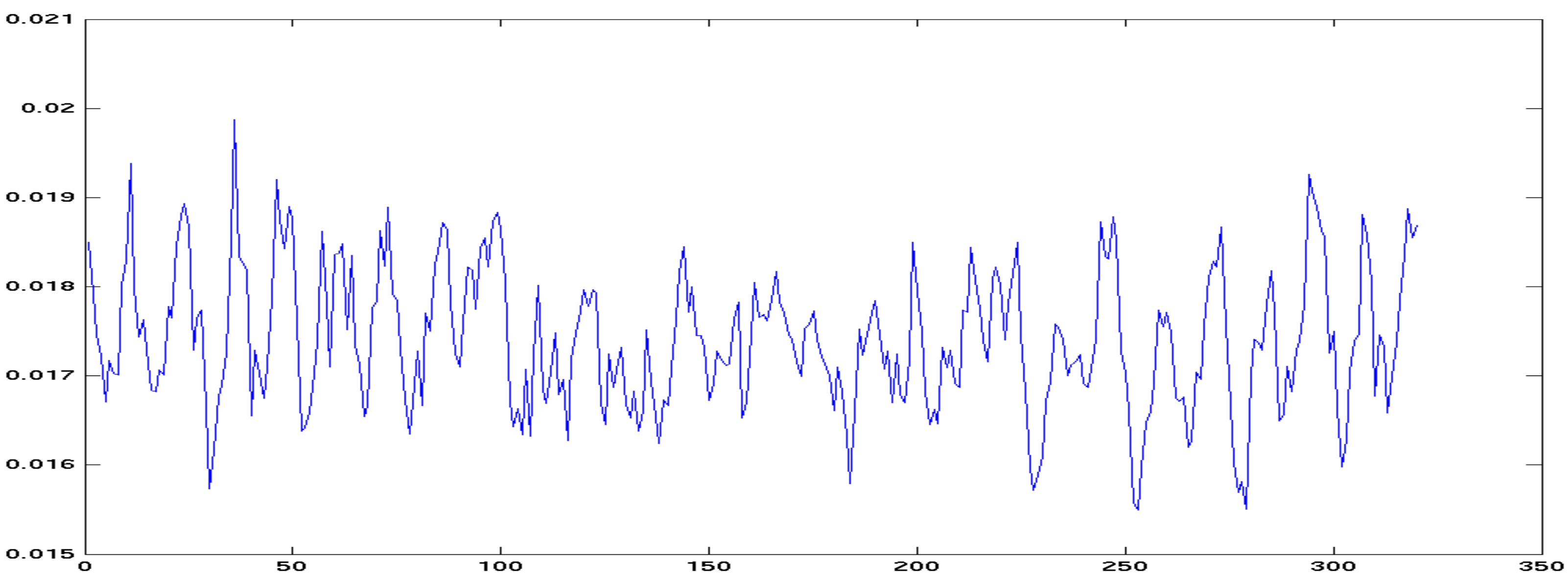
Stationary

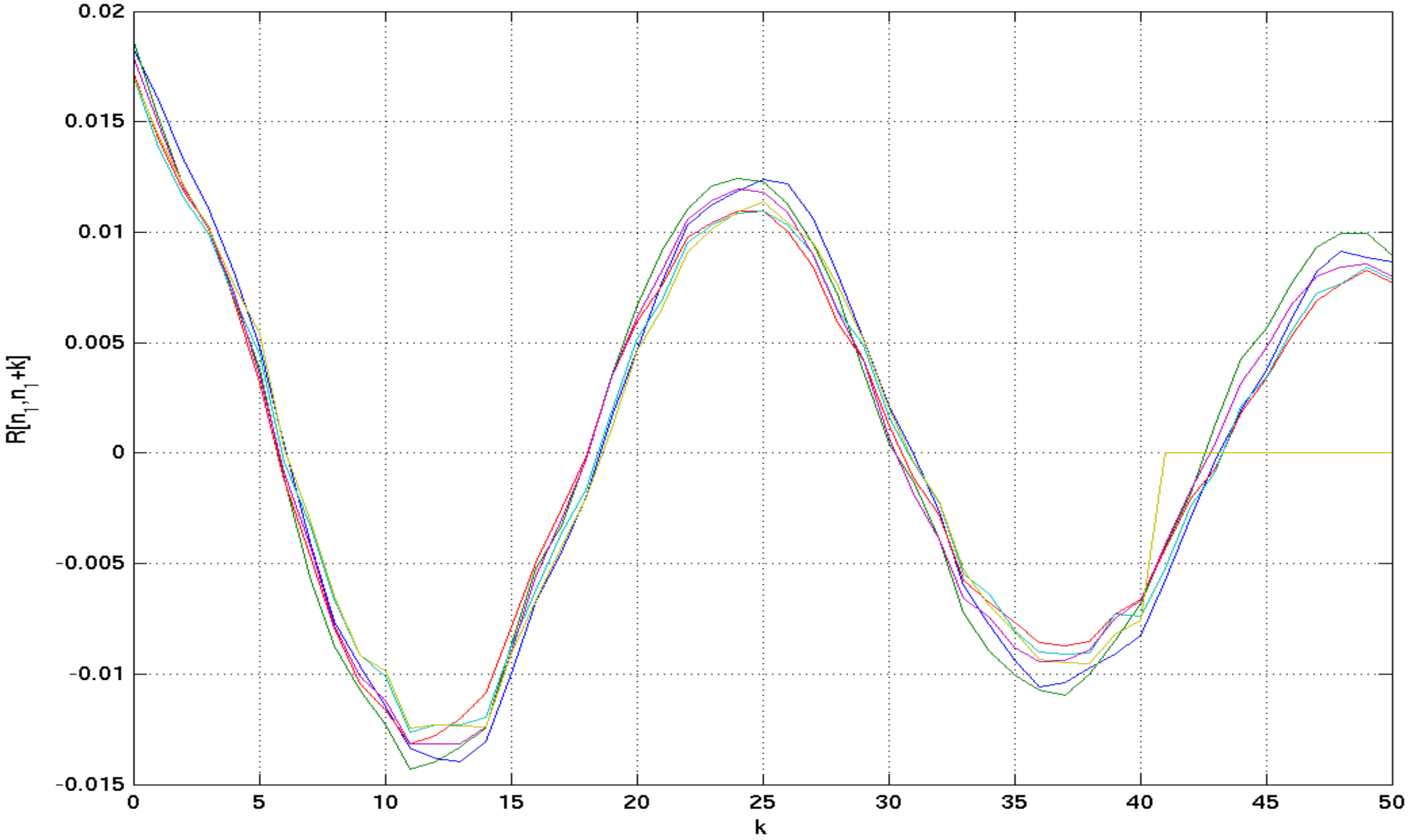
Is water stationary ?






