Graph Algorithms

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Outline

Introduction

References

Books

- Cormen, Leiserson, Rivest, Stein: *Introduction to algorithms.* The MIT Press and McGraw-Hill, 2001.
- Gibbons: *Algorithmic Graph Theory.* Cambridge University Press, 1985.

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Materials

- Lecture slides @ `https://www.fit.vutbr.cz/study/courses/GALe/public/`
- Text generated from lecture slides

Course Details

- lectures (2/3 + 0/1) – Zbyněk Křivka
- project (25 points) – Ľubica Genčúrová
- midterm test (15 points) – approx. middle of semester
- exam (60 points) — 3 terms, minimum 25 points
- consultations – `krivka@fit.vut.cz`, `igencurova@fit.vut.cz`

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About the Project

- individual
- implementation of two/more graph algorithms, experiments, comparison
- own assignment (suggestion of algorithms related to your thesis)
- presentation of your solutions during the last lecture
- implementation programming language - C/C++, Java, Python, Ruby (anything available at Merlin server or agreed by the teacher)

Algorithms and Complexity

Basic Notions

- Informally, algorithm is a well-defined procedure (sequence of computational steps) that transforms some input into the corresponding output.
- Data structure is a way of storage and organization of data optimized for access and/or modification.

Requirements on Algorithms

- Finiteness: Algorithm always ends for a valid (correct) input.
- Soundness, Correctness: The result is correct as well.

- Memory and time are limited!
- There is many solutions, we focus on the effective ones.

Algorithm Complexity

Time complexity of algorithm:

- Running time $T(n)$ – function giving the maximum number of "primitive" steps depending on the size of an input n, i.e. number of steps in the worst case.

Space complexity of algorithm:

- Memory consumption $S(n)$ – function giving the maximum number of used memory cells during the computation depending on the size of an input n. (including algorithm initialization or not?)

In general, n can be a vector (multidimensional).

Θ-notation

Let $g(n)$ be a function. Let $f(n)$ denote, for instance, $T(n)$ or $S(n)$.

- $\Theta(g(n)) = \{f(n):$ there exist $c_1, c_2, n_0 > 0$ such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.
- $\Theta(g(n))$ is a family of functions that can be "sandwiched" between $c_1 g(n)$ and $c_2 g(n)$, for sufficiently large n.
- Sometimes written as $f(n) = \Theta(g(n))$ instead $f(n) \in \Theta(g(n))$.
- We say that $g(n)$ is an asymptotically tight bound for $f(n)$.

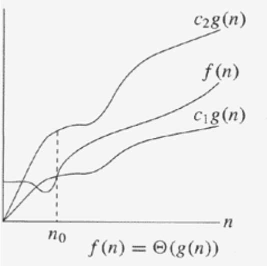

Figure: Θ-notation.

- $\frac{1}{2}n^2 - 3n = \Theta(n^2)$ – verify its properties for $c_1 = \frac{1}{14}, c_2 = \frac{1}{2}, n_0 = 7$.

O-notation

Let $g(n)$ be a function.

- $O(g(n)) = \{f(n) :$ there exist $c, n_0 > 0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$.
- $O(g(n))$ is a family of functions $f(n)$ such that $f(n)$'s value is on or below $cg(n)$ for all $n \geq n_0$.
- $f(n) = O(g(n))$ means some $cg(n)$ is an asymptotic upper bound on $f(n)$ (but not necessarily tight $\approx$ worst-case scenario).

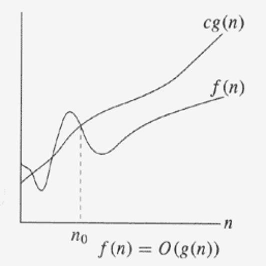

Figure: O-notation.

- $\Theta(g(n)) \subseteq O(g(n))$.
- $n = O(n^2)$, but $n \neq \Theta(n^2)$.

Ω-notation

Let $g(n)$ be a function.

- $\Omega(g(n)) = \{f(n) :$ there exist $c, n_0 > 0$ such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0\}$.
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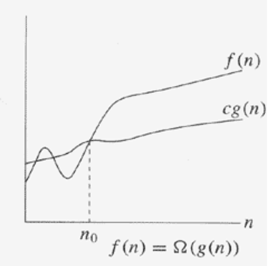

Figure: Ω-notation.

Theorem 1.
For any $f(n)$, $g(n)$, it holds $f(n) = \Theta(g(n))$ if and only if (iff) $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

o-notation and ω-notation

Let $g(n)$ be a function.

- $o(g(n)) = \{f(n) :$ for every $c > 0$ there exist $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0\}$.
  - upper bound that is NOT asymptotically tight

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- $2n = o(n^2)$, but $2n^2 \neq o(n^2)$.
- $f(n) = o(g(n))$, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

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- $f(n) = o(g(n))$, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

- $n^2/2 = \omega(n)$, but $n^2/2 \neq \omega(n^2)$.
- $f(n) = \omega(g(n))$, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

Properties

Let $f(n)$, $g(n)$, and $h(n)$ be (asymptotically positive) functions.

- Transitivity
  $f(n) = X(g(n))$ and $g(n) = X(h(n))$ imply $f(n) = X(h(n))$, for $X \in \{\Theta, O, \Omega, o, \omega\}$.

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  $f(n) = X(f(n))$, for $X \in \{\Theta, O, \Omega\}$.
- Symmetry
  $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$.

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- Transpose symmetry
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  $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$.
- Not always comparable
  n and $n^{1+\sin(n)}$ are incomparable.

Graphs

Graph Theory: The Beginning

- Leonhard Euler, *The Königsberg bridges problem*, 1736.
- Problem: Is it possible to cross all bridges, but everyone just once?
- `https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg`

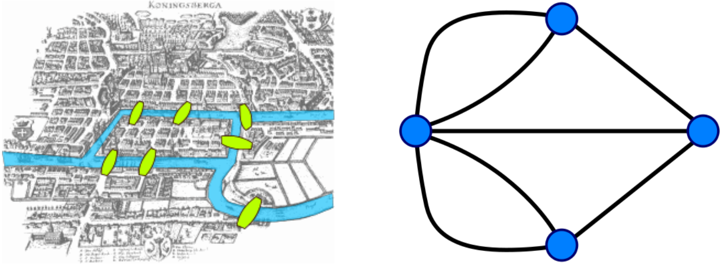

Figure: Map of bridges and its logical representation.

Definitions

Directed graph (digraph) G is a pair

$$G = (V, E),$$

where

- V is a finite set of vertices (nodes) and
- $E \subseteq V^2$ is a set of edges (arrows, arcs).

An edge (u, u) is called a self-loop.
If (u, v) is an edge, we say that (u, v) is incident from u and incident to v, that is v is adjacent to u.

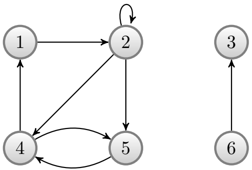

Figure: Digraph

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$, if

- $V' \subseteq V$ and $E' \subseteq E$.

Let $V'' \subseteq V$. Subgraph induced by V'' is graph $G'' = (V'', E'')$, where

- $E'' = \{(u, v) \in E : u, v \in V''\}$.

Let $E''' \subseteq E$. Factor subgraph of G is graph $G''' = (V, E''')$.

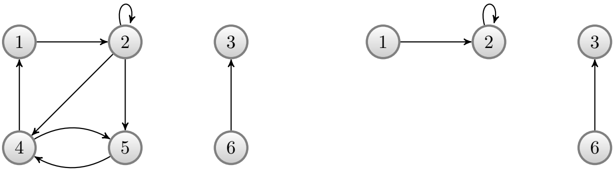

Figure: A graph and its subgraph induced by $\{1, 2, 3, 6\}$.

Definitions

Undirected graph G is a pair

$$G = (V, E),$$

where

- V is a finite set of vertices and
- $E \subseteq \binom{V}{2}$ is a set of edges.

Note

An edge is a set $\{u, v\}$, where $u, v \in V$ and $u \neq v$. Self-loops are forbidden.

Convention: $\{u, v\}$, (u, v), and (v, u) denote the same edge.

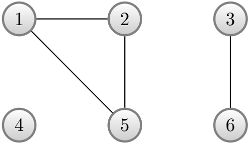

Figure: Undirected Graph

- Degree of vertex u in an undirected graph is the number of adjacent vertices, denoted by $d(u)$.
- $d(1) = d(2) = d(5) = 2$, $d(3) = d(6) = 1$, $d(4) = 0$.
- If $d(u) = 0$, u is called isolated vertex.

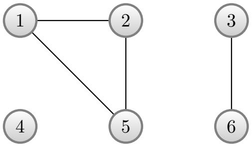

Figure: Undirected graph

- Out-degree of vertex u is the number of outcoming edges, denoted as $deg_-(u)$.
- In-degree of vertex u is the number of incoming edges, denoted as $deg_+(u)$.
- Degree of vertex u is the sum of its in-degree and out-degree, denoted as $deg(u)$.
- $deg_-(2) = 3$, $deg_+(2) = 2$, $deg(2) = 5$.

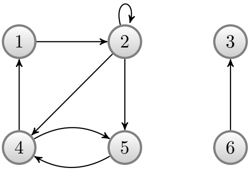

Figure: Digraph

Definitions

- A path $p = \langle v_0, v_1, v_2, \ldots, v_k \rangle$ is a connected sequence of vertices where $(v_{i-1}, v_i) \in E$ for all $i = 1, 2, \ldots, k$.

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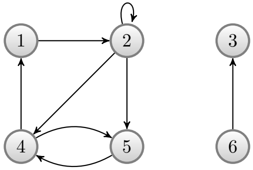

- Give some examples of a path and simple path.
- Give an example of unconnected sequence.

Definitions

- A subpath s of $p = \langle v_0, v_1, v_2, \ldots, v_k \rangle$ is a contiguous subsequence, $s = \langle v_i, v_{i+1}, v_{i+2}, \ldots, v_j \rangle$, for $0 \leq i \leq j \leq k$.

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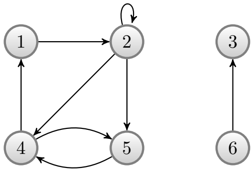
- Closed simple path is called simple cycle.



- What is $\langle 1, 2, 4, 5, 4, 1 \rangle$?
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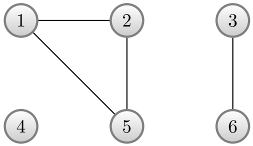
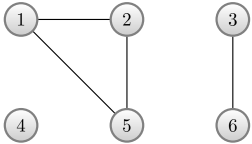
- What is $\langle 2, 2 \rangle$?



- $\langle 1, 2, 5, 1 \rangle$ is an undirected cycle.
- $\langle 3, 6, 3 \rangle$ is not a cycle



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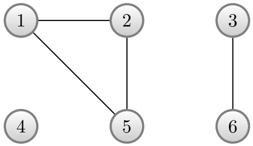

- $\langle 1, 2, 5, 1 \rangle$ is an undirected cycle.
- $\langle 3, 6, 3 \rangle$ is not a cycle , or is it?

- A digraph with no self-loops is simple.
- Acyclic graph contains no cycles.

Special Cases of Graphs

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- Regular graph: For every $u, v \in V$, $d(u) = d(v)$.
- Cycle graph: $n \geq 3$ and vertices are connected in a closed chain.

Tree, Forest

- An undirected graph is connected if every pair of vertices is connected by a path.
- An connected, acyclic, undirected graph is a tree.
  - Homework: Prove that $|E| = |V| - 1$.
- In a rooted tree, there is one special vertex called root (with no parents).
- An acyclic, undirected graph is a forest (several trees).

Bipartite Graph

- Let $G = (V, E)$ be a undirected graph.
- We call G bipartite if the vertex set V can be partitioned into
  $V = L \cup R$,

  where L and R are disjoint and all edges in E go between L and R.
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- Complete bipartite graph $K_{m,n}$: $|L| = m$, $|R| = n$, and $|E| = mn$.

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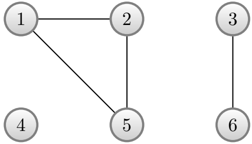

A graph with three connected components:

- $\{1, 2, 5\}$
- $\{3, 6\}$
- $\{4\}$

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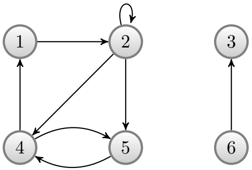

Graph has three strongly connected components:

- $\{1,2,4,5\}$
- $\{3\}$
- $\{6\}$

Graph Representation

Let $G = (V, E)$ be a graph. Denote:

- $n = |V|$
- $m = |E|$.

1. Adjacency-list representation
   - effective for sparse graphs ($m \ll n^2$);
   - we will use this representation in this talk.

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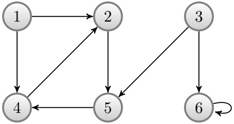
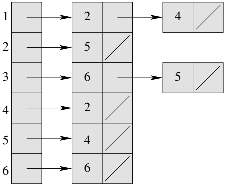
1. Adjacency-list representation
   - effective for sparse graphs ($m \ll n^2$);
   - we will use this representation in this talk.
2. Adjacency-matrix representation
   - effective for dense graphs (m close to n^2);
   - when we often need quick answer whether two given vertices are connected by an edge.

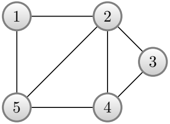
Adjacency-list representation

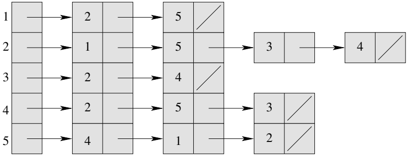
$G = (V, E)$ is represented as

- an array $Adj[1 \dots n]$ with n lists, one list for each vertex,
- where $Adj[u]$ stores all vertices v such that $(u, v) \in E$.









- Space complexity: $\Theta(m + n)$ (depends linearly on the size of the graph).

Weighted graph

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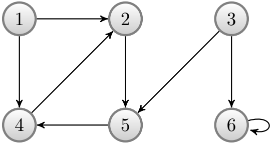
- Representation of $w(u, v)$ in adjacency list: extend the list item (a structure) for v in $Adj[u]$ with value $w(u, v)$.
- Disadvantage: Finding whether an edge (u, v) belongs to E requires the search of the whole list $Adj[u]$.

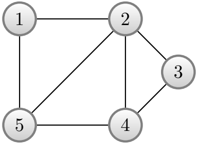
Adjacency-matrix representation

Let $G = (V, E)$ be a graph and assume $V = \{1, 2, \ldots, n\}$. Adjacency matrix $A = (a_{ij})$ is a matrix of size $n \times n$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$



	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1



	1	2	3	4	5
1	0	1	0	0	1
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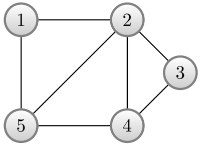
- Space complexity: $\Theta(n^2)$ (independent of the number of edges).



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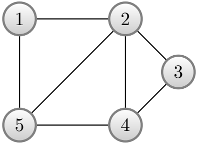

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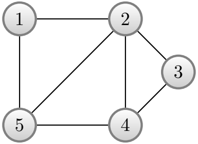

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- If A represents an undirected graph, then $A = A^T$. It is enough to store just one half of A.
- Let $G = (V, E)$ be a weighted graph, then

$$a_{ij} = \begin{cases} w(i,j) & \text{if } (i,j) \in E, \\ \text{NIL} & \text{otherwise,} \end{cases}$$

where NIL is a special value, mostly 0 or ∞.

Exercises

1. Given an adjacency-list representation of a directed graph and a vertex v, how long does it take to compute $deg_-(v)$ and $deg_+(v)$?
2. The transpose of a directed graph $G = (V, E)$ is the graph $G^T = (V, E^T)$, where $E^T = \{(v, u) \in V \times V : (u, v) \in E\}$. Thus, G^T is G with all its edges reversed. Describe an efficient algorithm for computing G^T from G for the adjacency-list representation of G. Analyze the time complexity of your algorithm.
3. The square of a directed graph $G = (V, E)$ is the graph $G^2 = (V, E^2)$ such that $(u, v) \in E^2$ if and only G contains a path with at most two edges between u and v. Describe an efficient algorithm for computing G^2 from G for the adjacency-list representation of G. Analyze the time complexity of your algorithm.

Breath-First Search

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- $color[u] \in \{$WHITE, GREY, BLACK$\}$.
- $\pi[u]$ denotes a predecessor of u at a path from s.
- $d[u]$ denotes a distance of u from s (the number of edges).

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Q ← ∅
ENQUEUE(Q, s)
while Q ≠ ∅
     do u ← DEQUEUE(Q)
        for each v ∈ Adj[u]
            do if color[v] = WHITE
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                       d[v] ← d[u] + 1
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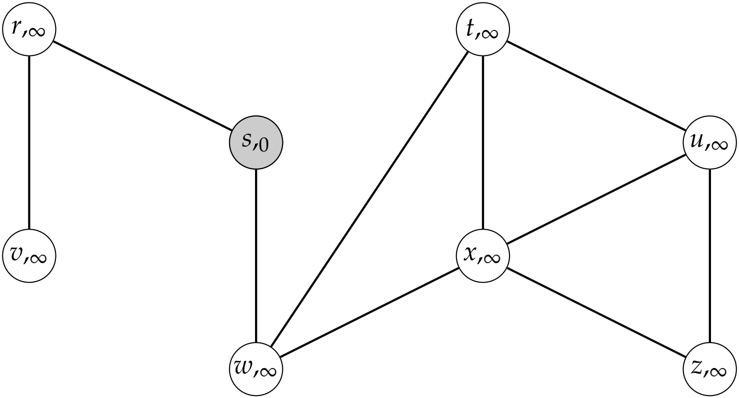
BFS – Example



$$Q = \langle s \rangle$$

Figure: Note: We use red color to show BLACK vertices.