0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
0
50
100
150
200
250
300
350


0.021
0.02
0.019
0.018
0.017
0.016
0.015
0
50
100
150
200
250
300
350


0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
R[n1,n1+k]
0
5
10
15
20
25
30
35
40
45
50
k

Stationary

Ergodicity

- The parameters can be estimated from one single realization

… or at least we hope

… most of the time, we'll have to do it anyway

$$\xi_\omega[n] \quad \Rightarrow \quad \xi[n]$$

Temporal estimates

$$\hat{a} = \frac{1}{N}\sum_{n=0}^{N-1}\xi[n] \qquad \hat{D} = \frac{1}{N}\sum_{n=0}^{N-1}[\xi[n] - \hat{a}]^2 \qquad \hat{\sigma} = \sqrt{\hat{D}}$$

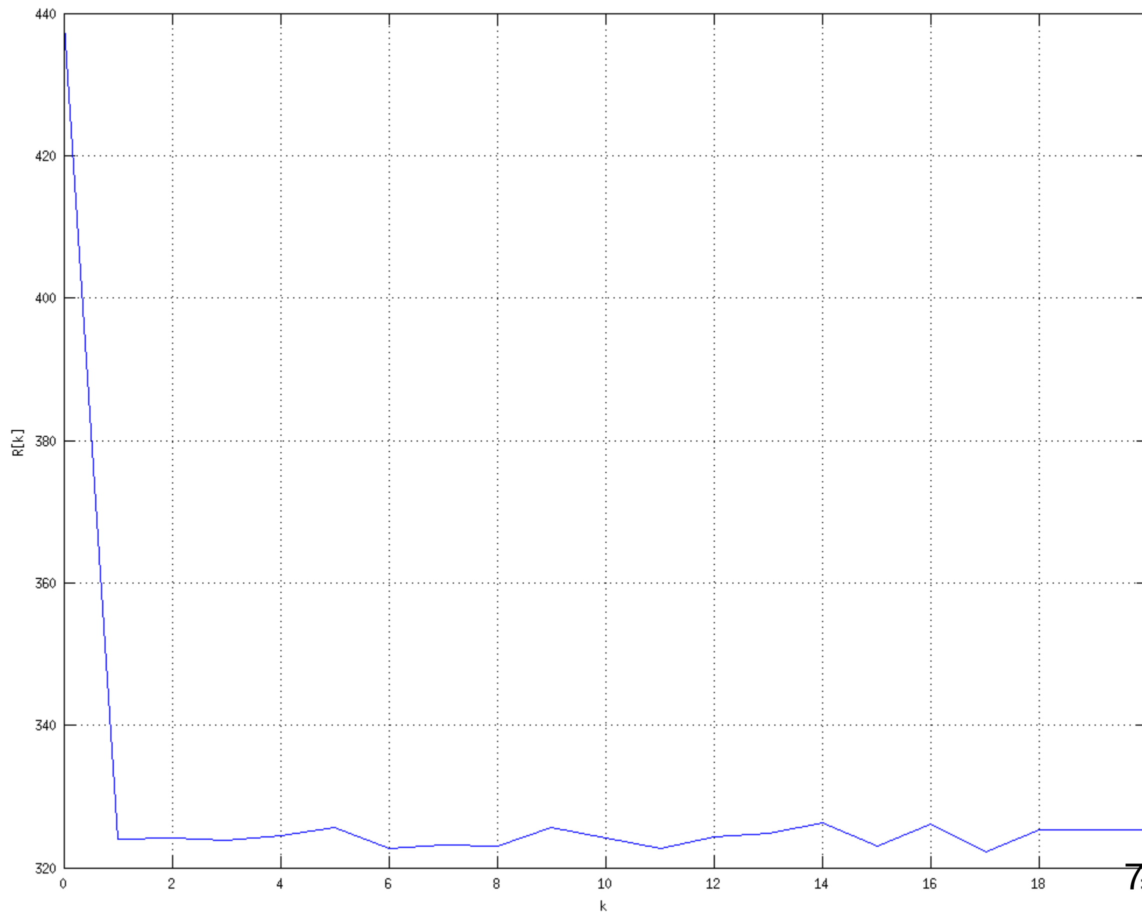
$$\hat{R}[k] = \frac{1}{N}\sum_{n=0}^{N-1}\xi[n]\xi[n+k]$$

Roulette

a = 18.0348

D = 114.4742

R[k]