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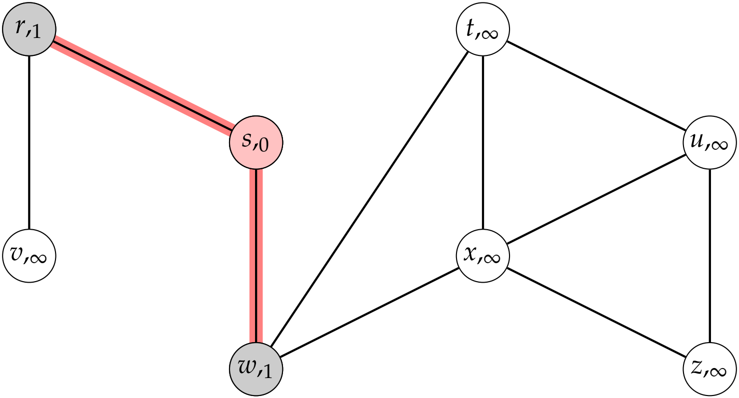

$$Q = \langle wr \rangle$$

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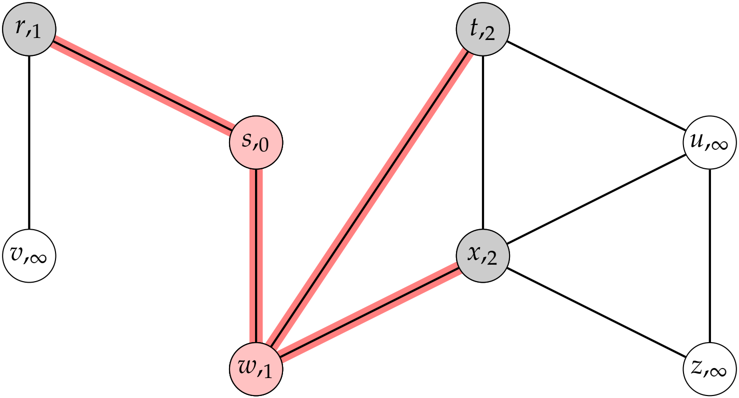

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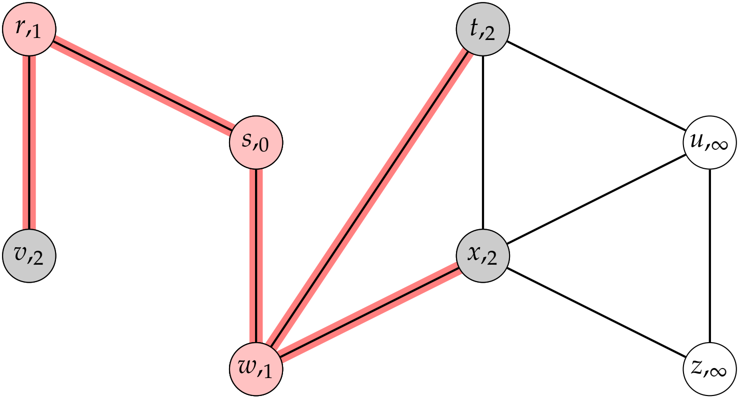

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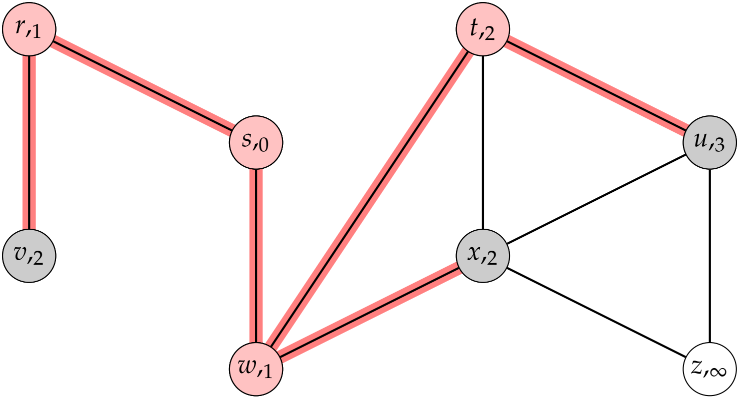

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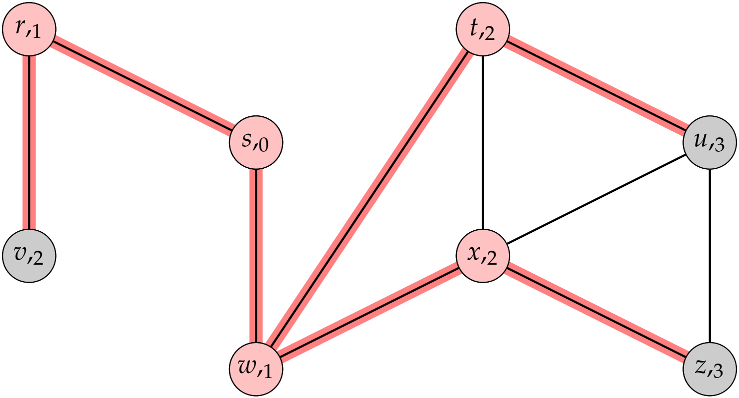

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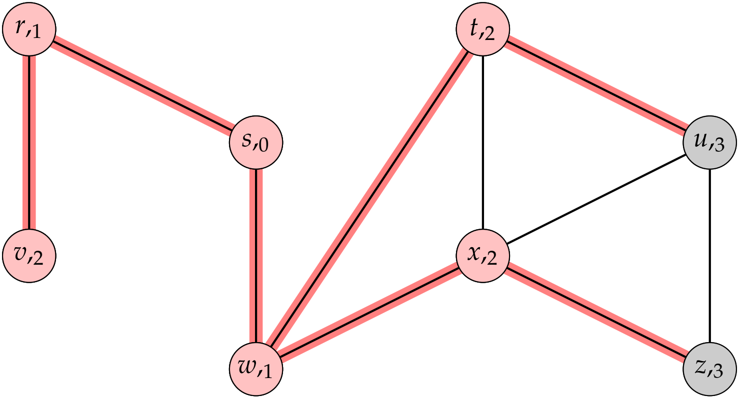

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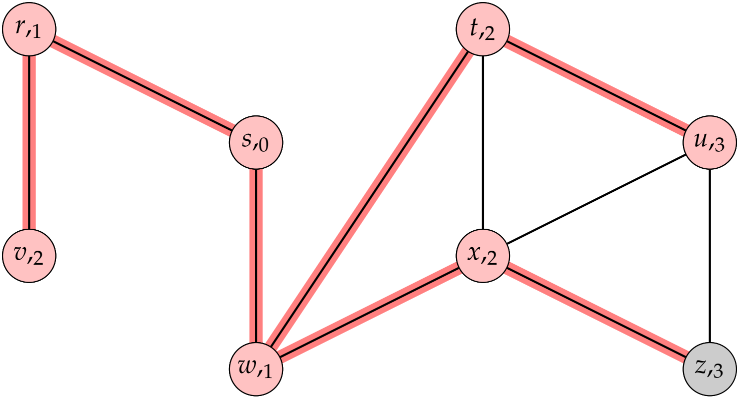

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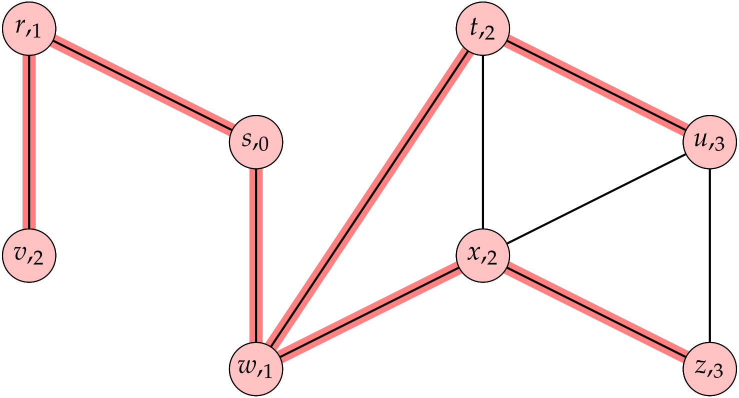

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- ENQUEUE and DEQUEUE takes $O(1)$, so the aggregation of all queue operations takes $O(n)$.
- Since it scans the adjacency list of each vertex only after it is dequeued, each adjacency list is scanned at most once.

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- Observe that the sum of the lengths of all the adjacency lists is $\Theta(m)$, the total time of scanning is $O(m)$.
- The overhead for initialization is $O(n)$, so the total running time of BFS is $O(m+n)$. Thus, it is linear in the size of G (adjacency-list representation).

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- A path of length $\delta(s, v)$ from s to v is called a shortest path from s to v.

Lemma 2.

Let $G = (V, E)$ be a (di)graph and $s \in V$ be a vertex. Then, for every edge $(u, v) \in E$,

$$\delta(s, v) \leq \delta(s, u) + 1 \, .$$

Proof.

- If vertex u is reachable from s, then vertex v is reachable from s as well. Therefore, the shortest path from s to v is no longer than a shortest path from s to u followed by edge (u, v). So inequality holds.

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Lemma 3.

Let $G = (V, E)$ be a (di)graph and assume that BFS is executed on G from vertex $s \in V$. Then, when BFS finishes, then $d[v] \geq \delta(s, v)$ for every $v \in V$.

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- By induction on the number of ENQUEUE operations. Induction Hypothesis (IH): Assume that $d[v] \geq \delta(s, v)$ for every $v \in V$.

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- Let v is WHITE vertex discovered during the exploration from u. By IH, we have $d[u] \geq \delta(s, u)$. By line 15 of BFS, IH, and the previous lemma,

  $$d[v] = d[u] + 1 \geq \delta(s, u) + 1 \geq \delta(s, v) .$$

  Since v is GREY now (and enqueued) and lines 14–17 are executed only for WHITE vertices, v cannot be enqueued again and its $d[v]$ value remains unchanged.

Lemma 4.

During the execution of BFS on $G = (V, E)$, *let queue* Q *contains vertices* $\langle v_1, v_2, \ldots, v_r \rangle$, *where* v_1 *is the front item of* Q *(leader) and* v_r *is the last item of* Q. *Then,* $d[v_r] \leq d[v_1] + 1$ *and* $d[v_i] \leq d[v_{i+1}]$ *for* $i = 1, 2, \ldots, r - 1$.

Proof.

- By induction on the number of queue operations. First, $Q = \langle s \rangle$, so lemma holds. It holds after execution of both queue operations:

□

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During the execution of BFS on $G = (V, E)$, *let queue* Q *contains vertices* $\langle v_1, v_2, \ldots, v_r \rangle$, *where* v_1 *is the front item of* Q *(leader) and* v_r *is the last item of* Q. *Then,* $d[v_r] \leq d[v_1] + 1$ *and* $d[v_i] \leq d[v_{i+1}]$ *for* $i = 1, 2, \ldots, r-1$.

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- By induction on the number of queue operations. First, $Q = \langle s \rangle$, so lemma holds. It holds after execution of both queue operations:
- v_1 is removed so v_2 is new leader (if Q is emptied, it holds trivially). By IH, $d[v_1] \leq d[v_2]$. But then, $d[v_r] \leq d[v_1] + 1 \leq d[v_2] + 1$ and the rest of inequalities is unchanged.

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- v_{r+1} is inserted into Q (line 17). In that time, u (whose adjacency list is being explored) is already removed from Q. By IH, $d[u] \le d[v_1]$. So, $d[v_{r+1}] = d[u] + 1 \le d[v_1] + 1$. Therefore, $d[v_r] \le_{IH} d[u] + 1 = d[v_{r+1}]$. The rest of inequalities is unchanged.

□

Corollary 5.

Let vertices v_i and v_j are stored in the queue during the computation of BFS such that v_i is inserted before v_j. Then, $d[v_i] \leq d[v_j]$ in the moment of insertion of v_j into the queue.

Proof.

By the previous lemma and the property that every vertex obtains final value of d at most once during the computation of BFS. □

Theorem 6 (Correctness of BFS).

Let $G = (V, E)$ be (di)graph and $s \in V$. Then, $BFS(G, s)$ explores all vertices $v \in V$ reachable from s and after it is finished $d[v] = \delta(s, v)$ for all $v \in V$. In addition, for every vertex $v \neq s$ reachable from s one of the shortest paths from s to v is a shortest path from s to $\pi[v]$ followed by edge $(\pi[v], v)$.

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- Altogether, $d[v] > \delta(s, v) = \delta(s, u) + 1 = d[u] + 1$.

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- v is GREY, then v is greyed during picking another vertex w that was dequeued from Q before u. In addition, $d[v] = d[w] + 1$. By Corollary 5, $d[w] \leq d[u]$, i.e. $d[v] \leq d[u] + 1$ – contradiction.

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- Therefore, $d[v] = \delta(s, v)$ for all $v \in V$. Furthermore, all vertices reachable from s must be visited, otherwise its d value is infinity.
- Finally, observe that if $\pi[v] = u$, then $d[v] = d[u] + 1$; that is, a shortest path from s to v can be obtained by addition of edge $(\pi[v], v)$ to the end of a shortest path from s to $\pi[v]$.

□

Breadth-First Search Tree (BFS Tree)

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- G_π is BFS tree, if V_π contains only vertices reachable from s and for all $v \in V_\pi$, there exists the only path from s to v that is the shortest path.
- Since G_π is connected and $|E_\pi| = |V_\pi| - 1$, G_π is a tree.

Lemma 7.

Let G be (di)graph. Procedure BFS constructs π such that G_π is BFS tree.

Proof.

- Line 16 of BFS sets $\pi[v] = u$ iff $(u, v) \in E$ and $\delta(s, v) < \infty$.

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- V_π contains only vertices reachable from s.
- Since G_π is tree, G_π contains only one path from s to each other vertex.
- By inductive application of Theorem 6, each such path is a shortest one.

□

How to print the shortest path from s to v?

```
PRINT-PATH(G, s, v)
if v = s
   then print s
   else if π[v] = NIL
           then print "No path from " s " to " v "!"
           else PRINT-PATH(G, s, π[v])
                print v
```

Its time complexity is $O(n)$.

Exercises

1. Given an example of a directed graph $G = (V, E)$, a source vertex $s \in V$, and a set of tree edges $E_\pi \subseteq E$ such that for each vertex $v \in V$, the unique simple path in the graph (V, E_π) from s to v is a shortest path in G, yet E_π cannot be produced by running $\mathrm{BFS}(G, s)$, no matter how the vertices are ordered in each adjacency list.
2. Give an efficient algorithm to compute whether the given undirected graph is bipartite.
3. The diameter of a tree $T = (V, E)$ is defined as $max_{u,v \in V}\delta(u, v)$, that is, the largest of all shortest-path distances in the tree. Give an efficient algorithm to compute the diameter of a tree, and analyze the running time of your algorithm.

Depth-First Search

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- On contrary to BFS, DFS visits all vertices.
- It colors the vertices with WHITE, GREY, and BLACK color as well.
- The array of predecessors π is in use.
- Creates a DFS forest that contains all vertices such that $G_\pi = (V, E_\pi)$, where

$$E_\pi = \{(\pi[v], v) : v \in V,\ \pi[v] \neq \text{NIL}\}.$$

- Graph representation – Adjacency-list representation.
- $color[u] \in \{$WHITE, GREY, BLACK$\}$.
- $d[u]$ is a timestamp of the first vertex discover (color changed to GREY).
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- *time* is a global variable (ticks after each *color* change).

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DFS(G)
for each vertex u ∈ V
    color[u] ← WHITE
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time ← 0
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color[u] ← GREY
time ← time + 1
d[u] ← time
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       then π[v] ← u
            DFS-VISIT(G, v)
color[u] ← BLACK
time ← time + 1
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```

DFS – Example

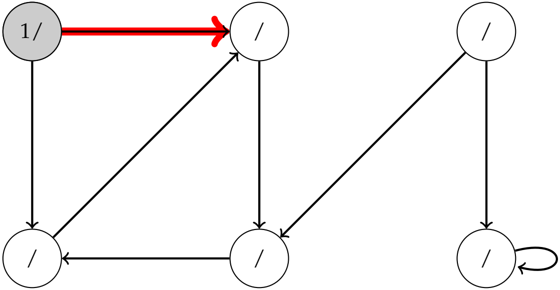

Figure: Vertex u is labeled by $d[u]/f[u]$. B, F, and C denote Back, Forward, and Cross edge, respectively.

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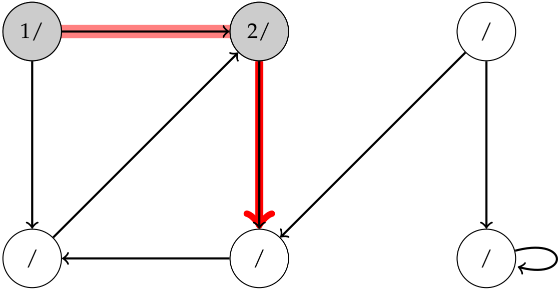

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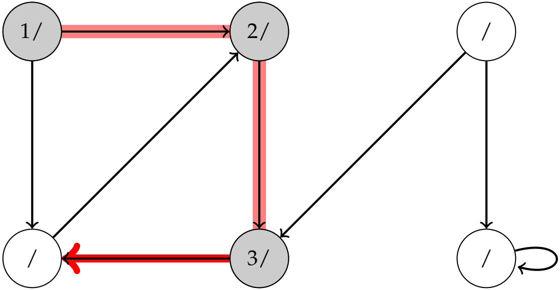

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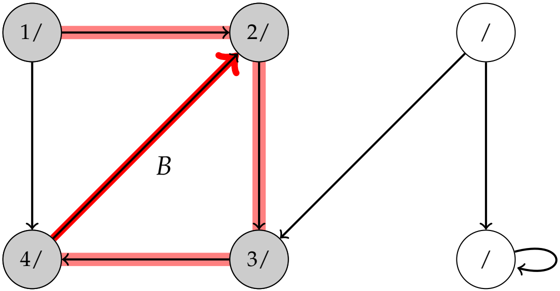

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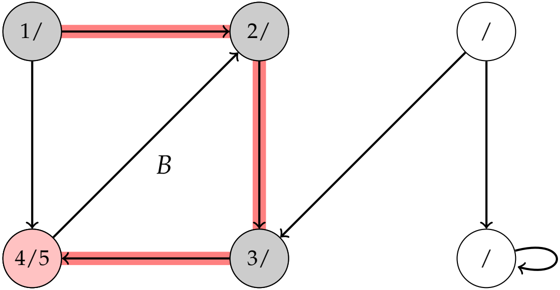

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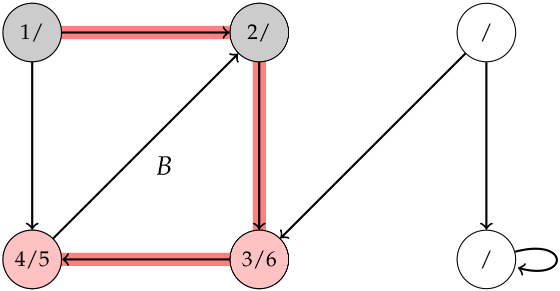

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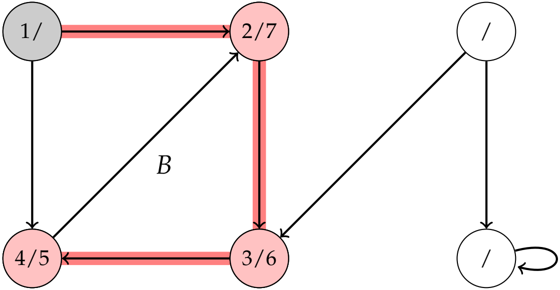

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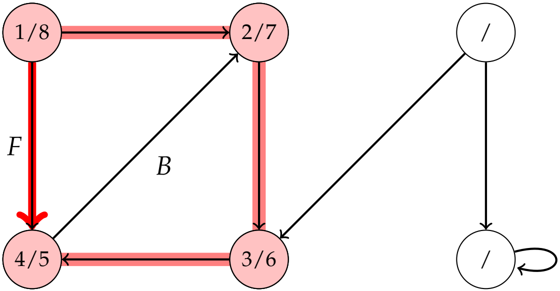

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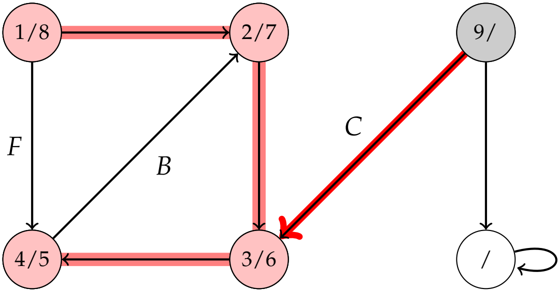

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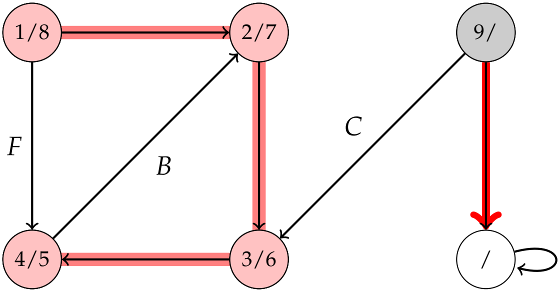

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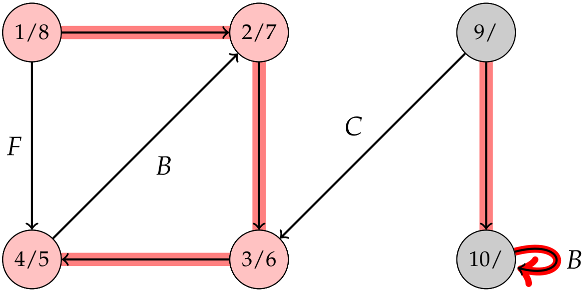

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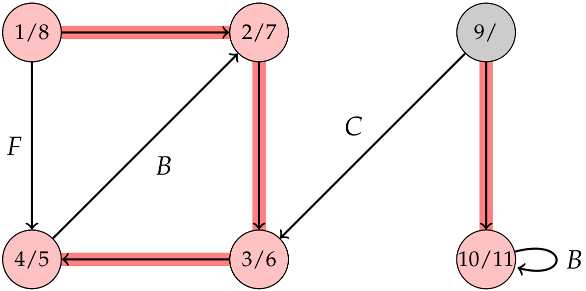

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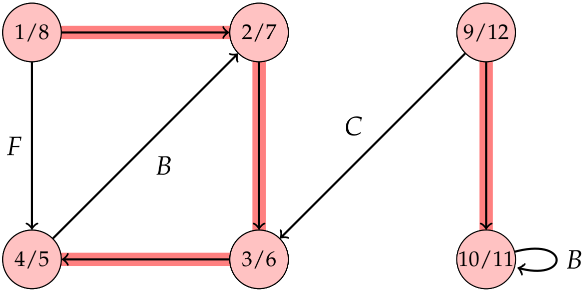

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7          then DFS-VISIT(G, u)
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- Loops at lines 1–3 and 5–7 without DFS-VISIT calls take $\Theta(n)$.

Time Complexity of DFS-VISIT

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DFS-VISIT(G, u)
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    d[u] ← time
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- Therefore, the running time is $\Theta(m + n)$.

Parenthesis Theorem

In any DFS of a graph $G = (V, E)$, for any two vertices u and v, exactly one of the following conditions holds:

- intervals $[d[u], f[u]]$ and $[d[v], f[v]]$ are disjoint, and neither u nor v is descendant of the other in DFS forest,
- interval $[d[u], f[u]]$ is contained within the interval $[d[v], f[v]]$ and u is a descendant of v in a DFS tree, or
- interval $[d[v], f[v]]$ is contained within the interval $[d[u], f[u]]$ and v is a descendant of u in a DFS tree.