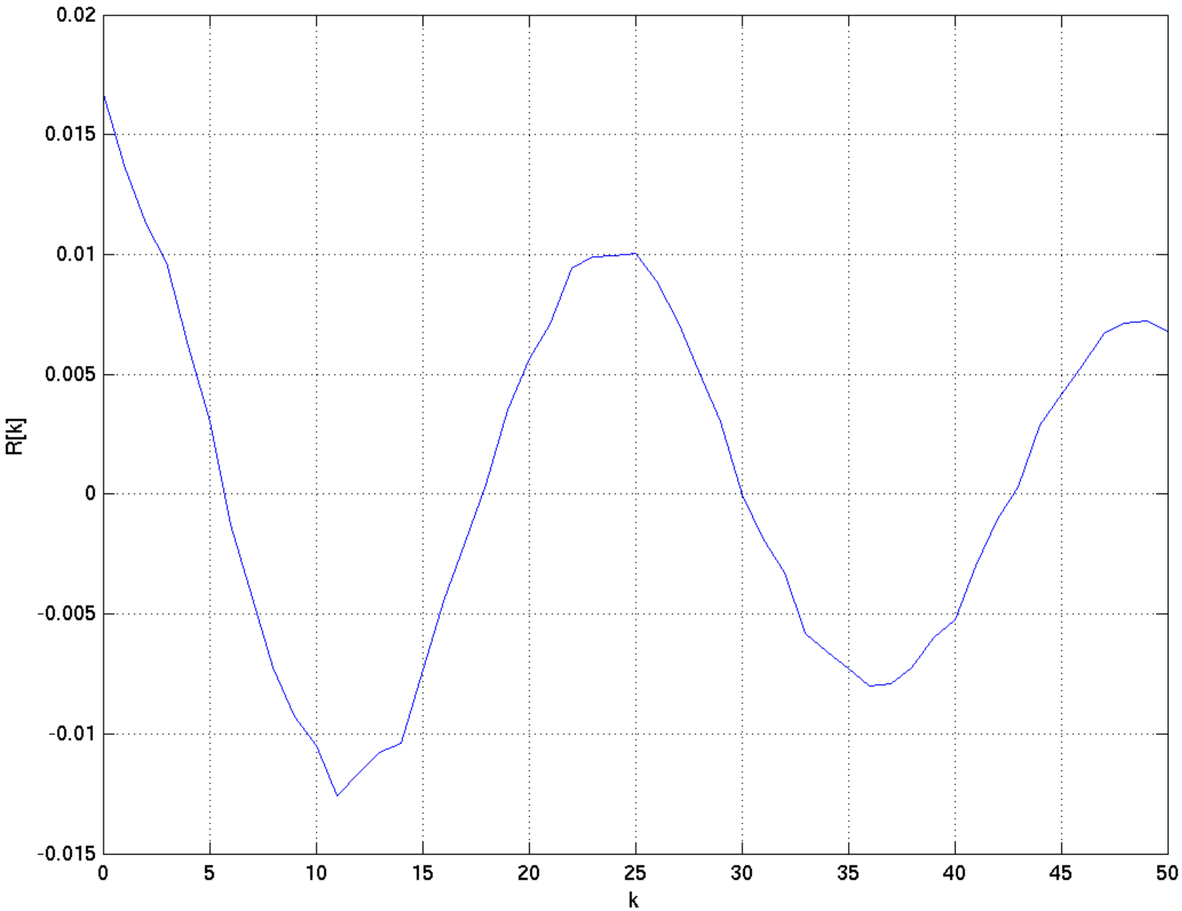


Water

a = -0.0035

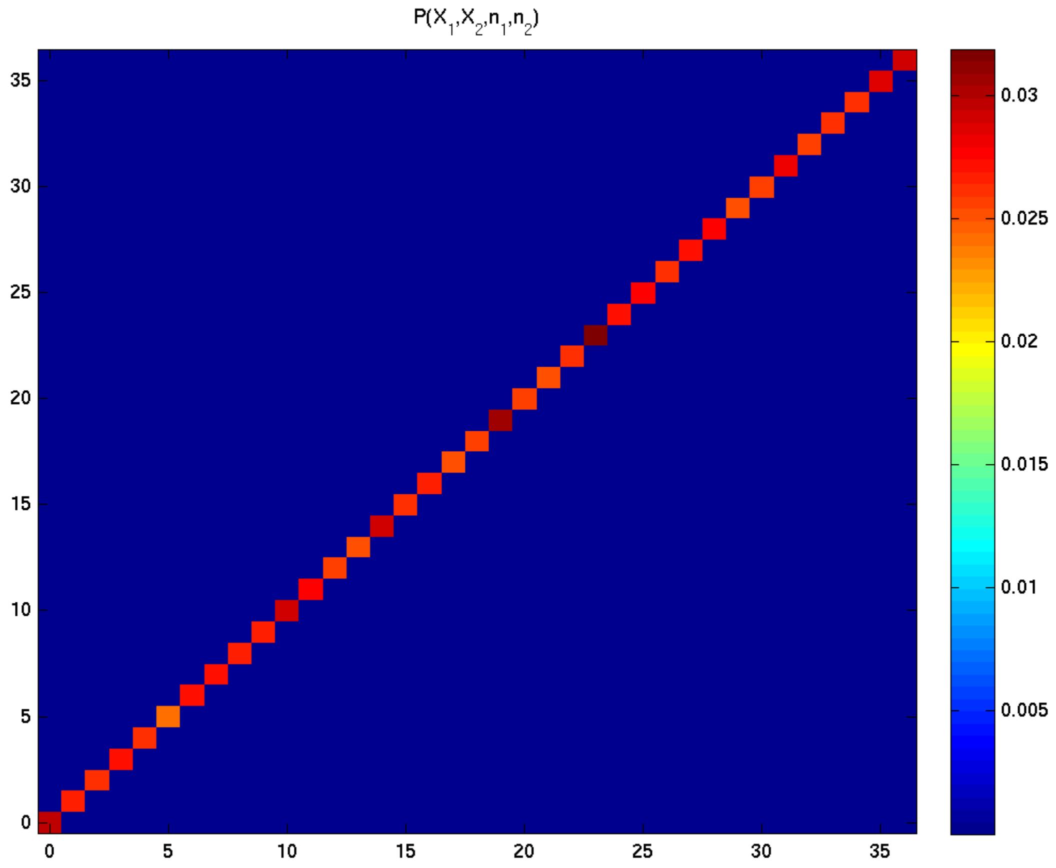
D = 0.0168

R[k]



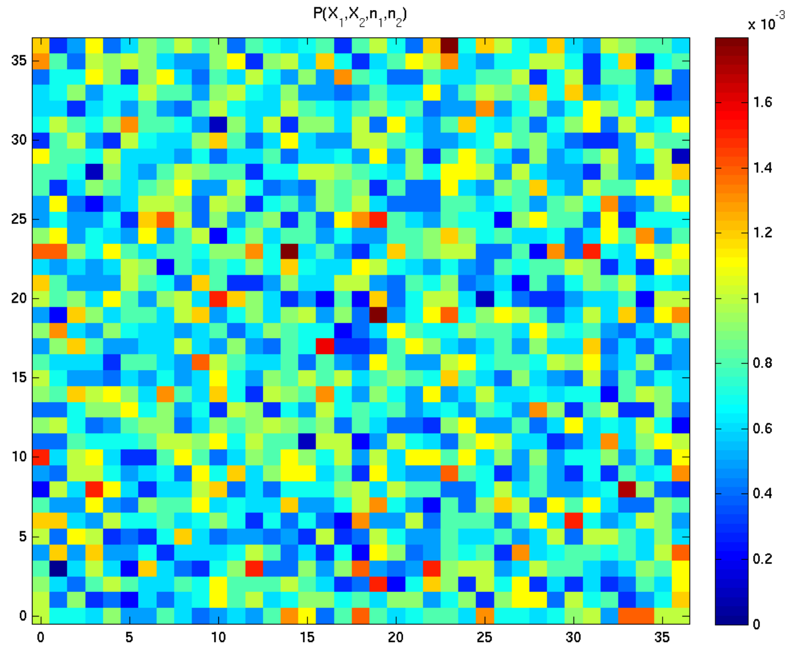
Temporal estimates of joint probabilities ?

$$\hat{\mathcal{P}}(X_i, X_j, k) = \frac{\text{count}(\xi[n] = X_1 \quad \textbf{AND} \quad \xi[n+k] = X_2)}{N}$$



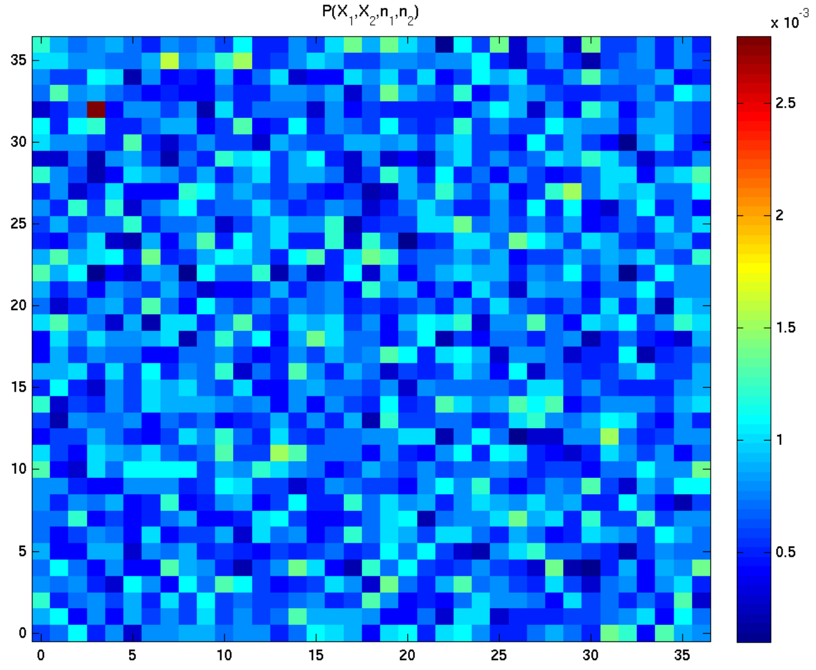
Roulette,

k = 0



Roulette,

k = 1



Roulette,

k = 3

Spectral analysis of random signals

- No idea on which frequencies they are
  - No fundamental frequency
  - No harmonics
- Phases have no sense
- The spectrum can tell us just the density of power at different frequencies.

=> Power spectral density, PSD

Computing PSD from correlation coefficients

$$G(\frac{k}{N}) = DFT\{R[n]\}$$

Normalized frequency

$$G(\frac{kF_s}{N}) = DFT\{R[n]\}$$

Real frequency

PSD water

1.4
1.2
1
0.8
0.6
0.4
0.2
0
G(f)
0
1000
2000
3000
4000
5000
6000
7000
8000
f
???

Estimation of PSD directly from signal

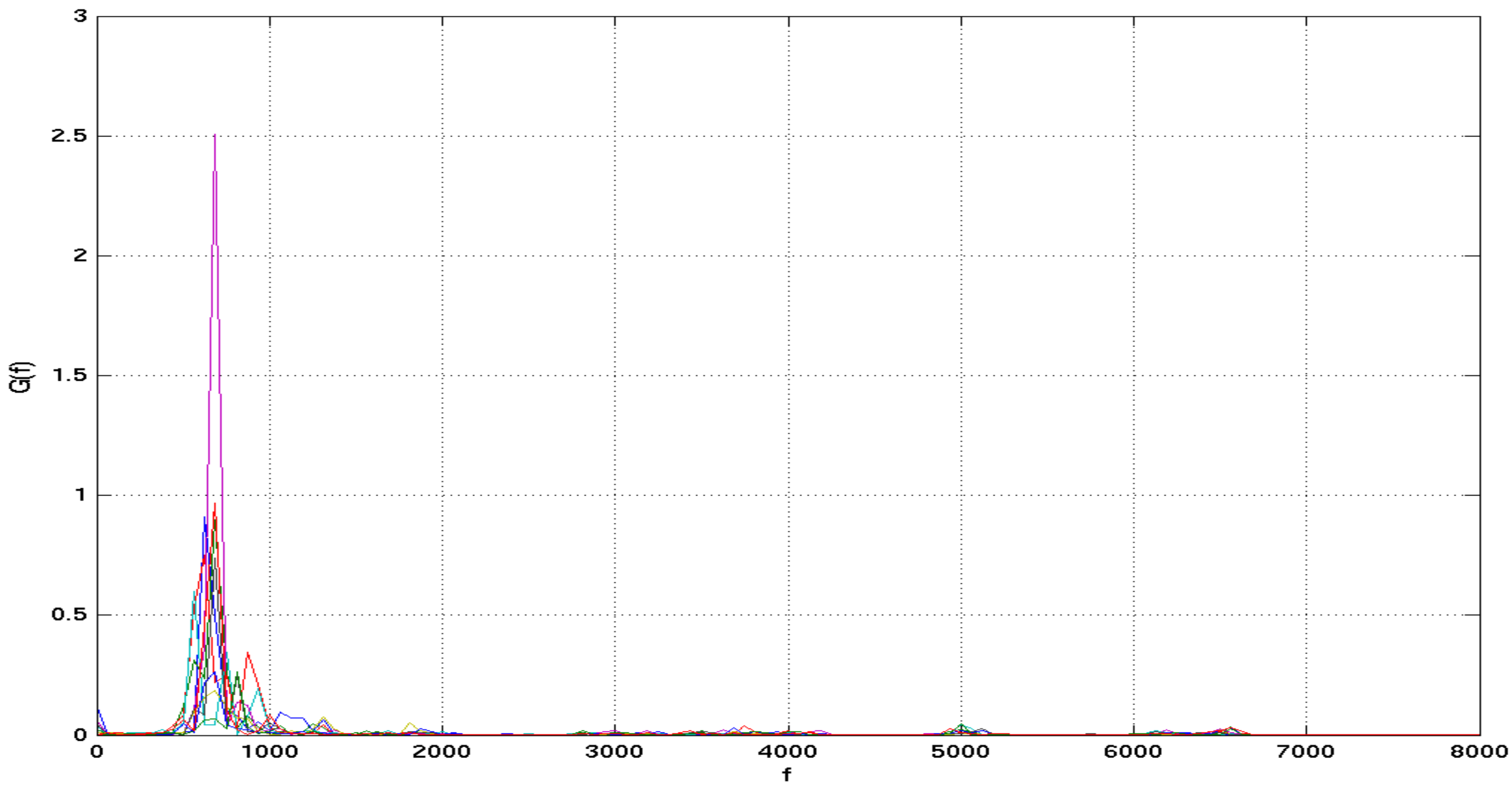
$$G(\frac{k}{N}) = \frac{|DFT\{\xi[n]\}|^2}{N}$$