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- intervals $[d[u], f[u]]$ and $[d[v], f[v]]$ are disjoint, and neither u nor v is descendant of the other in DFS forest,
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Proof for $d[u] < d[v]$ (Homework: prove case $d[v] < d[u]$).

- Subcase $d[v] < f[u]$: Then, v was discovered while u was still GREY. Since v was discovered later than u, v is finished before u. Hence, $f[v] < f[u]$.

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- Subcase $f[u] < d[v]$: Then, from the definition $d[u] < f[u]$ and $d[v] < f[v]$, so both intervals are disjoint. Moreover, neither vertex was discovered while the other was GREY, and so neither vertex is a descendant of the other.

Corollary 8.

Vertex v is descendant of vertex u in DFS forest of $G = (V, E)$ iff

$$d[u] < d[v] < f[v] < f[u] .$$

White Path Theorem

In DFS forest of graph $G = (V, E)$, vertex v is descendant of vertex u iff in time $d[u]$ there is a path from u to v from WHITE vertices only.

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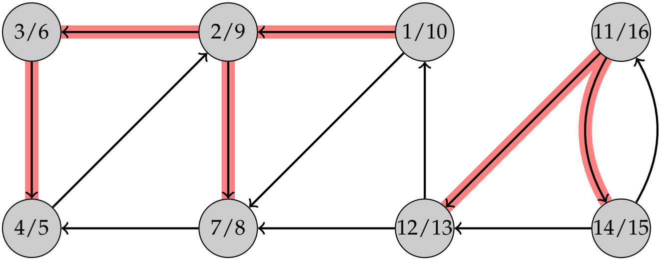
- Since v must be discovered after u but before finishing w, we have $d[u] < d[v] < f[w] \leq f[u]$.
- Parenthesis Theorem says that interval $[d[v], f[v]]$ is completely included in interval $[d[u], f[u]]$. And by the previous corollary, v is descendant of u.

Edge Classification

1. **Tree edges** are edges in DFS forest G_π. (u, v) is a tree edge if v was firstly discovered by exploring edge (u, v). These edges are highlighted using red color in the figures.
2. **Back edges** are edges (u, v) connecting u to its predecessor v in DFS forest. Self-loop is always back edge.
3. **Forward edges** are non-tree edges (u, v) connecting u to its descendant v in DFS forest.
4. **Cross edges** are all other edges.

Edge Classification – Example



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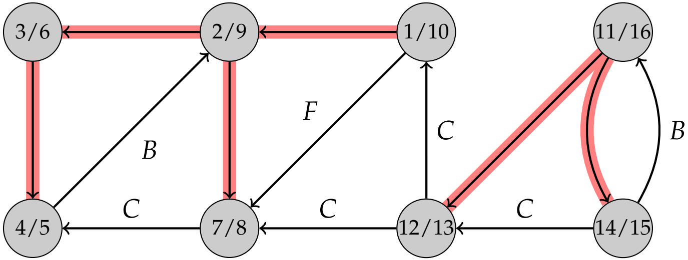
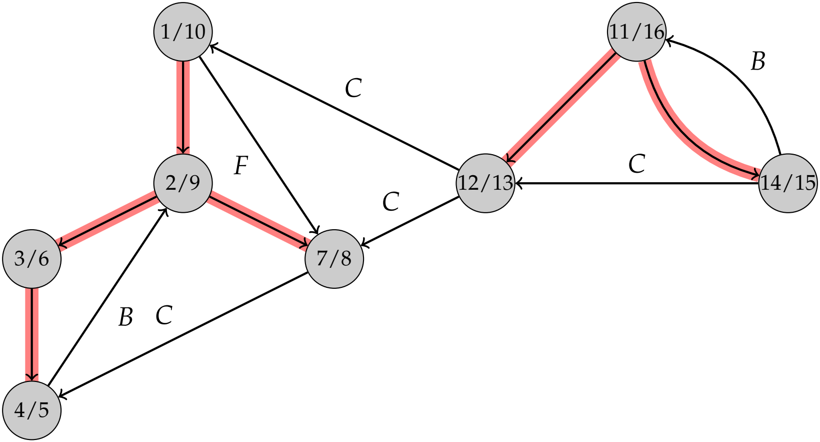


Drawing a Graph

We can draw every graph such that tree and forward edges lead downwards and back edges lead upwards.



DFS and Edge Classification

Let (u, v) be an edge. Then, using a color of v during DFS computation, we can classify (u, v) as follows:

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1. WHITE indicates a tree edge,
2. GREY indicates a back edge, and
3. BLACK indicates a forward or cross edge:
   - (u, v) is a forward edge, if $d[u] < d[v]$.
   - (u, v) is a cross edge, if $d[u] > d[v]$.

Edge Classification in Undirected Graph

Theorem 9.
During the DFS computation of undirected graph G, each edge is either a tree edge or a back edge.

Proof.

- Let (u,v) is an arbitrary edge of G and let $d[u] < d[v]$.

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- If (u, v) is firstly explored in the direction from u to v, then v is WHITE – otherwise we would have explored (u, v) in the other direction (from v to u). Thus, (u, v) is a tree edge.

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- If (u, v) is firstly explored in the direction from v to u, u is still GREY – since u is still GREY at the time the edge is explored for the first time, then (u, v) is a back edge.

□

Exercises

1. Give an efficient algorithm to find whether a given directed graph contains a cycle, and analyze the running time of your algorithm.

2. Let G be an undirected graph. Show how to modify $\textsc{Dfs}$ so that it assigns to each vertex v an integer label between 1 and k in array cc, where k is the number of connected components of G, such that $cc[u] = cc[v]$ if and only if u and v are in the same connected component.

Topological sort

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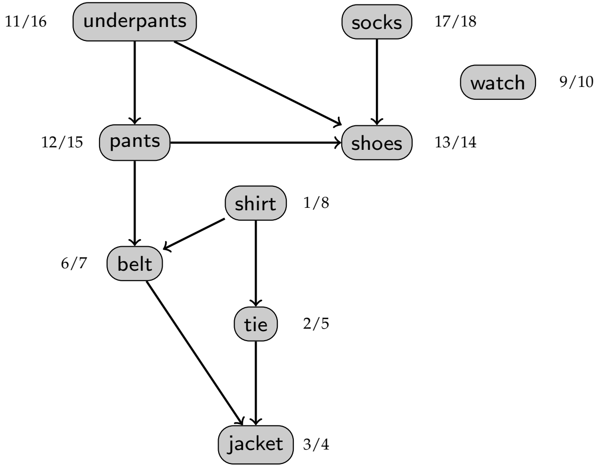
- Time complexity: DFS is $\Theta(m+n)$, add a vertex to the list is constant, so, in total, $\Theta(m+n)$.

Topological sort – Example



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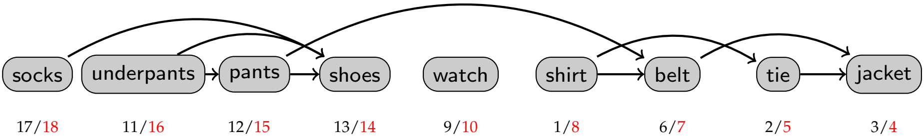

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□

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- If v is WHITE, then v is descendant of u in DFS forest, so $f[v] < f[u]$.
- If v is BLACK, then $f[v]$ is already set. Since u is still in exploration process (grey), its $f[u]$ is not set yet, so $f[v] < f[u]$.

□

Exercises

1. Give a linear-time algorithm that takes as input a directed acyclic graph $G = (V, E)$ and two vertices s and t, and returns the number of simple paths from s to t in G.

2. Prove or disprove: If a directed graph G contains cycles, then $\textsc{Topological-Sort}(G)$ produces a vertex ordering that minimizes the number of "bad" edges that are inconsistent with the ordering produced.

Strongly Connected Components

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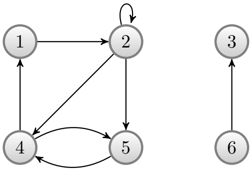

Graph with 3 SCCs:

- $\{1, 2, 4, 5\}$
- $\{3\}$
- $\{6\}$

- The transpose graph of $G = (V, E)$ is $G^T = (V, E^T)$, where $E^T = \{(u, v) : (v, u) \in E\}$.

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3  call modified DFS(G^T) such that DFS's main iteration takes vertices
           in the decreasing order according to f[u]
4  output all vertices of each DFS tree computed in line 3
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- How to create G^T from G in the adjacency-lists representation in time $O(m + n)$?
- G and G^T has the same SCCs – u and v are mutually reachable in G if and only if they are mutually reachable in G^T.

SCC – Example

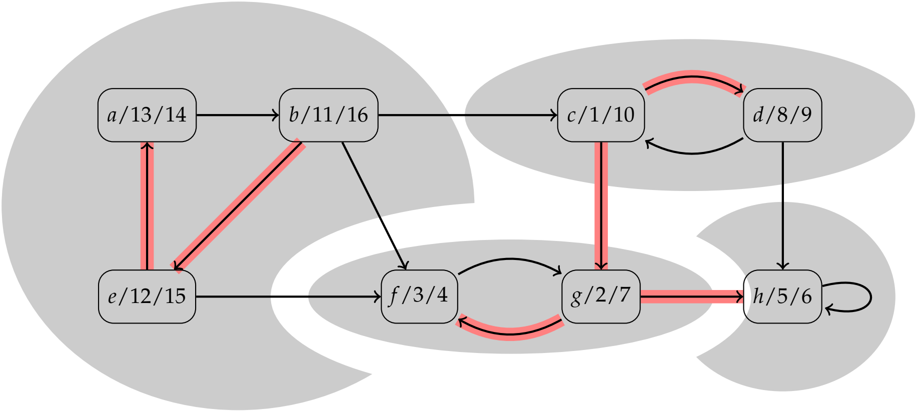


Figure: Result of line 1 of $\text{SCC}(G)$. Tree edges are red. Grey background forms the boundary of SCCs.

SCC – Example

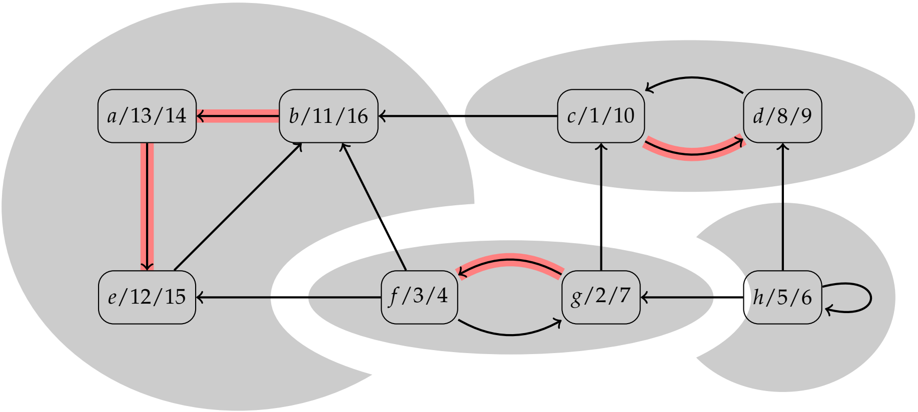
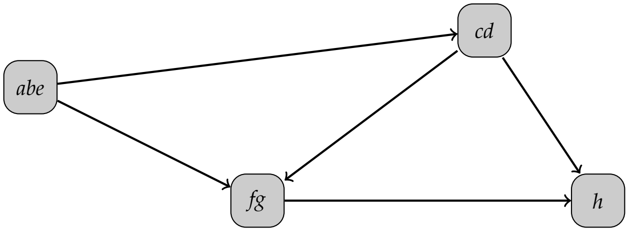


Figure: Graph G^T and result of line 3 of $\textsc{Scc}(G)$. b, c, g and h – roots in DFS forest. Each tree $\approx$ one SCC.

- The component graph of $G = (V, E)$ is graph $G^{scc} = (V^{scc}, E^{scc})$ defined as follows:
  - Let $C_1, C_2, \ldots, C_k$ be SCCs of G.
  - $V^{scc} = \{v_1, v_2, \ldots, v_k\} \subseteq V$, $V^{scc} \cap C_i \neq \emptyset$, $i = 1, 2, \ldots, k$.
  - $(v_i, v_j) \in E^{scc}$, if there exist $x \in C_i$ and $y \in C_j$ such that $(x, y) \in E$.
  - Informally: By contracting all edges incident to the vertices of the same SCCs, we get G^{SCC}.



Properties of Component Graph

Lemma 12.
Let C, C' *be two different SCCs of a digraph* $G = (V, E)$. *Let* $u, v \in C$, $u', v' \in C'$ *and* $u \rightsquigarrow u'$ *in* G. *Then, it DOES NOT hold that* $v' \rightsquigarrow v$.

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- In what follows, consider only times $d[u]$ and $f[u]$ computed by the first call of DFS procedure.
- If necessary, the values from the second call of DFS are denotes as $d_3[u]$ and $f_3[u]$.

- Let $U \subseteq V$. Then, $d(U) = \min_{u \in U}\{d[u]\}$ and $f(U) = \max_{u \in U}\{f[u]\}$.

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Proof

- 1) $d(C) < d(C')$ – let x be the first discovered vertex in C. In time $d[x]$, all vertices from $C \cup C'$ are WHITE. For $w \in C'$ there exists a WHITE path $x \rightsquigarrow u \to v \rightsquigarrow w$. By WHITE path theorem, all vertices from $C \cup C'$ are descendants of x in its DFS tree. Then, collorary from Parenthesis theorem says that $f[x] = f(C) > f(C')$.

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- 1) $d(C) < d(C')$ – let x be the first discovered vertex in C. In time $d[x]$, all vertices from $C \cup C'$ are WHITE. For $w \in C'$ there exists a WHITE path $x \rightsquigarrow u \to v \rightsquigarrow w$. By WHITE path theorem, all vertices from $C \cup C'$ are descendants of x in its DFS tree. Then, collorary from Parenthesis theorem says that $f[x] = f(C) > f(C')$.
- 2) $d(C) > d(C')$ – let y be the first discovered in C'. In time $d[y]$, all vertices from C' are WHITE and there exists a WHITE path from y to every vertex of C'. By WHITE path theorem and corollary of Parenthesis theorem, we have $f[y] = f(C')$.

- Let $U \subseteq V$. Then, $d(U) = \min_{u \in U}\{d[u]\}$ and $f(U) = \max_{u \in U}\{f[u]\}$.

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Corollary 14.

Let C*,* C' *be two different SCCs of a digraph* $G = (V, E)$*. Let* $(u, v) \in E^T$*,* $u \in C$*,* $v \in C'$*. Then,* $f(C) < f(C')$*.*

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$(u,v) \in E^T$ implies that $(v,u) \in E$. Since SCCs of G and SCCs of G^T coincide, the previous lemma implies $f(C) < f(C')$. □

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Closing times of the second DFS

Observe that $f_3(C) > f_3(C')$ so $(u, v) \in E^T$ is a cross edge according to the classification from the second DFS.

Theorem 15.
$\textsc{Scc}(G)$ *procedure is correct.*

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- By IH and the previous corollary, every edge of G^T leads from C to some already visited SCC.
- So no vertex from another SCC (different from C) is descendant of u during DFS of G^T. Therefore, the vertices of the tree form an SCC.

Exercises

1. How can the number of strongly connected components of a graph change if a new edge is added?

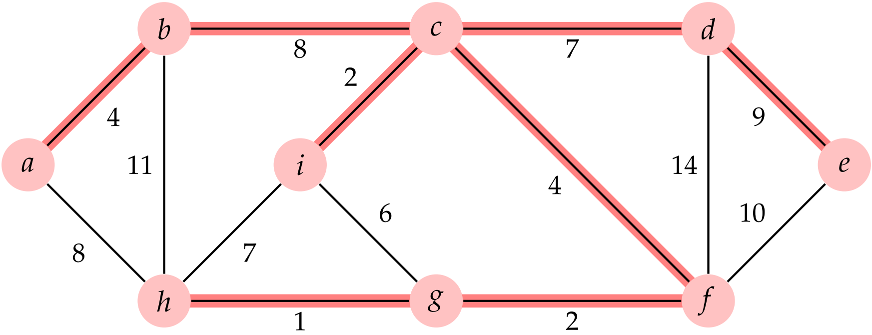
2. Give an $O(n+m)$-time algorithm to compute the component graph of digraph $G = (V, E)$. Make sure that there is at most one edge between two vertices in the resulting graph (E is not a multiset).

Minimum Spanning Trees

Minimum Spanning Tree (MST)

- The first algorithm by mathematician from Brno, O. Borůvka, 1926 (in Czech).
- Let $G = (V, E)$ be a connected undirected graph with weight function
$$w : E \to \mathbb{R}\,.$$
- Goal: Find a subset of edges $T \subseteq E$ such that subgraph (V, T) is connected, acyclic and
$$w(T) = \sum_{(u,v) \in T} w(u, v)$$
is minimal.

Minimum Spanning Tree – Example



Generic Algorithm

```
GENERIC-MST(G, w)
A ← ∅
while A does not form a spanning tree
      do find an edge (u, v) ∈ E that is safe for A
          A ← A ∪ {(u, v)}
return A
```

- Loop invariant: Prior to each iteration, A is a subset of some MST.
- Edge $(u, v) \in E$ is safe edge for A, since $A \cup \{(u, v)\}$ maintains the invariant.
- Note: Greedy algorithm – making choice that is the best at the moment.

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- A cut respects a set of edges A if no edge from A crosses the cut.
- An edge is a light edge crossing a cut if its weight is the minimum of any edge crossing the cut.

Theorem 16.

- *Let* $G = (V, E)$ *be a connected, undirected graph with real-valued weight function* w*.*
- *Let* $A \subseteq E$ *is included in some MST for* G*.*
- *Let* $(S, V - S)$ *be any cut of* G *that respects* A*.*
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- Let (x, y) lies on $u \rightsquigarrow v$ in T crossing $(S, V - S)$. Since, the cut respects A, $(x, y) \notin A$.
- $T' = (T - \{(x, y)\}) \cup \{(u, v)\}$ is a spanning tree of G. Is T' minimal?

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- T is a MST, therefore $w(T) \leq w(T')$.

- Since $A \subseteq T$ and $(x, y) \notin A$, $A \subseteq T'$.
- Finally, $A \cup \{(u, v)\} \subseteq T'$. Since T' is MST as well, (u, v) is safe for A.

□

Exercises

1. Give a simple example of a connected graph $G = (V, E)$ such that the set of edges $\{(u, v) :$ there exists a cut $(S, V - S)$ such that (u, v) is a light edge crossing $(S, V - S)\}$ does not form a MST for G.

2. Show that a graph has a unique MST if, for every cut of the graph, there is a unique light edge crossing the cut. Show that the converse is not true by giving a counterexample.

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- Based on the generic greedy algorithm.
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- Kruskal: Set A forms a forest. Safe edge for A is an edge with the smallest weight connecting two different connected components.

- Prim (Jarník): Set A is a tree. Safe edge for A is an edge with the smallest weight connecting tree A with a (yet) non-tree vertex.

Kruskal Algorithm

Disjoint Dynamic Sets

- Set of non-empty sets $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$
- Each set S_i identified by a representative (some member of S_i)
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Implementation (Data structure)

- Linked-list representation (with weight-union heuristic; $O(m + n \log n)$)
- Rooted trees (with heuristics "union by rank" and "path compression"; $O(m\alpha(n))$, where α grows very slowly ($\alpha(n) \leq 4$))

Kruskal Algorithm

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KRUSKAL-MST(G, w)
1  A ← ∅
2  for each vertex v ∈ V
3       do MAKE-SET(v)
4  sort the edges of E into nondescreasing order by weight w
5  for each edge (u, v) ∈ E, taken in the order from step 4
6       do if FIND-SET(u) ≠ FIND-SET(v)
7              then A ← A ∪ {(u, v)}
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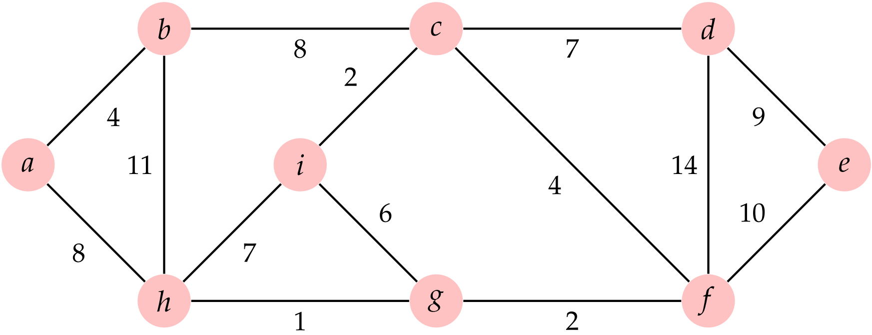
- G is connected, so $m \geq n - 1$. Then, sets operations take $O(m\alpha(n))$. Since $\alpha(n) = O(\log n) = O(\log m)$, sorting outweighs by $O(m \log m)$.
- Notice that $m < n^2$, so $\log m = O(\log n)$. Therefore, $O(m \log n)$.

Kruskal Algorithm – Example



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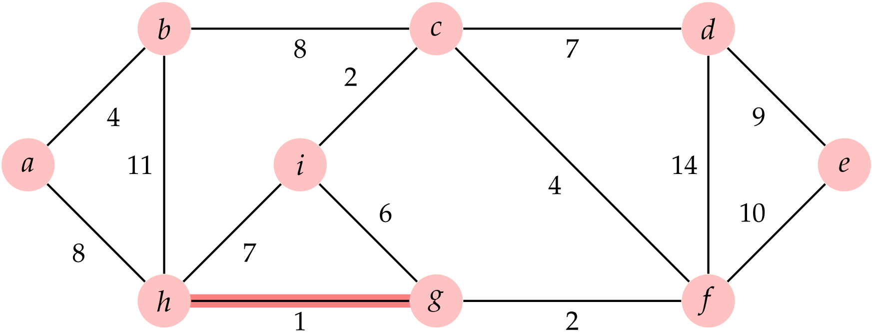

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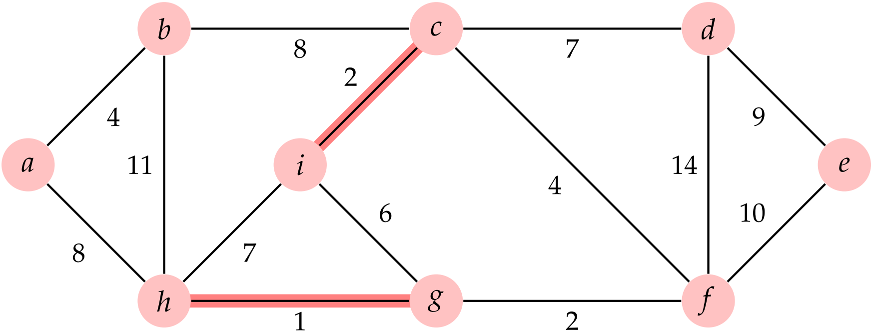

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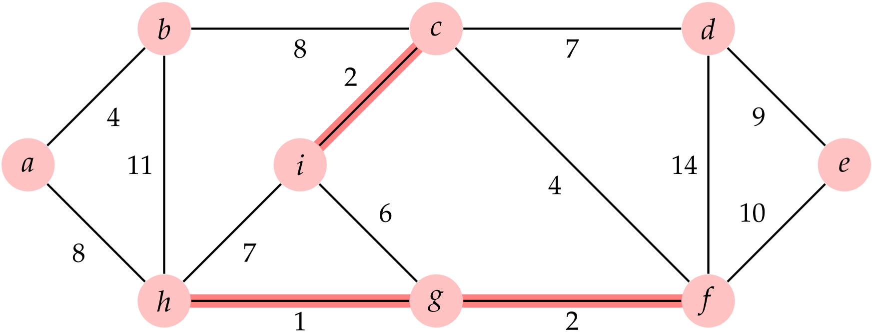

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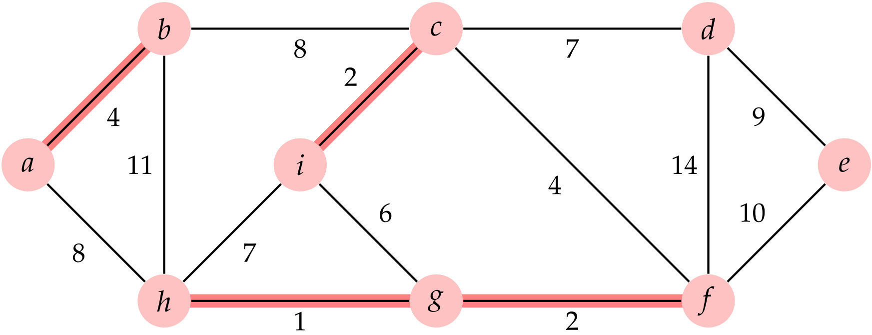

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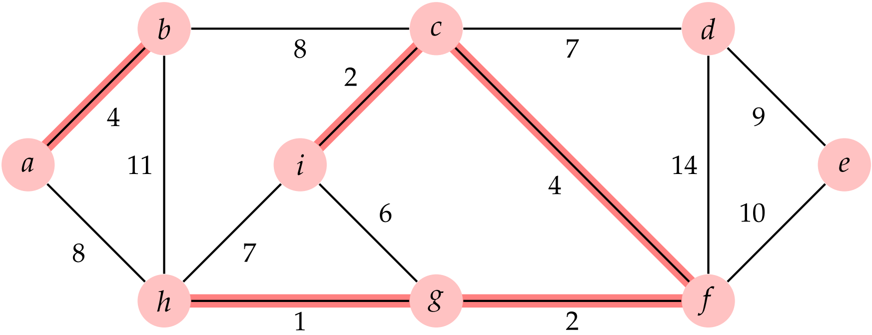

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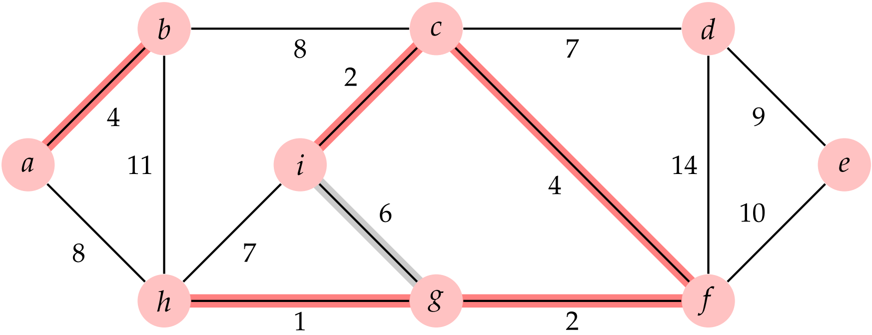

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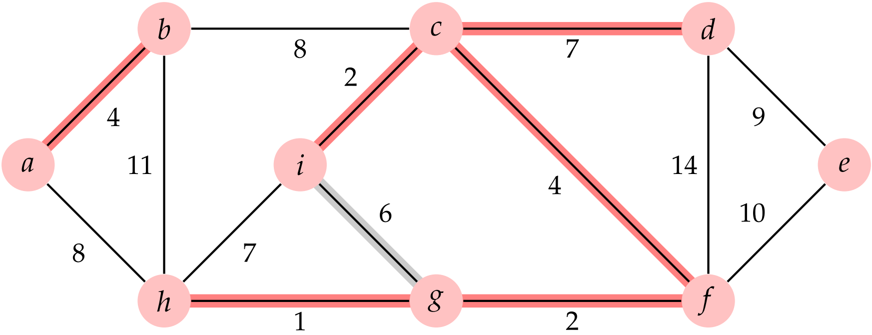

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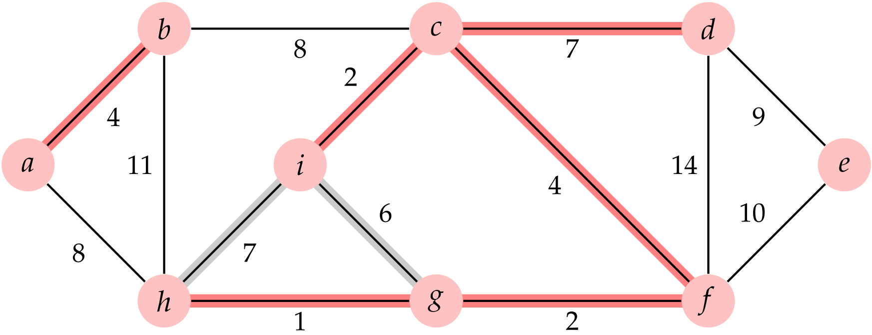

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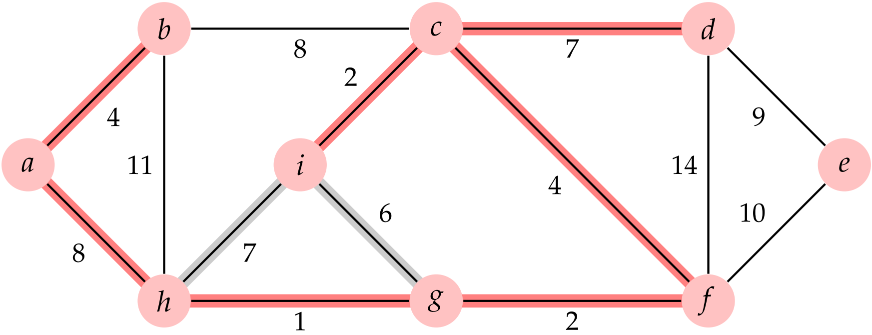

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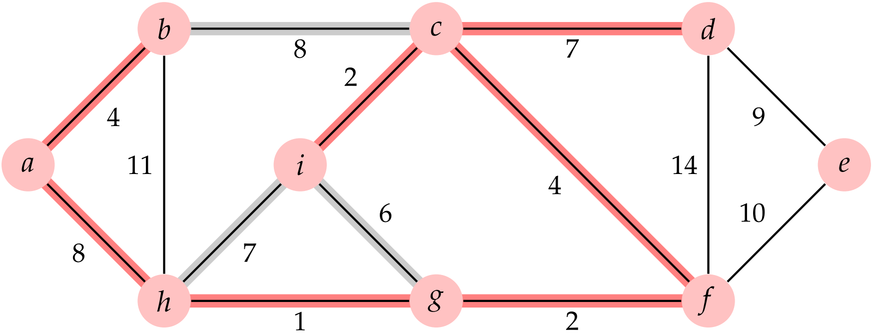

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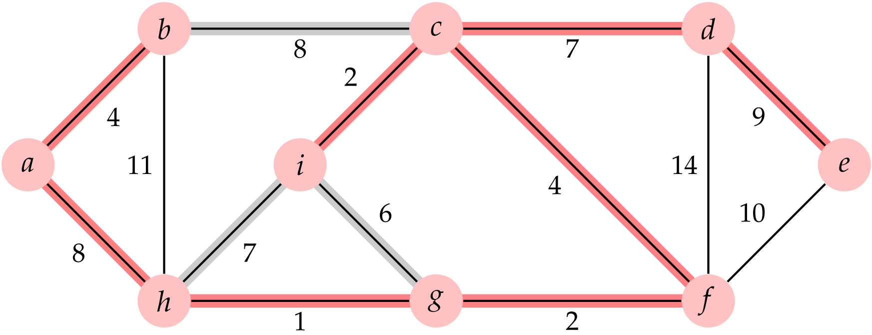

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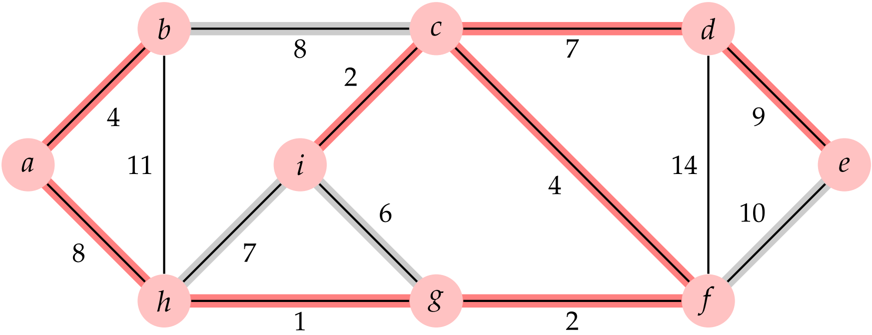

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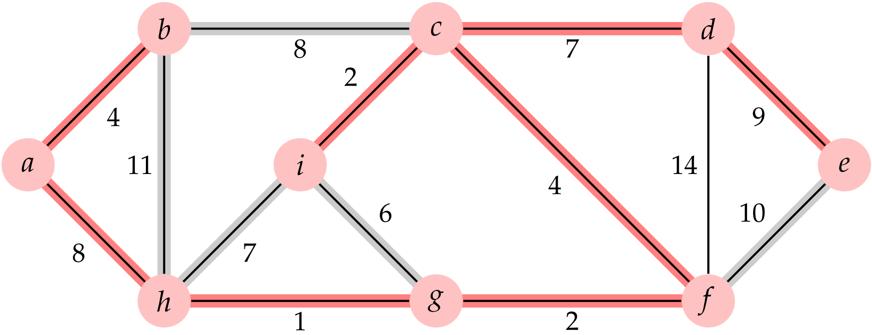

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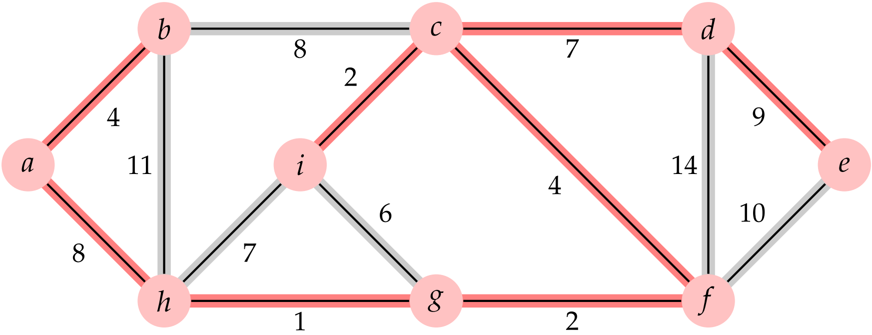

Prim Algorithm

Min-Priority Queue

- Data structure for maintaining a set of elements, each with an associated key (priority)
- Duality with max-priority queue
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- $\textsc{Insert}(Q, v)$ inserts vertex v into queue Q ($Q = Q \cup \{v\}$).
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Implementation (Data structure)

- Binary heap in array $A[1..n]$ with $A[\text{PARENT}(i)] \leq A[i]$ (each operation: $O(\log n)$)
- Fibonacci heap (DECREASE-KEY only $O(1)$)

Prim algorithm

```
PRIM-MST(G, w, r)
for each vertex u ∈ V
     do key[u] ← ∞
        π[u] ← NIL
key[r] ← 0
Q ← V
while Q ≠ ∅
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        for each v ∈ Adj[u]
            do if v ∈ Q and w(u, v) < key[v]
                  then π[v] ← u
                       DECREASE-KEY(Q, v, w(u, v))
```

Invariant:

- $A = \{(v, \pi[v]) : v \in V - \{r\} - Q\}$.
- If v belongs to a MST, then $v \in V - Q$.
- For all $v \in Q$, if $\pi[v] \neq$ NIL, then $key[v] < \infty$ and $key[v]$ is the weight of light edge $(v, \pi[v])$ that connects v to some vertex in $V - Q$.

Prim algorithm – Time Complexity (Binary Heap)

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- Line 9 can be done in $O(1)$. Why?
- Line 11 takes $O(\log n)$.
- In total, $O(n \log n + m \log n) = O(m \log n)$.

Prim Algorithm – Time Complexity

Implementation of Q by Fibonacci heap:

- EXTRACT-MIN operation takes $O(\log n)$ amortized time.
- DECREASE-KEY operation takes only $O(1)$ amortized time.

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- DECREASE-KEY operation takes only $O(1)$ amortized time.
- Together, we have $O(m + n \log n)$.

Prim Algorithm – Example

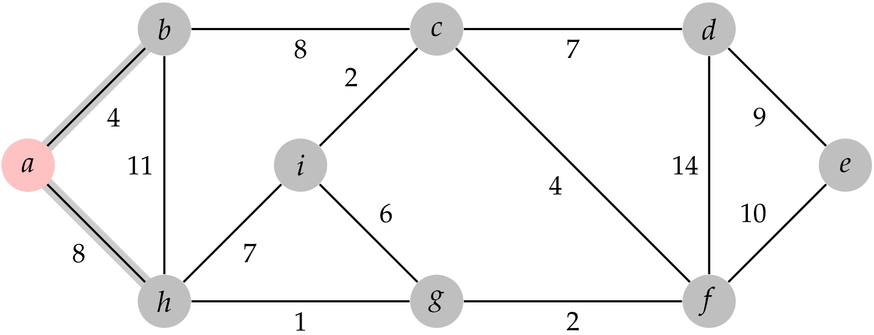

Figure: Gray edges crosses the cut $(V - Q, Q)$.

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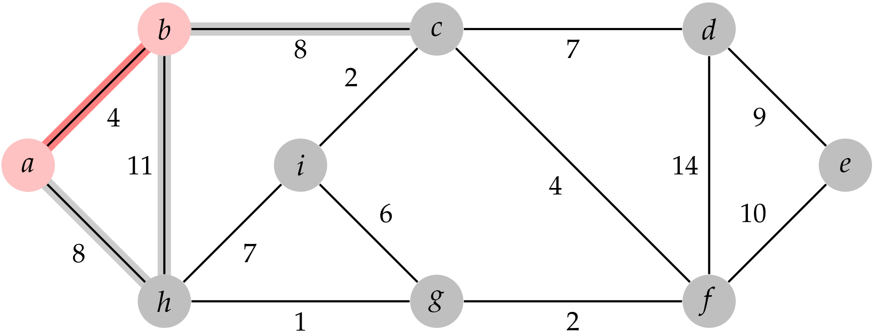

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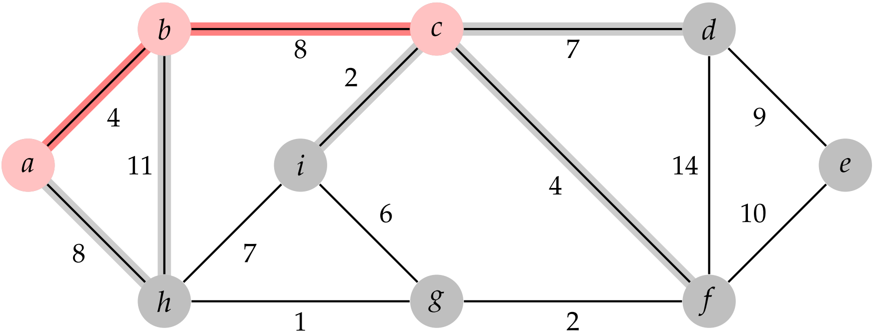

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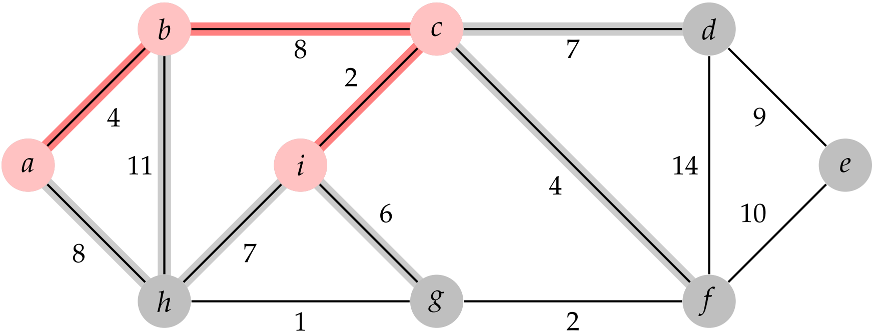

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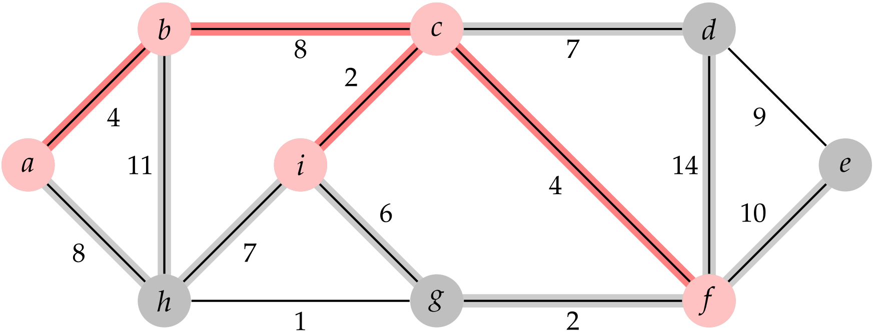