$$G(\frac{kF_s}{N}) = \frac{|DFT\{\xi[n]\}|^2}{N}$$

Normalized frequency

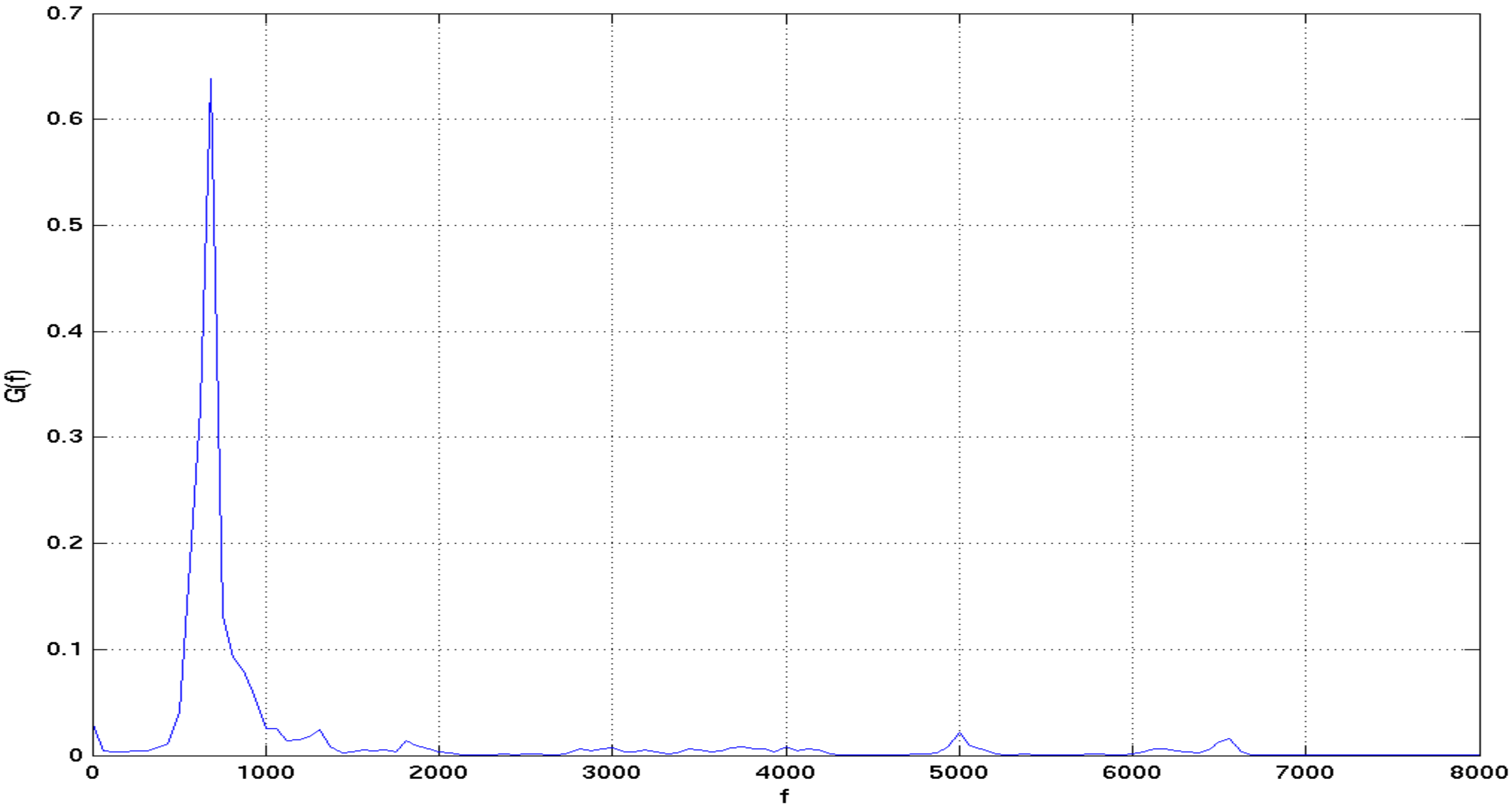
Real frequency

PSD estimate from signal – water



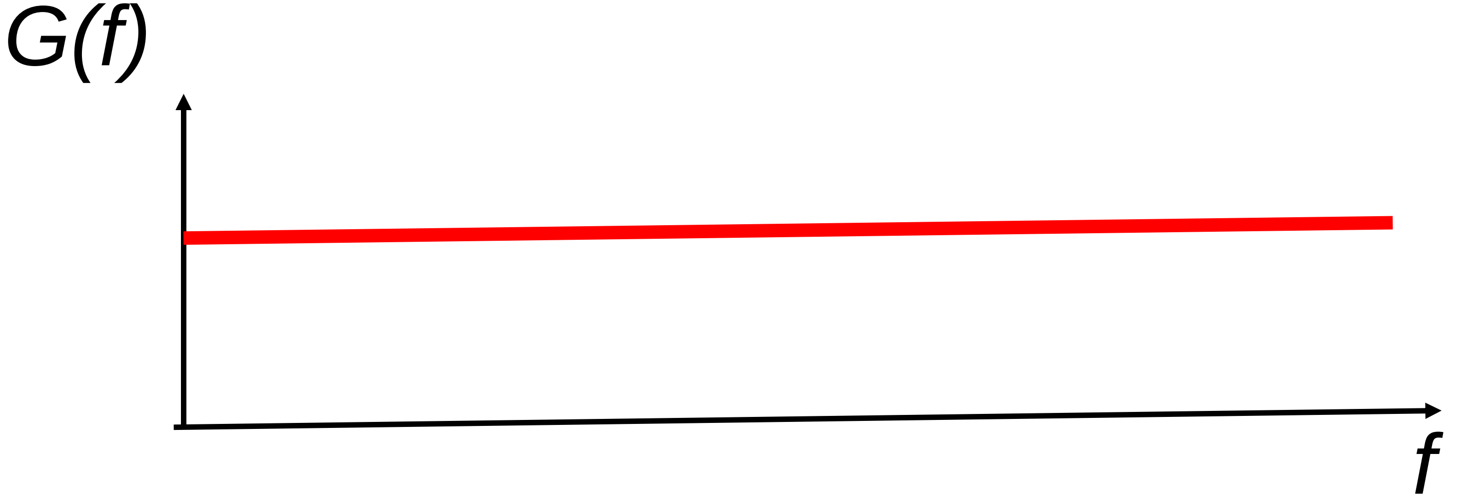
Welch's technique – improving the robustness of estimate

- Averaging over several segments of signal



White noise

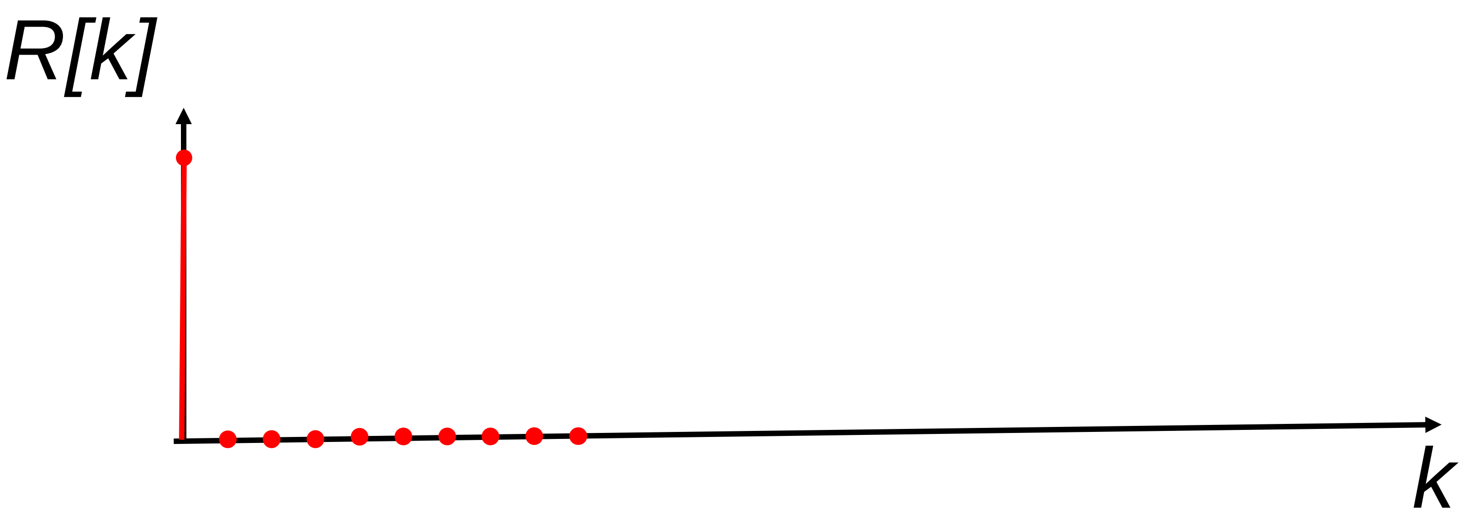
- Spectrum of white light is flat
- Power spectral density *G(f)* of a white noise should be also flat.