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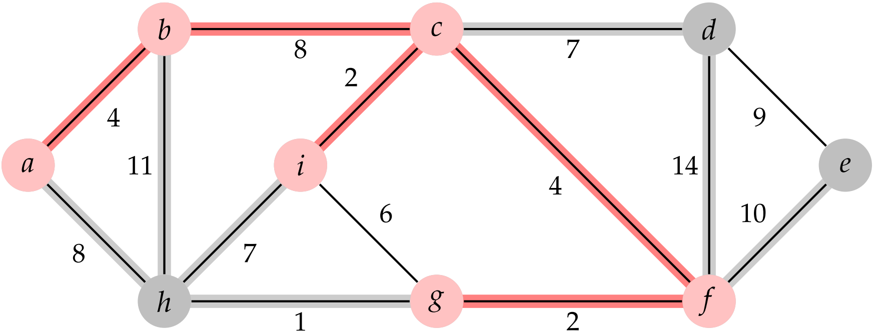

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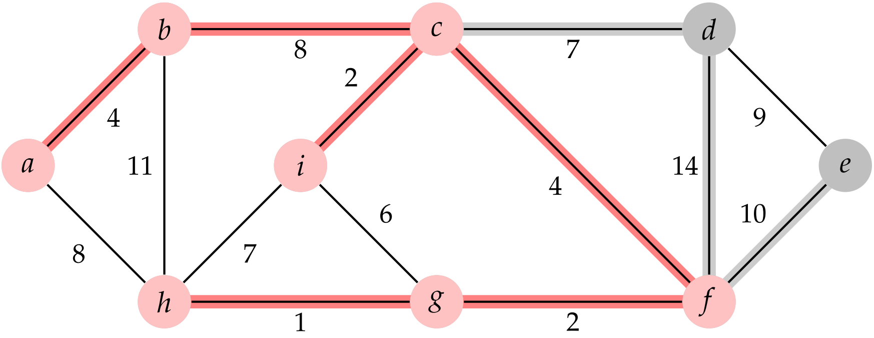

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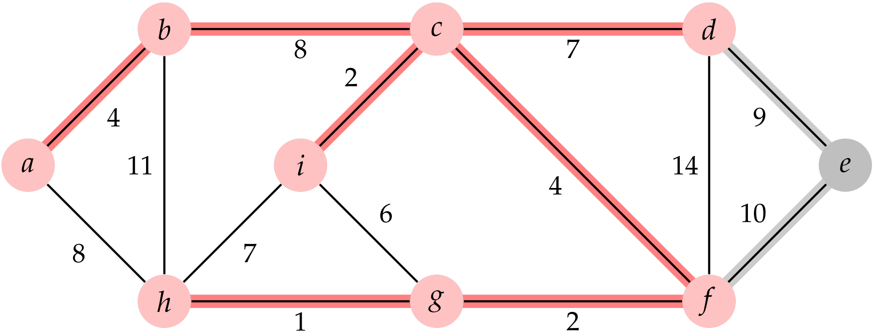

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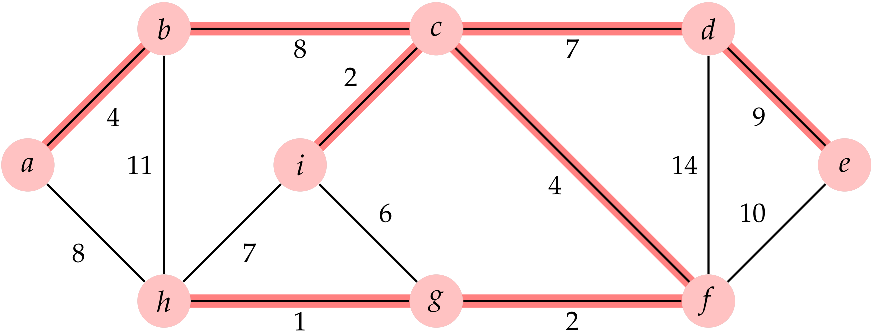

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Exercises

1. Show that for each MST T of G, there is a way to sort the edges of G in Kruskal's algorithm so that it returns T.

2. Suppose that we represent the graph $G = (V, E)$ as an adjacency matrix. Give a simple implementation of Prim's algorithm for this case that runs in $O(n^2)$ time.

Single-Source Shortest Paths

Shortest Paths

- Given weighted directed graph $G = (V, E)$ and
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$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- The shortest-path weight from u to v is

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\leadsto} v\} & \text{if there is a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

- A shortest path from u to v is any path p from u to v with $w(p) = \delta(u,v)$.

Shortest Paths – Variants

- Single-source shortest-paths problem
- Single-destination shortest-paths problem – by reversing the direction of each edge
- Single-pair shortest-path problem – is there faster solution?
- All-pairs shortest-paths problem – single-source from each vertex or faster?

Subpaths of Shortest Paths

Lemma 17.
Let $G = (V, E)$ *be directed graph with weight function* $w : E \to \mathbb{R}$*. Let* $p = \langle v_1, v_2, \ldots, v_k \rangle$ *be a shortest path from* v_1 *to* v_k*.*
For any $1 \leq i \leq j \leq k$*, let* $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ *be the subpath of* p *from* v_i *to* v_j*.*
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Proof.

□

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- Assume that there is p'_{ij} from v_i to v_j with $w(p'_{ij}) < w(p_{ij})$.

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- Assume that there is p'_{ij} from v_i to v_j with $w(p'_{ij}) < w(p_{ij})$.
- Then, $v_1 \overset{p_{1i}}{\rightsquigarrow} v_i \overset{p'_{ij}}{\rightsquigarrow} v_j \overset{p_{jk}}{\rightsquigarrow} v_k$, where $w(p_{1i}) + w(p'_{ij}) + w(p_{jk}) < w(p)$. Contradiction.

□

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- Note: There is always the shortest simple path, but not path. The algorithms work with paths $\Rightarrow$ problem.

Representing Shortest Paths

- Let $G = (V, E)$ be a graph.
- $\pi[v]$ is set to a predecessor to v; that is, a vertex or NIL.
- Use procedure PRINT-PATH(G, s, v) to print the path from s to v stored in π

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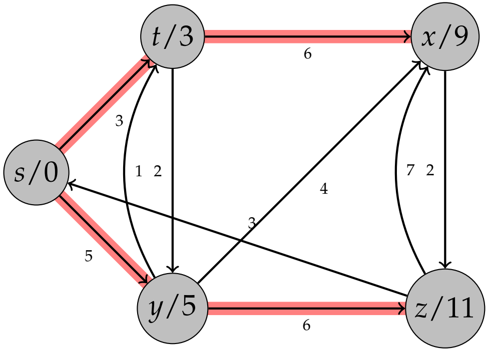
- After the algorithm is finished, G_π is a shortest-paths tree rooted at s containing shortest paths from s to all other reachable vertices.

Shortest paths are not necessarily unique – Example



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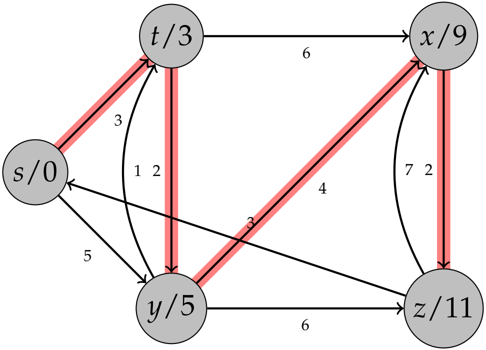

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Relaxation

- $d[v]$ – shortest-path estimate (upper bound of weight)

INITIALIZE-SINGLE-SOURCE(G, s)

for each vertex $v \in V$
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- Time complexity: $\Theta(n)$.

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```
RELAX(u, v, w)
if d[v] > d[u] + w(u, v)
   then d[v] ← d[u] + w(u, v)
        π[v] ← u
```

Bellman-Ford Algorithm

Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i ← 1 to n − 1
    do for each edge (u, v) ∈ E
           do RELAX(u, v, w)
for each edge (u, v) ∈ E
    do if d[v] > d[u] + w(u, v)
          then return FALSE
return TRUE
```

- If it returns FALSE, G contains negative-weight cycles reachable from s.
- If it returns TRUE, π contains the shortest paths.

Bellman-Ford – Example

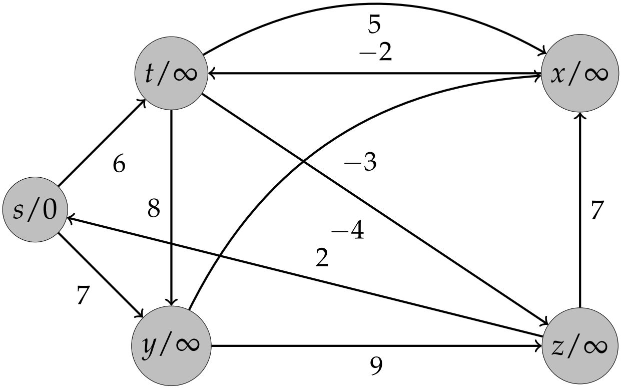

Figure: Computation by Bellman-Ford Algorithm.

- If $(u, v) \in E$ is highlighted, then $\pi[v] = u$.
- Edges are relaxed in the following order: $(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$.

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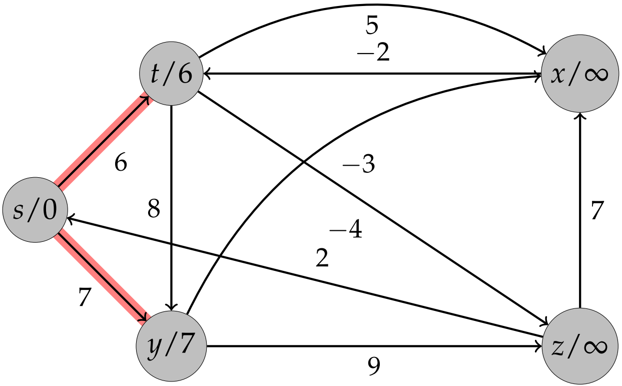

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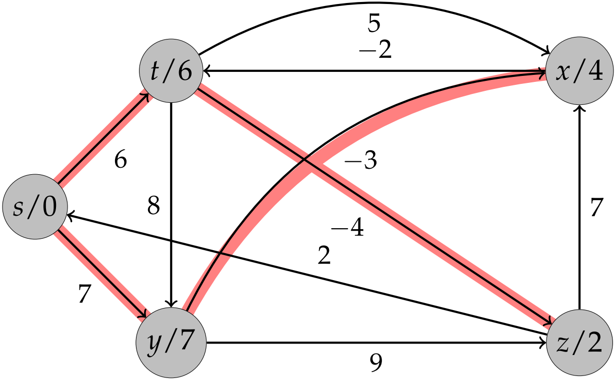

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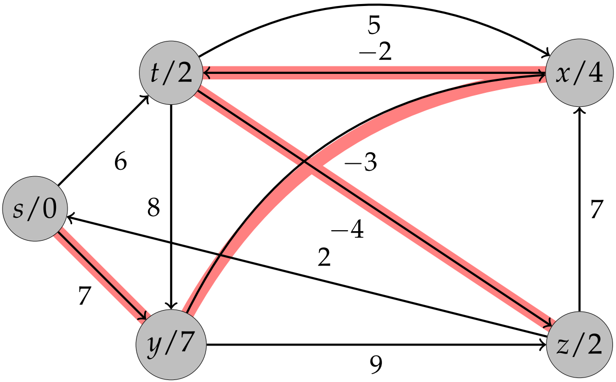

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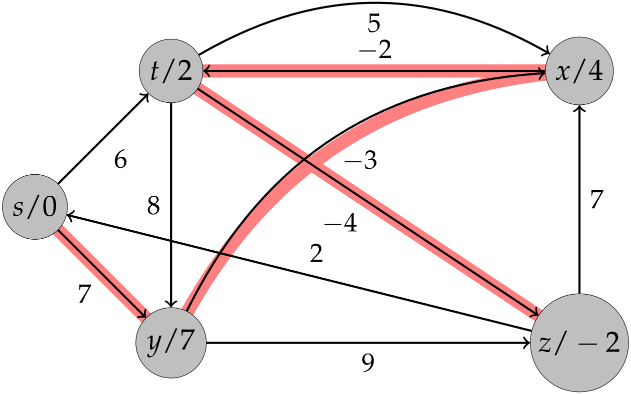

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Bellman-Ford Algorithm – Time Complexity

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5  for each edge (u, v) ∈ E
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7             then return FALSE
8  return TRUE
```

- Line 1 takes $\Theta(n)$.

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3      do for each edge (u, v) ∈ E
4              do RELAX(u, v, w)
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6      do if d[v] > d[u] + w(u, v)
7            then return FALSE
8  return TRUE
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- Line 1 takes $\Theta(n)$.
- Lines 2-4 take $(n-1)$-times $\Theta(m)$.

Bellman-Ford Algorithm – Time Complexity

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- In total, $\Theta(mn)$.

Bellman-Ford Algorithm – Correctness

Lemma 18.
Let $G = (V, E)$ be weighted digraf with source s and weight function $w : E \to \mathbb{R}$. Assume that G contains no negative cycle reachable from s. Then after $n - 1$ iterations of **for***-cycle (lines 2-4), $d[v] = \delta(s, v)$ for all $v \in V$ reachable from s.* **Note:** $d[v] = \infty$ *implies* $s \not\rightsquigarrow v$.

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Proof.

- Let $v \in V$ be reachable from s.
- Let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from s to v; $s = v_0$ and $v = v_k$.

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- Each of $n - 1$ iterations on lines 2-4 relaxes all m edges.

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- Each of $n-1$ iterations on lines 2-4 relaxes all m edges.
- Amongst the relaxed edges in i-th iteration, there is edge (v_{i-1}, v_i) and then $d[v_i] = \delta(s, v_i)$. (Prove by induction.)

Bellman-Ford Algorithm – Correctness

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- p contains at most $n-1$ edges, so $k \leq n-1$.
- Each of $n-1$ iterations on lines 2-4 relaxes all m edges.
- Amongst the relaxed edges in i-th iteration, there is edge (v_{i-1}, v_i) and then $d[v_i] = \delta(s, v_i)$. (Prove by induction.)
- Therefore, after k-th iteration, $d[v_k] = \delta(s, v_k)$.

Bellman-Ford Algorithm – Correctness

Theorem 19 (Correctness I).

- *If* G *contains* *no negative cycle* *reachable from* s*, the algorithm returns* $\textsc{True}$ *and* $d[v] = \delta(s, v)$ *for all* $v \in V$*.*

Bellman-Ford Algorithm – Correctness

Theorem 19 (Correctness I).

- *If G contains no negative cycle reachable from s, the algorithm returns $\textsc{True}$ and $d[v] = \delta(s, v)$ for all $v \in V$.*

Proof.

- Let G contains no negative cycle reachable from s.

□

Bellman-Ford Algorithm – Correctness

Theorem 19 (Correctness I).

- *If G contains no negative cycle reachable from s, the algorithm returns* $\textsc{True}$ *and $d[v] = \delta(s, v)$ for all $v \in V$.*

Proof.

- Let G contains no negative cycle reachable from s.
- When the algorithms is finished, $d[v] = \delta(s, v)$ for all $v \in V$ (Lemma 18)

□

Bellman-Ford Algorithm – Correctness

Theorem 19 (Correctness I).

- *If G contains no negative cycle reachable from s, the algorithm returns* $\textsc{True}$ *and $d[v] = \delta(s,v)$ for all $v \in V$.*

Proof.

- Let G contains no negative cycle reachable from s.
- When the algorithms is finished, $d[v] = \delta(s,v)$ for all $v \in V$ (Lemma 18)
- Moreover, $d[v] = \delta(s,v) \leq \delta(s,u) + w(u,v) = d[u] + w(u,v)$. So the algorithm returns $\textsc{True}$.

□

Bellman-Ford Algorithm – Correctness

Theorem 20 (Correctness II).

- *If G contains a negative-weight cycle reachable from s, the algorithm returns* FALSE*.*

Bellman-Ford Algorithm – Correctness

Theorem 20 (Correctness II).

- *If G contains a negative-weight cycle reachable from s, the algorithm returns* FALSE.

Proof.

- Let $c = \langle v_0, v_1, \ldots, v_k \rangle$, $v_0 = v_k$, be negative-weight cycle reachable from s.

Bellman-Ford Algorithm – Correctness

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- *If G contains a negative-weight cycle reachable from s, the algorithm returns* FALSE.

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- Let $c = \langle v_0, v_1, \ldots, v_k \rangle$, $v_0 = v_k$, be negative-weight cycle reachable from s.
- Then, $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$.

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- By contradiction – alg. returns TRUE, so $d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i)$ for $i = 1, 2, \ldots, k$.
- But then $\sum_{i=1}^{k} d[v_i] \leq \sum_{i=1}^{k} d[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$.

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- But then $\sum_{i=1}^{k} d[v_i] \leq \sum_{i=1}^{k} d[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$.
- Since $v_0 = v_k$, we have $\sum_{i=1}^{k} d[v_i] = \sum_{i=1}^{k} d[v_{i-1}]$.
- Because for $i = 1, 2, \ldots, k$ $d[v_i] < \infty$, we have $0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i)$. Contradiction.

Single-Source Shortest Paths in Directed Acyclic Graphs

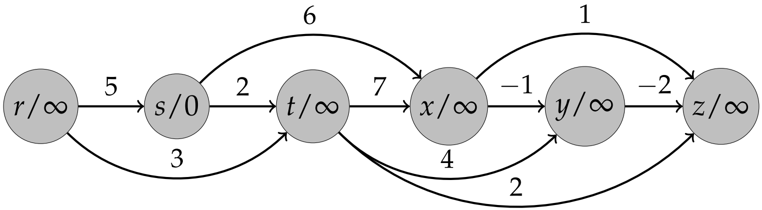
Shortest Paths in Directed Acyclic Graphs

- For DAG, there is significantly faster method than Bellman-Ford.

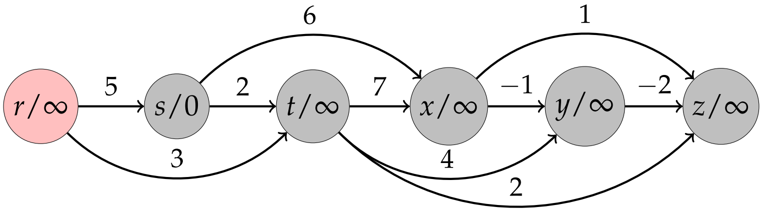
```
DAG-SHORTEST-PATHS(G, w, s)
1  Topologically sort the vertices of G
2  INITIALIZE-SINGLE-SOURCE(G, s)
3  for each vertex u, taken in topologically sorted order
4       do for each vertex v ∈ Adj[u]
5              do RELAX(u, v, w)
```

- Time complexity: $\Theta(n+m)$.
  - We get a topological order in $\Theta(n+m)$.
  - Line 2 takes $\Theta(n)$.
  - Lines 3-5 checks every edge exactly once; that is, the iteration is executed m-times. RELAX takes $\Theta(1)$.

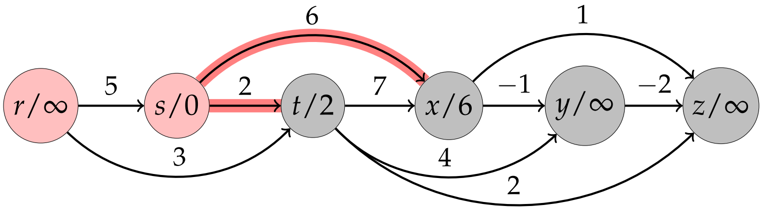
Example



Example



Example



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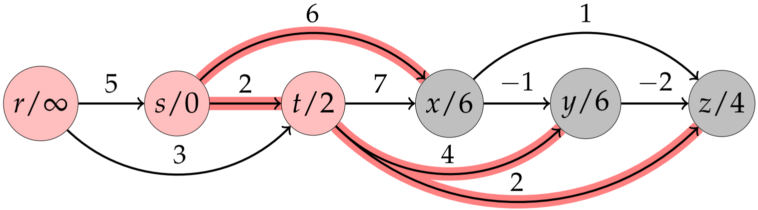

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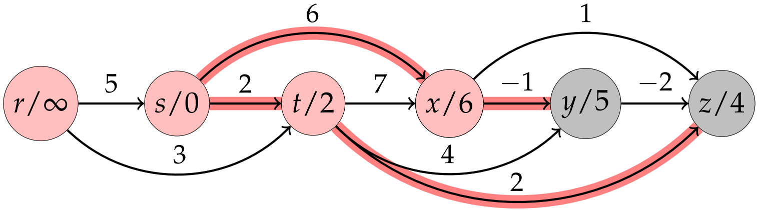

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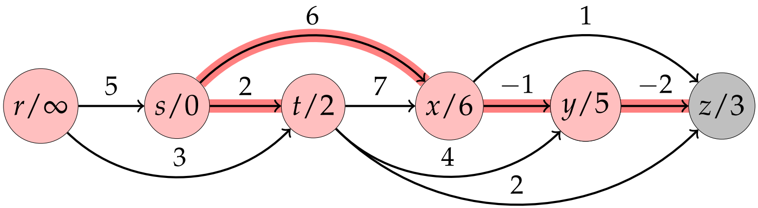

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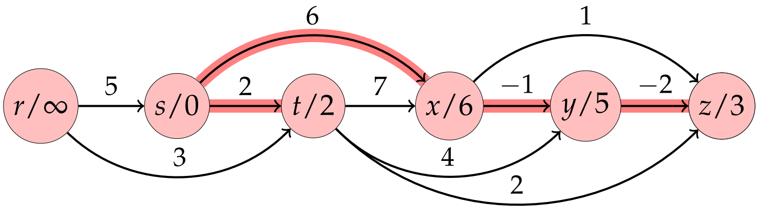

Correctness

Theorem 21.
If a weighted, digraph $G = (V, E)$ *has source vertex* s *and no cycles, then* DAG-SHORTEST-PATHS *computes* $d[v] = \delta(s, v)$ *for all* $v \in V$.

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If a weighted, digraph $G = (V, E)$ has source vertex s and no cycles, then DAG-SHORTEST-PATHS *computes $d[v] = \delta(s, v)$ for all $v \in V$.*

Proof.

- If v is not reachable from s, then $d[v] = \delta(s, v) = \infty$.

□

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- If v is not reachable from s, then $d[v] = \delta(s, v) = \infty$.
- Suppose there is a shortest path $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $s = v_0$ and $v = v_k$.

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If a weighted, digraph $G = (V, E)$ has source vertex s and no cycles, then $\textsc{Dag-Shortest-Paths}$ *computes $d[v] = \delta(s, v)$ for all $v \in V$.*

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- Because we process the vertices in topological order, we relax edges on p in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$.
- That implies that $d[v_i] = \delta(s, v_i)$ at termination for $i = 0, 1, \ldots, k$.

□

Dijkstra Algorithm

Dijkstra Algorithm

- Only for weighted, directed graphs without negative edges:
- $w(u, v) \geq 0$ for each edge $(u, v) \in E$.

Dijkstra Algorithm

- Only for weighted, directed graphs without negative edges:
- $w(u, v) \geq 0$ for each edge $(u, v) \in E$.

- Can we implement it with lower time complexity than Bellman-Ford algorithm?

Dijkstra Algorithm

DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
$S \leftarrow \emptyset$
$Q \leftarrow V$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        $S \leftarrow S \cup \{u\}$
        for each vertex $v \in Adj[u]$
            do RELAX(u, v, w)

- S is a set of finished vertices (their shortest distance from s is already computed).
- Q is a min-priority queue; the vertex with the lowest d-value is at the beginning of Q.

Dijkstra Algorithm – Example

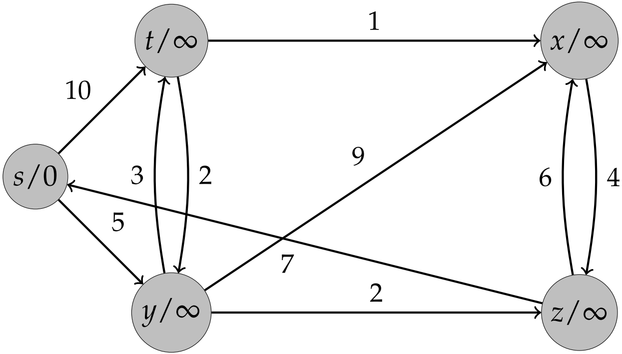
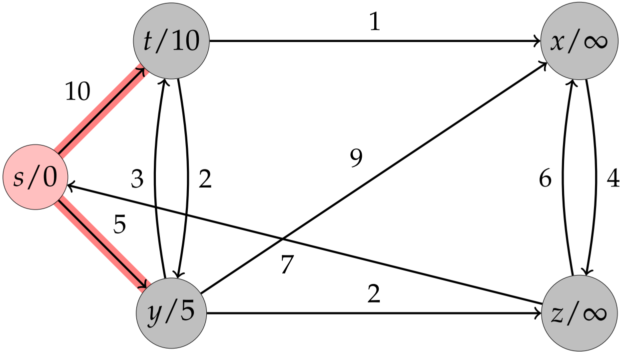


Figure: The computation by Dijkstra Algorithm. Highlighted vertices belong to set S.

Dijkstra Algorithm – Example



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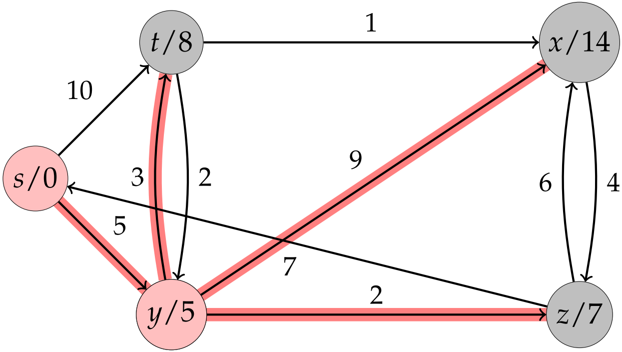

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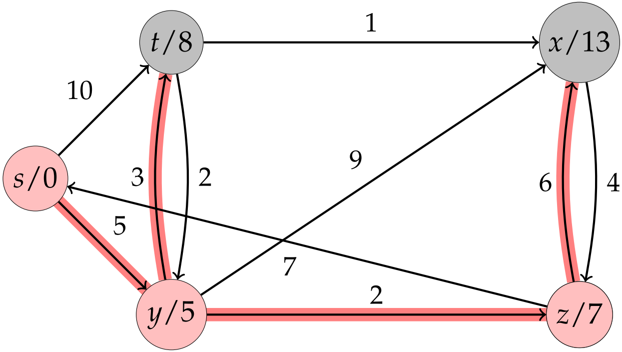

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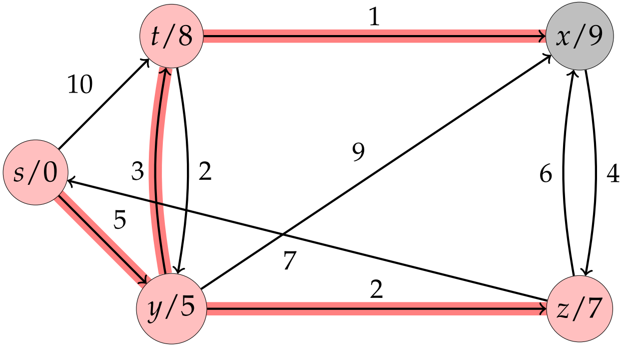

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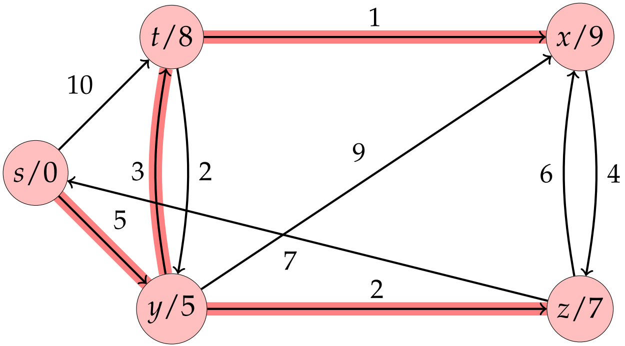

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Correctness

Theorem 22.
Dijkstra algorithm on weighted digraph $G = (V, E)$ *without negative-weight edges and with source* s *finishes with* $d[v] = \delta(s, v)$ *for all* $v \in V$.

Correctness

Theorem 22.
Dijkstra algorithm on weighted digraph $G = (V, E)$ without negative-weight edges and with source s finishes with $d[v] = \delta(s, v)$ for all $v \in V$.

Proof.

- Invariant: In the beginning of each **while**-iteration, $d[v] = \delta(s, v)$ for all $v \in S$.

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Theorem 22.
Dijkstra algorithm on weighted digraph $G = (V, E)$ *without negative-weight edges and with source* s *finishes with* $d[v] = \delta(s, v)$ *for all* $v \in V$.

Proof.

- Invariant: In the beginning of each **while**-iteration, $d[v] = \delta(s, v)$ for all $v \in S$.
- It holds for $S = \emptyset$.

Correctness

Theorem 22.
Dijkstra algorithm on weighted digraph $G = (V, E)$ *without negative-weight edges and with source* s *finishes with* $d[v] = \delta(s, v)$ *for all* $v \in V$.

Proof.

- Invariant: In the beginning of each **while**-iteration, $d[v] = \delta(s, v)$ for all $v \in S$.
- It holds for $S = \emptyset$.
- Let u be first vertex such that $d[u] \neq \delta(s, u)$ in the moment of its inclusion into S.

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Theorem 22.
Dijkstra algorithm on weighted digraph $G = (V, E)$ without negative-weight edges and with source s finishes with $d[v] = \delta(s, v)$ for all $v \in V$.

Proof.

- Invariant: In the beginning of each **while**-iteration, $d[v] = \delta(s, v)$ for all $v \in S$.
- It holds for $S = \emptyset$.
- Let u be first vertex such that $d[u] \neq \delta(s, u)$ in the moment of its inclusion into S.
- Then, necessarily $u \neq s$, because s is included as the first into S and $d[s] = \delta(s, s) = 0$ holds in the moment of inclusion of s into S.

Correctness

Theorem 22.
Dijkstra algorithm on weighted digraph $G = (V, E)$ *without negative-weight edges and with source* s *finishes with* $d[v] = \delta(s, v)$ *for all* $v \in V$*.*

Proof.

- Invariant: In the beginning of each **while**-iteration, $d[v] = \delta(s, v)$ for all $v \in S$.
- It holds for $S = \emptyset$.
- Let u be first vertex such that $d[u] \neq \delta(s, u)$ in the moment of its inclusion into S.
- Then, necessarily $u \neq s$, because s is included as the first into S and $d[s] = \delta(s, s) = 0$ holds in the moment of inclusion of s into S.
- Since $u \neq s$, $S \neq \emptyset$ right before inclusion of u.

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Dijkstra algorithm on weighted digraph $G = (V, E)$ without negative-weight edges and with source s finishes with $d[v] = \delta(s, v)$ for all $v \in V$.

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- It holds for $S = \emptyset$.
- Let u be first vertex such that $d[u] \neq \delta(s, u)$ in the moment of its inclusion into S.
- Then, necessarily $u \neq s$, because s is included as the first into S and $d[s] = \delta(s, s) = 0$ holds in the moment of inclusion of s into S.
- Since $u \neq s$, $S \neq \emptyset$ right before inclusion of u.
- The assumption $d[u] \neq \delta(s, u)$ implies that $s \rightsquigarrow u$ – otherwise $d[u] = \delta(s, u) = \infty$.

Correctness

Theorem 22.
Dijkstra algorithm on weighted digraph $G = (V, E)$ without negative-weight edges and with source s finishes with $d[v] = \delta(s, v)$ for all $v \in V$.

Proof.

- Invariant: In the beginning of each **while**-iteration, $d[v] = \delta(s, v)$ for all $v \in S$.
- It holds for $S = \emptyset$.
- Let u be first vertex such that $d[u] \neq \delta(s, u)$ in the moment of its inclusion into S.
- Then, necessarily $u \neq s$, because s is included as the first into S and $d[s] = \delta(s, s) = 0$ holds in the moment of inclusion of s into S.
- Since $u \neq s$, $S \neq \emptyset$ right before inclusion of u.
- The assumption $d[u] \neq \delta(s, u)$ implies that $s \rightsquigarrow u$ – otherwise $d[u] = \delta(s, u) = \infty$.
- So there is a shortest path p from s to u.

Correctness

Part II of the Proof.

- There is a shortest path p from s to u.

□

Correctness

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- Since edge (x, y) was already relaxed in that moment, we have $d[y] = \delta(s, y)$ in the moment of inclusion of u into S. (Prove it!)

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- $Q = \emptyset$ when alg. finishes. Since $Q = V - S$ (Do the reasoning!), we have $S = V$. So $d[v] = \delta(s, v)$ for all $v \in V$.

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- Done!. . .

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Time Complexity of Dijkstra algorithm

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- INSERT and DECREASE-KEY take $O(1)$.
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- In general, using Fibonacci heap we get the time complexity $O(n \log n + m)$.

Exercises

1. Modify the Bellman-Ford algorithm so that it sets $d[v]$ to $-\infty$ for all vertices v for which there is a negative-weight cycle on some path from the source s to v.
2. A critical path is a *longest* path through the DAG. Modify the DAG-SHORTEST-PATHS procedure to find a critical path in the given DAG.
3. Give a simple example of a digraph with negative-weight edge(s) for which Dijkstra's algorithm produces incorrect answers. Why?

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- Let us examine methods based on dynamic programming...

Adjacency-matrix Representation

- This time, we prefer to use an adjacency matrix $W = (w_{ij})$, where

$$w_{ij} = \begin{cases} 0 & \text{for } i = j, \\ w(i,j) & \text{for } i \neq j \text{ and } (i,j) \in E, \\ \infty & \text{for } i \neq j \text{ and } (i,j) \notin E \end{cases}$$

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  2. predecessor of j on some shortest path from i.

Printing All-Pairs Shortest Paths

```
PRINT-ALL-SHORTEST-PATH(Π, i, j)
if i = j
   then print i
   else if π_ij = NIL
           then print "No path from " i " to " j " exists!"
           else PRINT-ALL-SHORTEST-PATH(Π, i, π_ij)
                print j
```

Matrix Multiplication

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  where p' has $m' - 1$ edges.
- p' is a shortest path from i to k – HOMEWORK – so $\delta(i,j) = \delta(i,k) + w_{kj}$.

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- $l_{ij}^{(m)} = \min(l_{ij}^{(m-1)}, \min_{1 \leq k \leq n}\{l_{ik}^{(m-1)} + w_{kj}\}) = \min_{1 \leq k \leq n}\{l_{ik}^{(m-1)} + w_{kj}\}.$

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- A path from i to j with no more then $n - 1$ edges, so

$$\delta(i,j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = l_{ij}^{(n+1)} = \ldots.$$

(No negative-weight cycle.)

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- $L^{(n-1)}$, then it contains weights of shortest paths.
- $l_{ij}^{(1)} = w_{ij}$, i.e. $L^{(1)} = W$.

Algorithm Core

EXTEND-SHORTEST-PATHS(L, W)

$n \leftarrow rows[L]$
let $L' = (l'_{ij})$ be an $n \times n$ matrix
for $i \leftarrow 1$ **to** n
    do for $j \leftarrow 1$ **to** n
        do $l'_{ij} \leftarrow \infty$
            for $k \leftarrow 1$ **to** n
                do $l'_{ij} \leftarrow \min(l'_{ij}, l_{ik} + w_{kj})$
return L'

- $rows[L]$ denotes the line number of L.
- Time complexity $\Theta(n^3)$.

All-Pairs Shortest Paths Vs. Matrix Multiplication

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- For the comparison:

$$l_{ij}^{(m)} = \min_{1 \leq k \leq n} \{ l_{ik}^{(m-1)} + w_{kj} \}$$

Find 3 differences (skip the naming and names of variables)

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MATRIX-MULTIPLY(A, B)

$n \leftarrow rows[A]$
let $C = (c_{ij})$ be an $n \times n$ matrix
for $i \leftarrow 1$ **to** n
    do for $j \leftarrow 1$ **to** n
        do $c_{ij} \leftarrow 0$
            for $k \leftarrow 1$ **to** n
                do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$
return C

Matrix multiplication revisited

- Notation $X \cdot Y$ represents a matrix computed by $\textsc{Extend-Shortest-Paths}(X, Y)$.

Matrix multiplication revisited

- Notation $X \cdot Y$ represents a matrix computed by $\textsc{Extend-Shortest-Paths}(X, Y)$.
- Then, we compute the whole sequence of matrices

$$
\begin{array}{ccccc}
L^{(1)} & = & L^{(0)} \cdot W & = & W \\
L^{(2)} & = & L^{(1)} \cdot W & = & W^2 \\
L^{(3)} & = & L^{(2)} \cdot W & = & W^3 \\
 & & \vdots & & \\
L^{(n-1)} & = & L^{(n-2)} \cdot W & = & W^{n-1}
\end{array}
$$

where W^{n-1} contains the weights of shortest paths.

Slow method

```
SLOW-ALL-SHORTEST-PATHS(W)
n ← rows[W]
L^(1) ← W
for m ← 2 to n − 1
    do L^(m) ← EXTEND-SHORTEST-PATHS(L^(m−1), W)
return L^(n−1)
```

- Time complexity $\Theta(n^4)$.

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- Therefore, instead of $n-1$ multiplications, only $\lceil \log n - 1 \rceil$ suffice.
- We compute the following sequence of matrices

$$\begin{array}{cccc} L^{(1)} & = & W & \\ L^{(2)} & = & W^2 & \\ L^{(4)} & = & W^4 & = & W^2 \cdot W^2 \\ L^{(8)} & = & W^8 & = & W^4 \cdot W^4 \\ & & \vdots & & \\ L^{(2^{\lceil \log n-1 \rceil})} & = & W^{(2^{\lceil \log n-1 \rceil})} & = & W^{2^{\lceil \log n-1 \rceil - 1}} \cdot W^{2^{\lceil \log n-1 \rceil - 1}} \end{array}$$

Since $2^{\lceil \log n-1 \rceil} \geq n-1$, we have $L^{(2^{\lceil \log n-1 \rceil})} = L^{(n-1)}$.

Faster method