Correlation coefficients of white noise

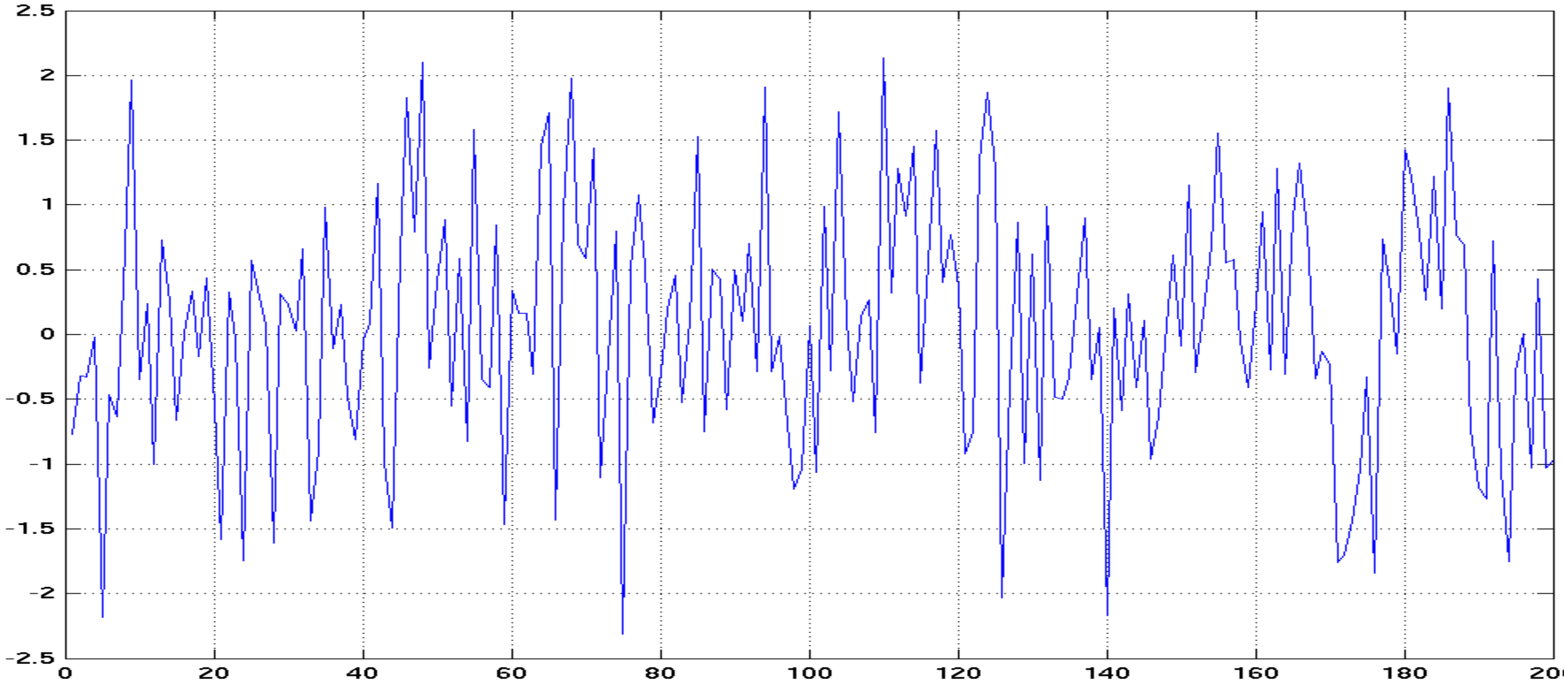
$$G(\frac{kF_s}{N}) = DFT\{R[n]\}$$

- How must *R[k]* look, so that their DFT is a constant ?