```
FAST-ALL-SHORTEST-PATHS(W)
n ← rows[W]
L^(1) ← W
m ← 1
while m < n − 1
        do L^(2m) ← EXTEND-SHORTEST-PATHS(L^(m), L^(m))
           m ← 2m
return L^(m)
```

- Time complexity $\Theta(n^3 \log n)$.

The Floyd-Warshall algorithm

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- Negative-weight edges are allowed,
- but we assume, there are no negative-weight cycle.

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- Since for $k = n$ all inner vertices are from $V = \{1, 2, \ldots, n\}$, the matrix $D^{(n)} = (d_{ij}^{(n)})$ contains $d_{ij}^{(n)} = \delta(i, j)$ for $i, j \in V$.

Computation

FLOYD-WARSHALL(W)

$n \leftarrow rows[W]$
$D^{(0)} \leftarrow W$
for $k \leftarrow 1$ **to** n
    do for $i \leftarrow 1$ **to** n
        do for $j \leftarrow 1$ **to** n
            do $d_{ij}^{(k)} \leftarrow \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$
return $D^{(n)}$

- Time complexity $\Theta(n^3)$.

Construction of shortest paths

$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{for } i = j \text{ or } w_{ij} = \infty \\ i & \text{for } i \neq j \text{ and } w_{ij} < \infty \end{cases}$$

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- We can improve a little bit . . . .

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- Logical operations with bits are usually faster than arithmetical operations with integers (not asymptotically). Moreover, lower space complexity (bits vs. bytes).

Flow Networks

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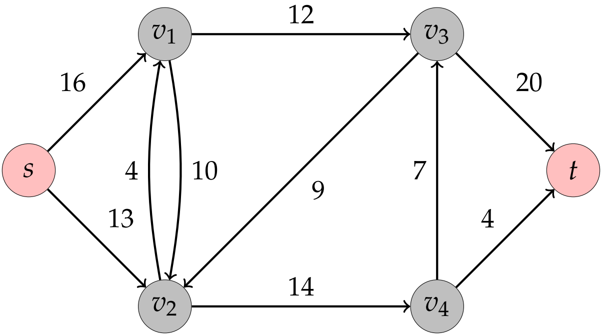
- Therefore, a flow network is connected graph with $m \geq n - 1$.

Flow network – Example



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- The value of a flow f is defined as

$$|f| = \sum_{v \in V} f(s,v) .$$

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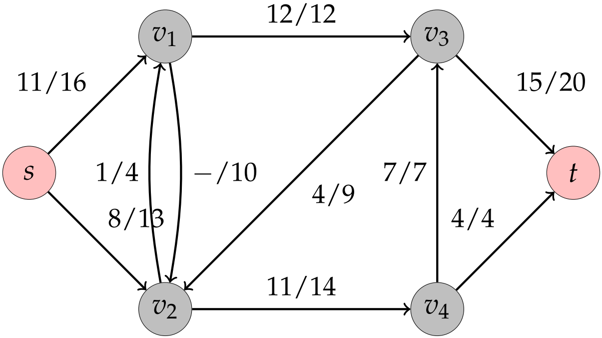

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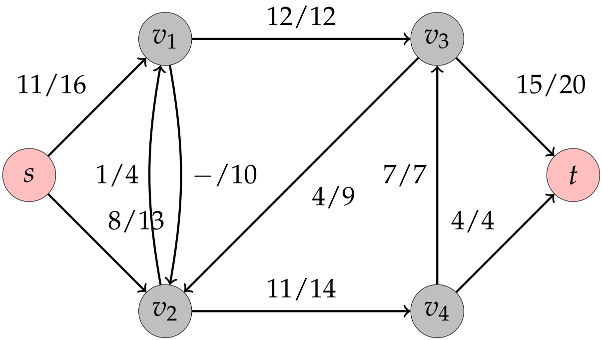

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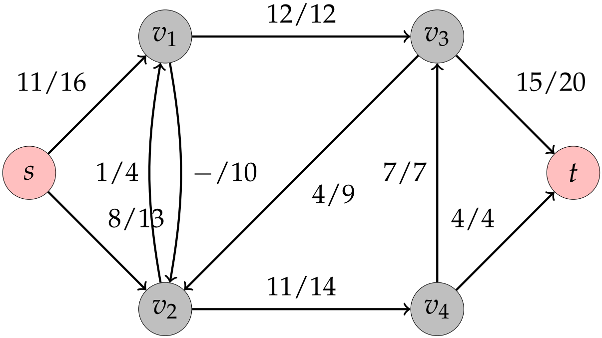

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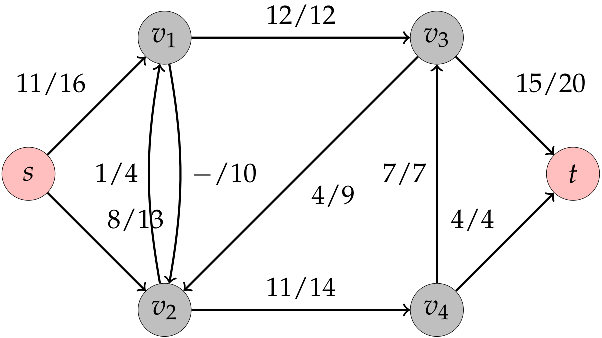

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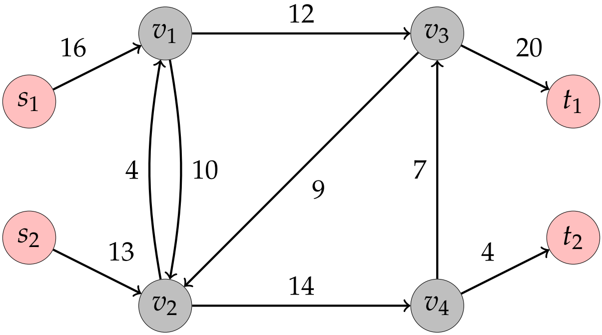
- $|f| = 19$.

Maximum-flow Problem

- We are given a flow network G with source s and sink t,
- we wish to find a flow of maximum value.

Networks with multiple sources and sinks



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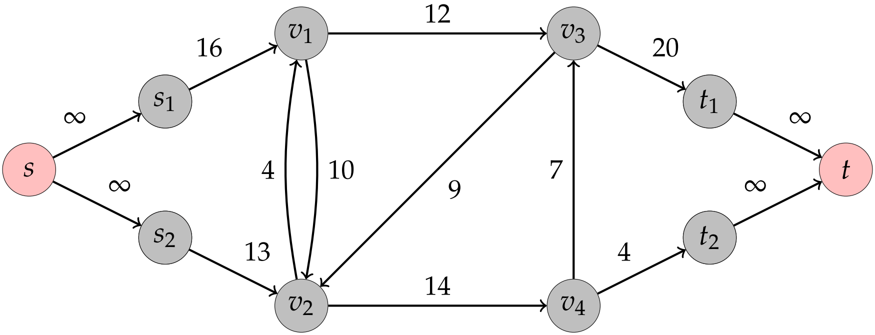

- How to deal with it?
- Create a new supersource s and a new supersink and set the capacity to ∞ for these new edges.

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- For all $X, Y, Z \subseteq V$, $X \cap Y = \emptyset$,

  $$f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$$

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  $$f(Z, X \cup Y) = f(Z,X) + f(Z,Y).$$

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- Thus, $|f| = f(V, t)$.

□

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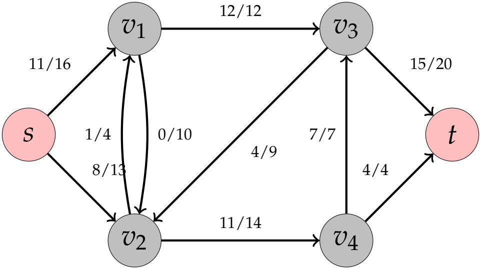
- Augmenting path is a simple path from s to t along which the flow can be increased.

Residual Network(s)



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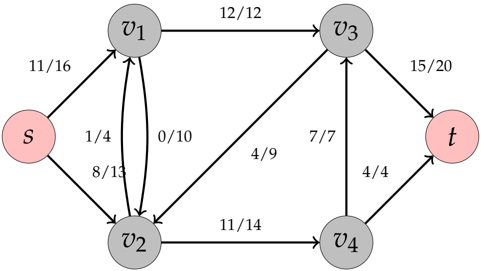

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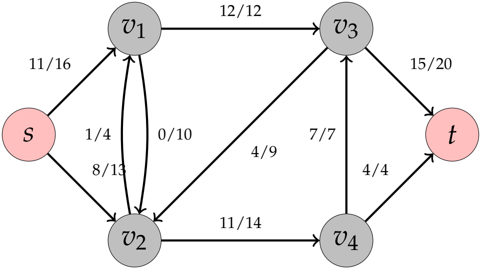
- For example, $c_f(s, v_1) = 16 - 11 = 5$.

Residual Network(s)



- Residual capacity of (u, v) is

$$c_f(u, v) = c(u, v) - f(u, v)\,.$$

- For example, $c_f(s, v_1) = 16 - 11 = 5$.
- Flow $f(u, v)$ can be increased by 5 units.

Residual Network

- Let $G = (V, E)$ be a network and f be a flow in G.

Residual Network

- Let $G = (V, E)$ be a network and f be a flow in G.
- The residual network of G inducted by flow f is a network $G_f = (V, E_f)$, where

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}.$$

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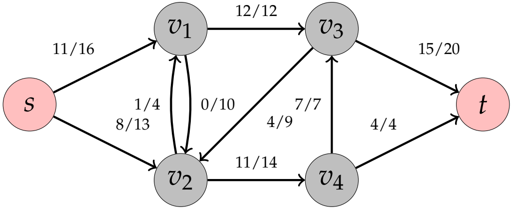
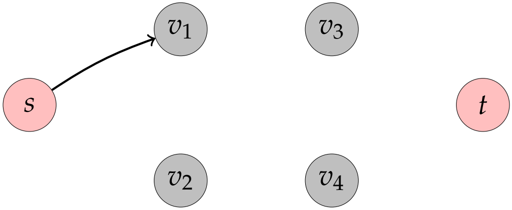
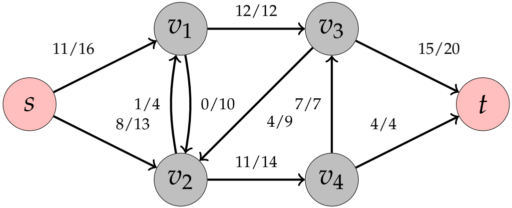
- It holds that $|E_f| \leq 2|E|$ – Think about it!

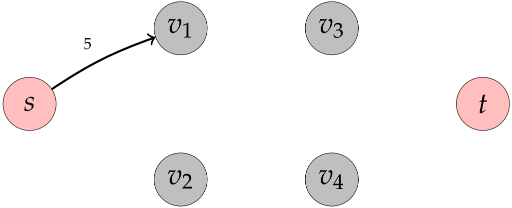
Network and its residual network





Network and its residual network





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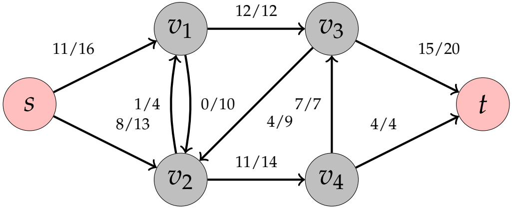

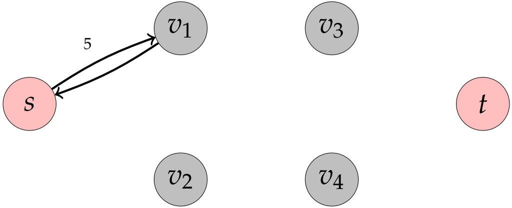

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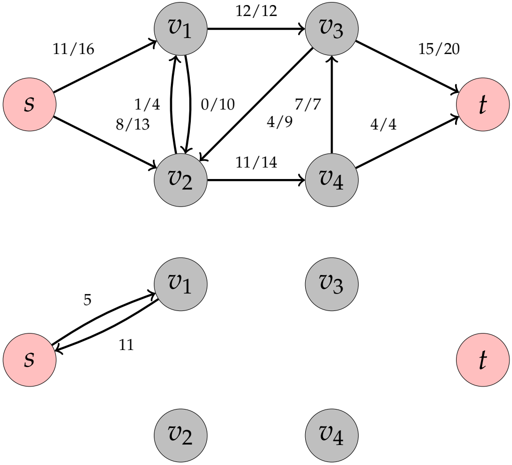

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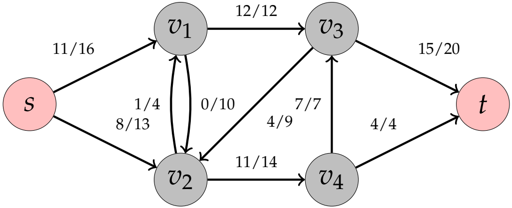

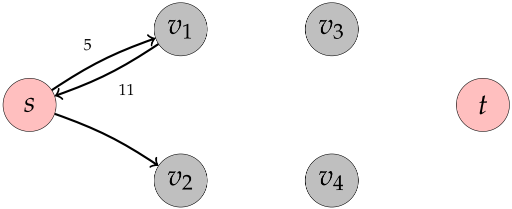

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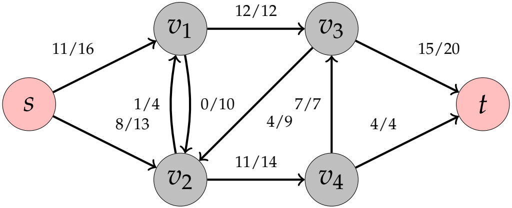

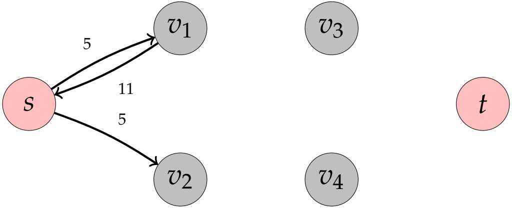

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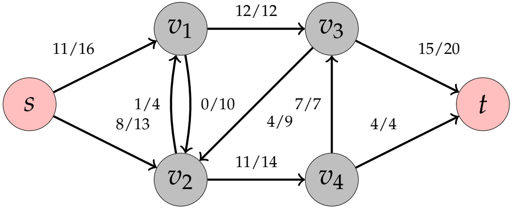
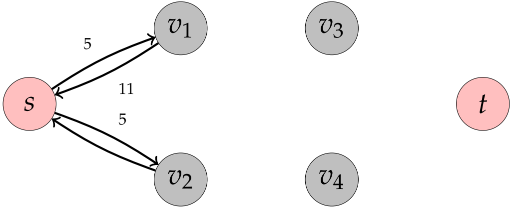
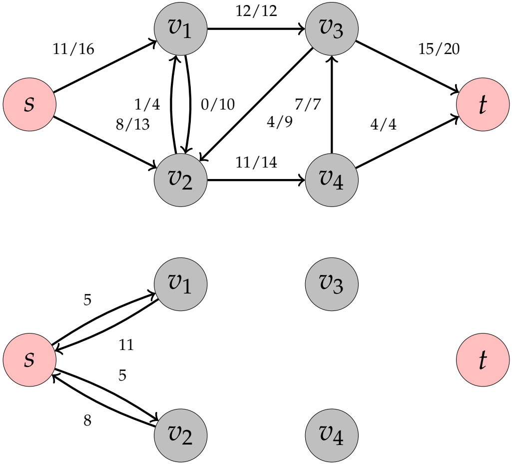




Network and its residual network



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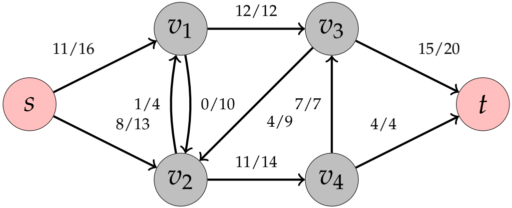

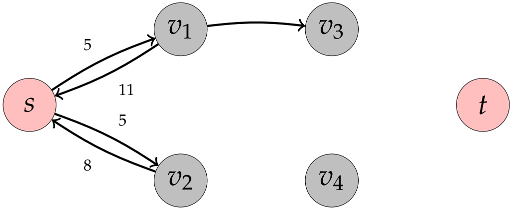

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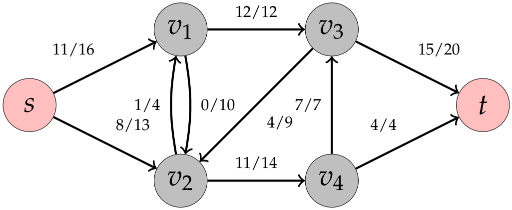

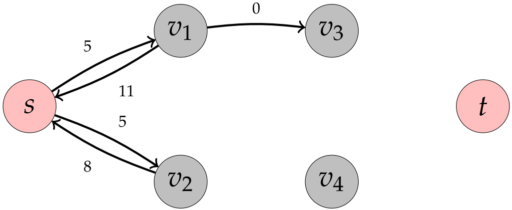

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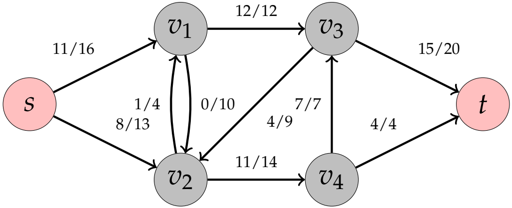

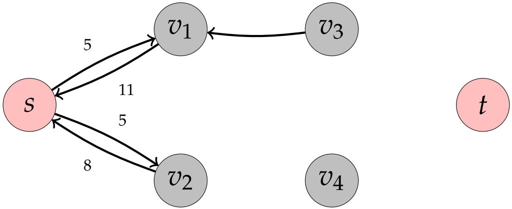

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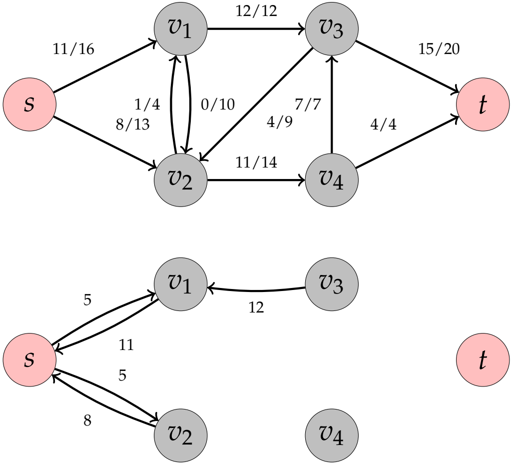

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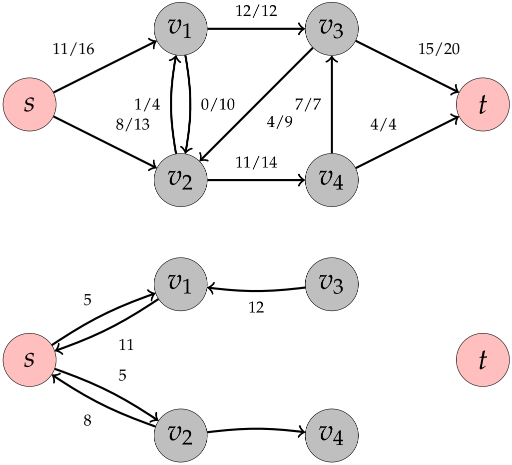

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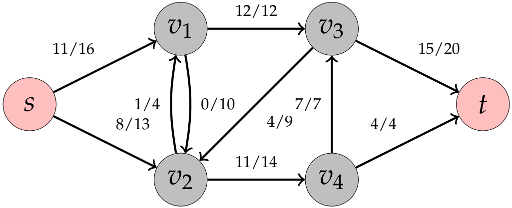

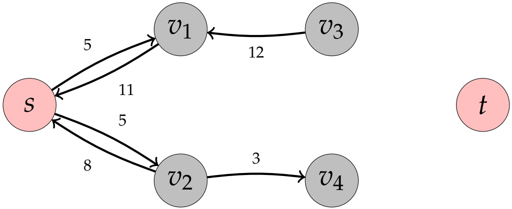

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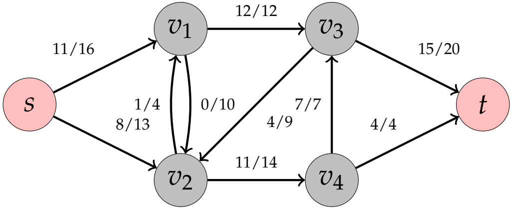

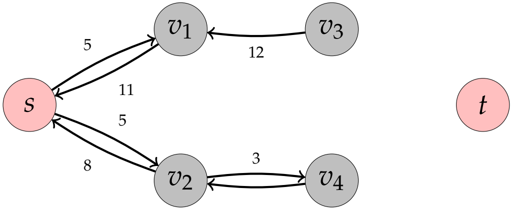

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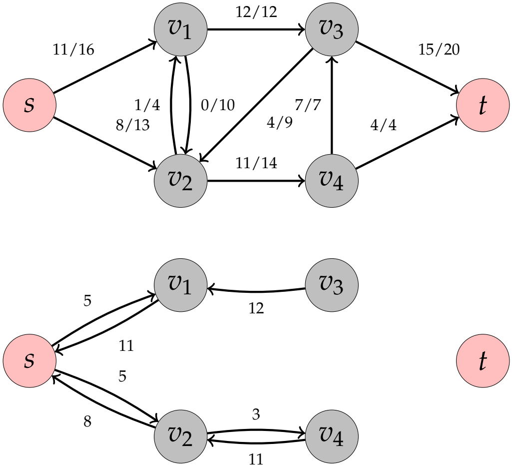

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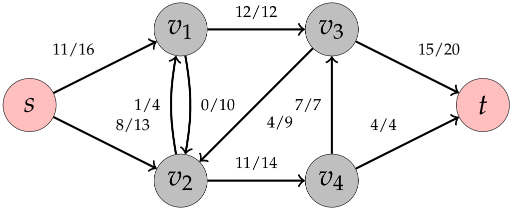

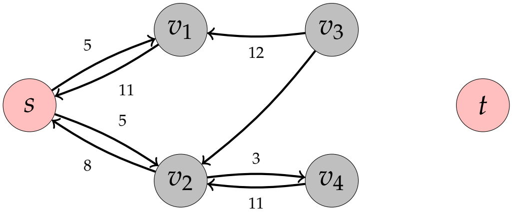

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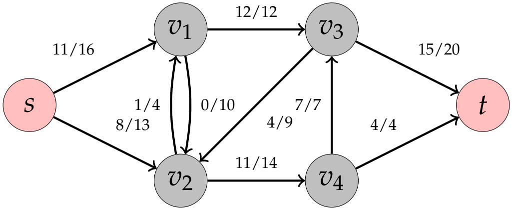

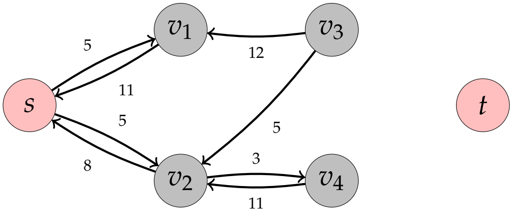

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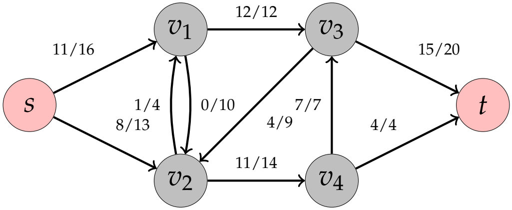