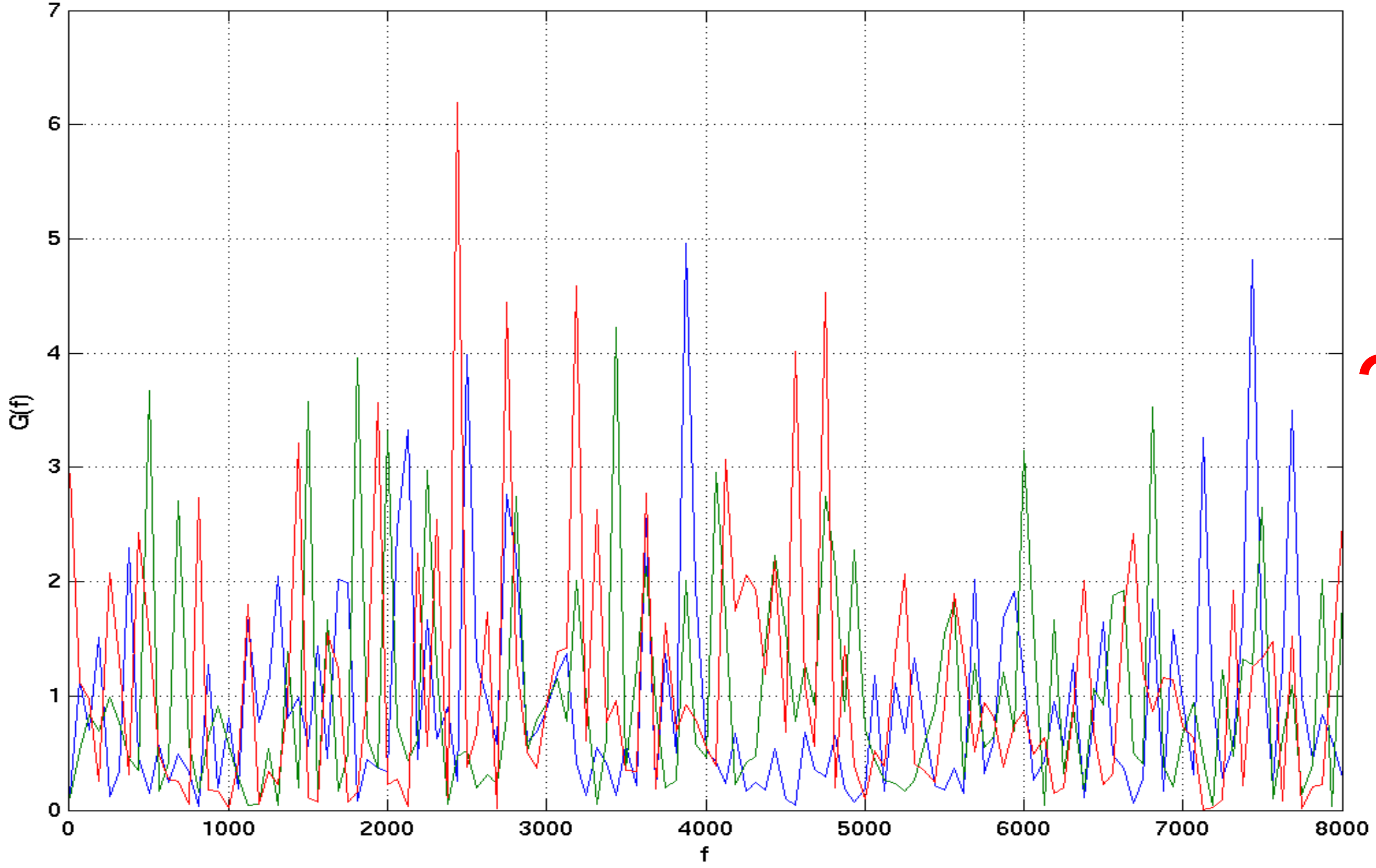
White noise

- Signal having only *R[0]* non-zero
- … has **no dependencies between samples**



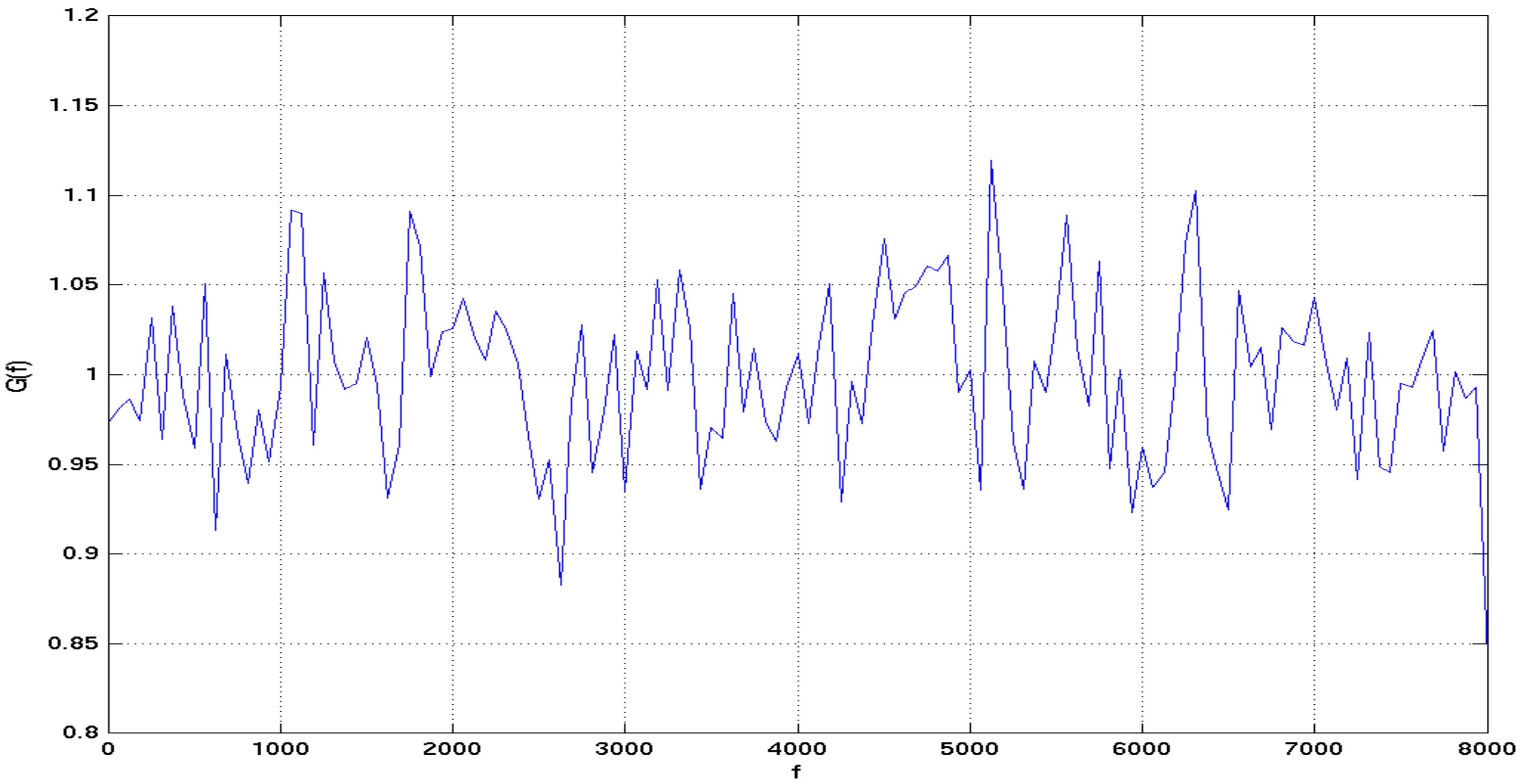
Determining PSD of white noise

$$G\left(\frac{kF_s}{N}\right) = \frac{|DFT\{\xi[n]\}|^2}{N}$$



???

Welch … help …



1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8
G(f)
0
1000
2000
3000
4000
5000
6000
7000
8000
f

SUMMARY

- Random signals are of high interest
  - Everywhere around us
  - Carry information
- Discrete vs. continuous range
- Can not precisely define them, other means of description
  - Set of realizations
  - Functions – cumulative distribution, probabilities, probability density
  - Scalars – moments
  - Behavior between two times – correlation coefficients

SUMMARY II.

- Counts
  - of an event „how many times did you see the water signal in interval 5 to 10?“
- Probabilities
  - Estimated as *count / total*.
- Probability density
  - Estimated as *Probability / size of interval* (1D or 2D)
- In case we have a set of realizations – ensemble estimates.

SUMMARY III.

- Stationarity – behavior not depending on time.
- Ergodicity – everything can be estimated from one relazation
  - Temporal estimates
- Spectral analysis
  - Power spectral density – PSD
  - From correlation coefficients
  - Or directly from the signal, often improving the estimate by averaging.

SUMMARY IV

- White noise
  - No dependencies of samples (uncorrelated samples)
  - So that only *R[0]* is non-zero, the others zero.
  - So that DFT is constant
  - White light has constant spectrum too.

NOT COVERED…

- Can we model generation of random signals ?
- What to do for temporal estimates of correlation coefficients – less and less samples to work with as *k* increases!
- Can we color a white noise ?
- How exactly is power spectral density defined?
- Can we use all this for recognition / classification / detection ?

The END