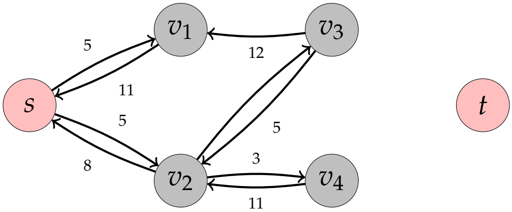

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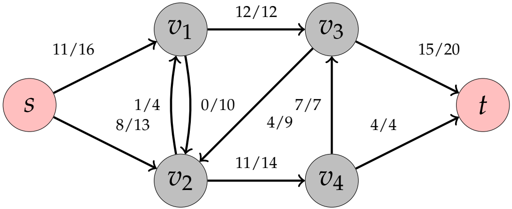

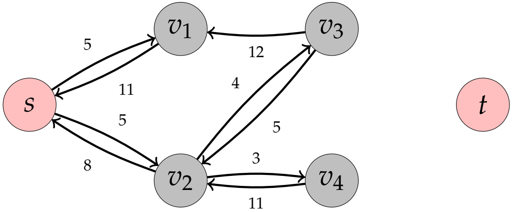

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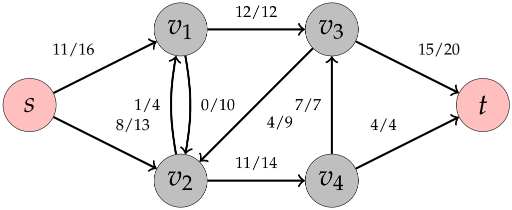

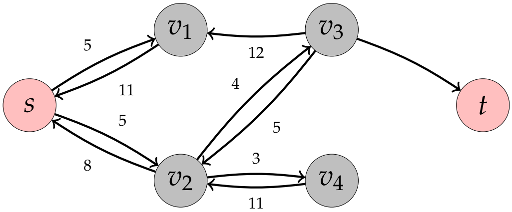

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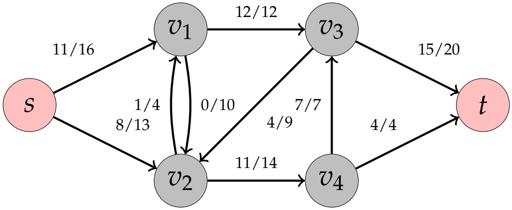

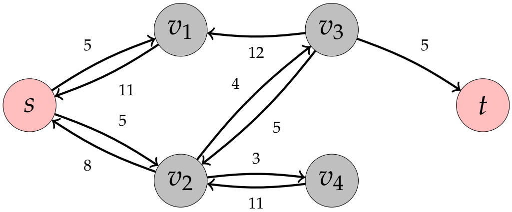

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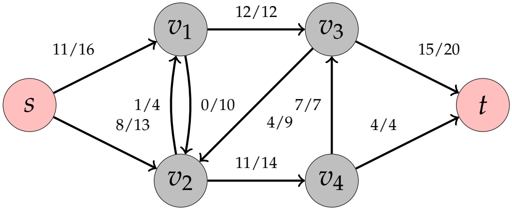

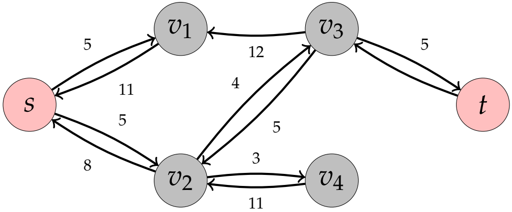

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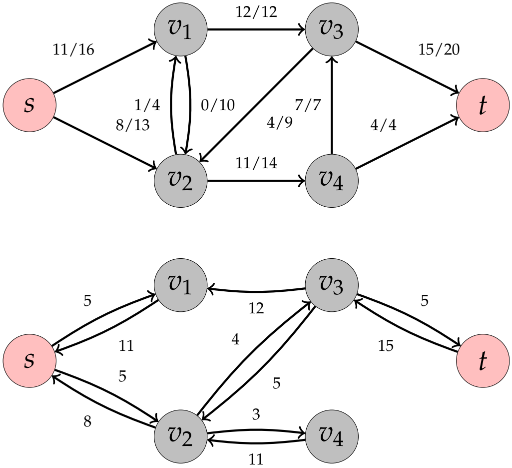

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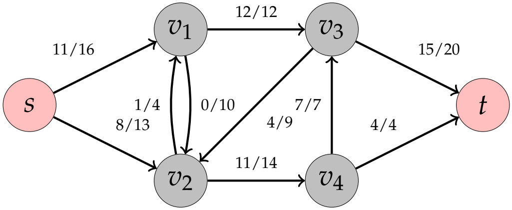

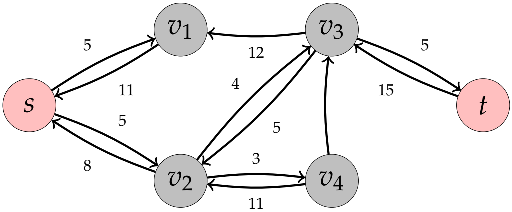

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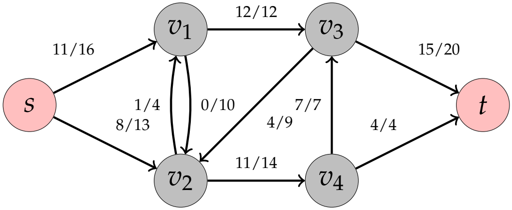

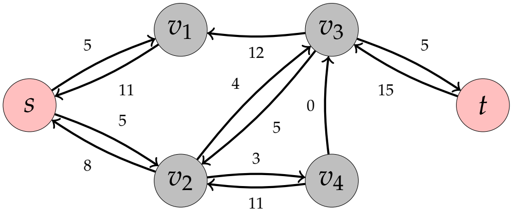

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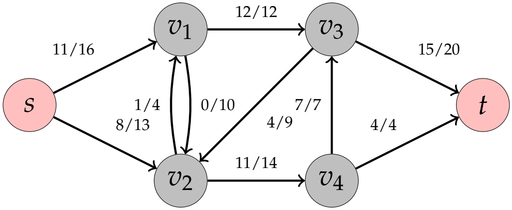

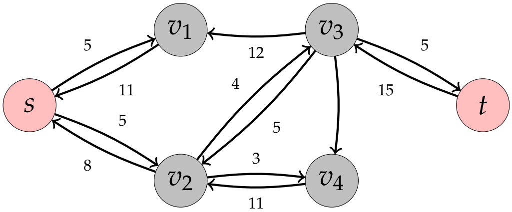

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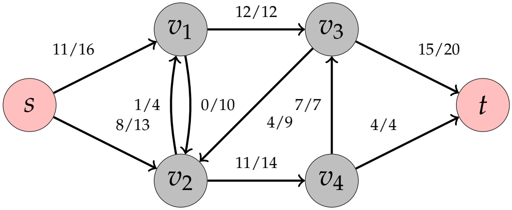

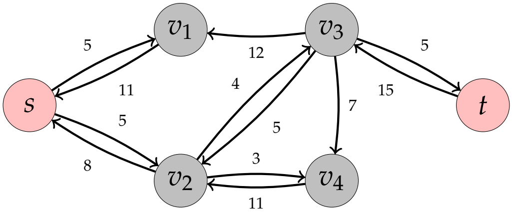

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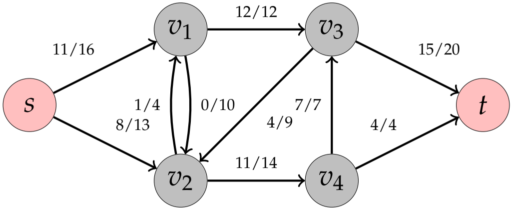

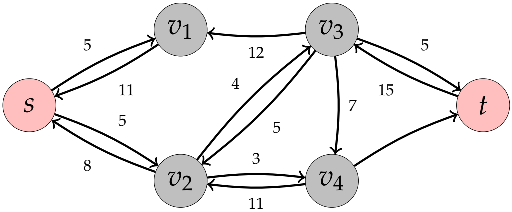

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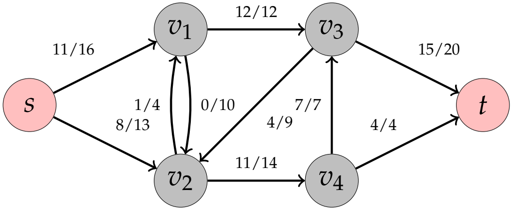

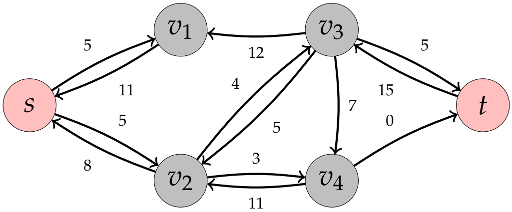

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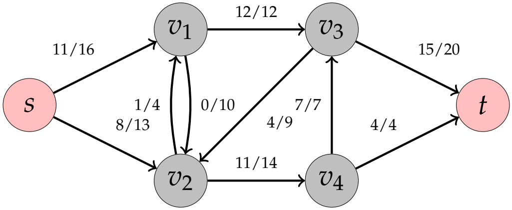

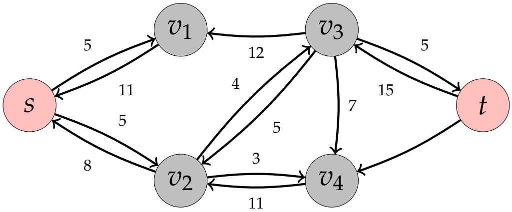

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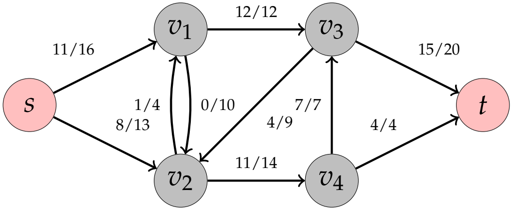

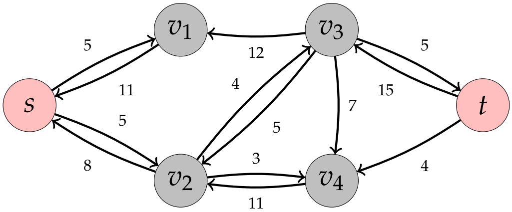

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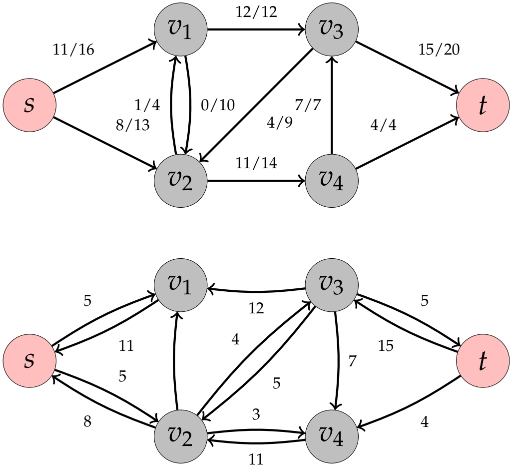

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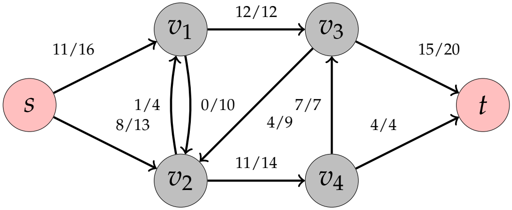
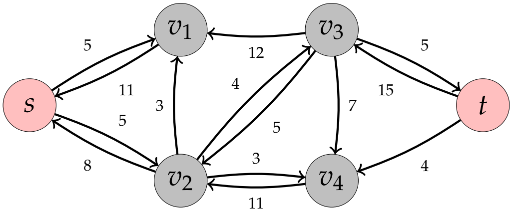
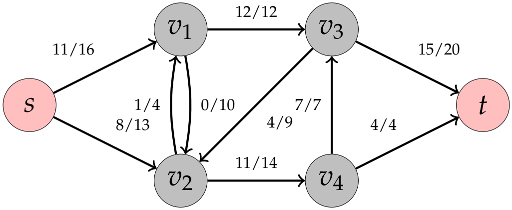
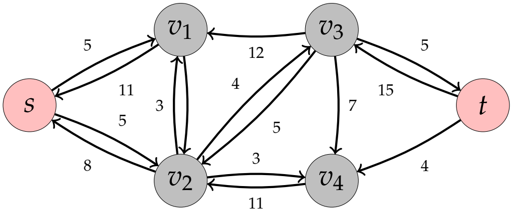




Network and its residual network





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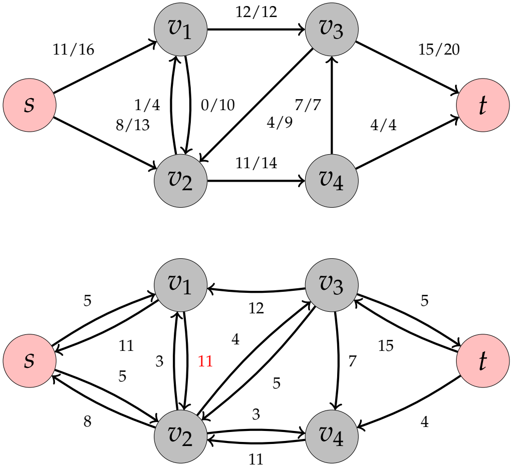
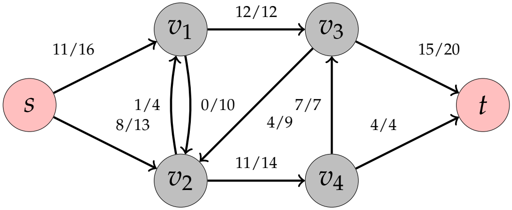
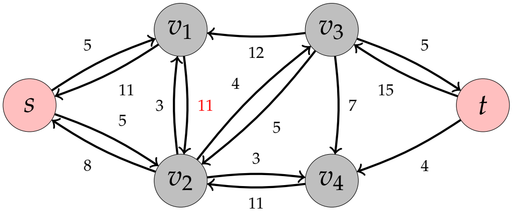


Network and its residual network





- Attention! $f(v_1, v_2) = 0 + (-1)$ so $c_f(v_1, v_2) = 10 - (-1) = 11$.

Residual network

Lemma 23.
Let $G = (V, E)$ *be a network and* f *be a flow in* G. *Let* G_f *be a residual network of* G *induced by* f *and let* f' *be a flow in* G_f.
Then, $f + f'$ *is a flow in* G *with the value of* $|f + f'| = |f| + |f'|$.

Proof.

- We must verify that tree conditions from the definition of a flow.

□

Condition 1: Capacity constraint

Demonstrate that $(f + f')(u,v) \leq c(u,v)$.

Proof.

- $f'(u,v) \leq c_f(u,v)$.

□

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Proof.

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- $(f+f')(u,v) = f(u,v) + f'(u,v)$

□

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Demonstrate that $(f+f')(u,v) \le c(u,v)$.

Proof.

- $f'(u,v) \le c_f(u,v)$.
- $$\begin{aligned}(f+f')(u,v) &= f(u,v) + f'(u,v) \\ &\le f(u,v) + (c(u,v) - f(u,v))\end{aligned}$$

□

Condition 1: Capacity constraint

Demonstrate that $(f + f')(u,v) \leq c(u,v)$.

Proof.

- $f'(u,v) \leq c_f(u,v)$.
- $$\begin{aligned}(f + f')(u,v) &= f(u,v) + f'(u,v) \\ &\leq f(u,v) + (c(u,v) - f(u,v)) \\ &= c(u,v).\end{aligned}$$

□

Condition 2: Skew symmetry

Demonstrate that $(f + f')(u, v) = -(f + f')(v, u)$.

Proof.

- $(f + f')(u, v) = f(u, v) + f'(u, v)$

□

Condition 2: Skew symmetry

Demonstrate that $(f + f')(u, v) = -(f + f')(v, u)$.

Proof.

- $$\begin{aligned}(f + f')(u, v) &= f(u, v) + f'(u, v) \\ &= -f(v, u) - f'(v, u)\end{aligned}$$

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- $$\begin{aligned}(f+f')(u,v) &= f(u,v) + f'(u,v) \\ &= -f(v,u) - f'(v,u) \\ &= -(f(v,u) + f'(v,u)) \\ &= -(f+f')(v,u).\end{aligned}$$

□

Condition 3: Flow conservation

Demonstrate that for $u \in V - \{s,t\}$, $\sum_{v \in V} (f + f')(u,v) = 0$.

Proof.

- $\sum_{v \in V} (f + f')(u,v) = \sum_{v \in V} (f(u,v) + f'(u,v))$

□

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□

Value of the resulting flow

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Augmenting path – Example

- Let $G = (V, E)$ be a network and f be a flow.

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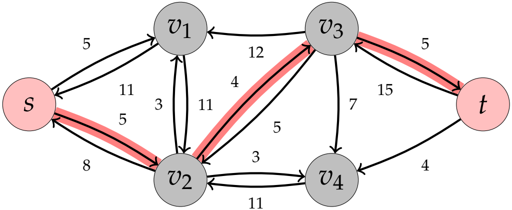

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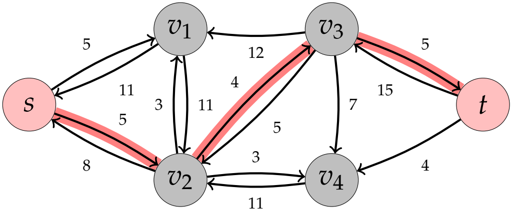

- Using this path, we can increase flow by 4 units.

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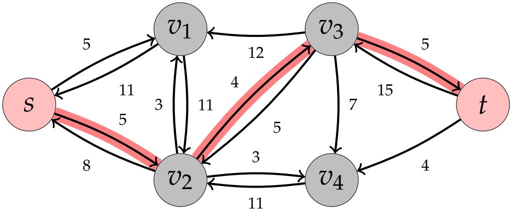

- Using this path, we can increase flow by 4 units.
- Residual capacity of augmenting path p is

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ lies on path } p\}.$$

Lemma 24.
Let $G = (V, E)$ be a network, f be its flow and p be an augmenting path in G_f. Let define a function

$$f_p(u,v) = \begin{cases} c_f(p) & \text{for } (u,v) \text{ on } p \\ -c_f(p) & \text{for } (v,u) \text{ on } p \\ 0 & \text{otherwise} \end{cases}$$

Then, f_p is the flow in G_f of size $|f_p| = c_f(p) > 0$.

Proof.
Homework. □

Lemma 24.

Let $G = (V, E)$ be a network, f be its flow and p be an augmenting path in G_f. Let define a function

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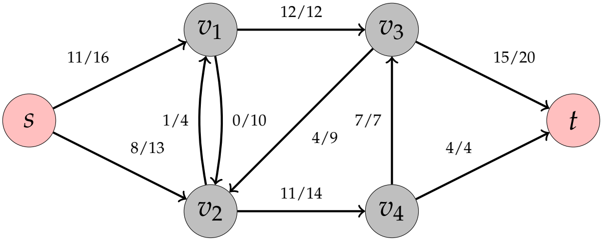
Proof.

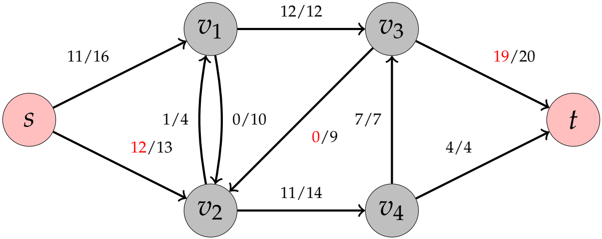
Homework. □

Corollary 25.

Let $f' = f + f_p$. Then, f' is a flow in G of size $|f'| = |f| + |f_p| > |f|$.

Residual network improved by 4 along $s \rightsquigarrow v_2 \rightsquigarrow v_3 \rightsquigarrow t$





Cut in Network

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- Network cut in $G = (V, E)$ is a partition of V to S and $T = V - S$ such that $s \in S$ and $t \in T$.

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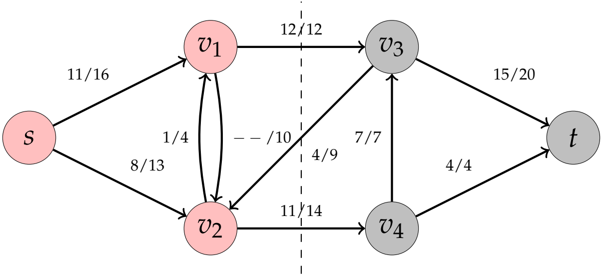
- Cut capacity (S, T) is $c(S, T)$.
- Minimal cut is a cut with minimal capacity.

Cut in Network – Example



- Flow through a cut: $f(\{s, v_1, v_2\}, \{v_3, v_4, t\}) = f(v_1, v_3) + f(v_2, v_3) + f(v_2, v_4) = 12 + (-4) + 11 = 19.$

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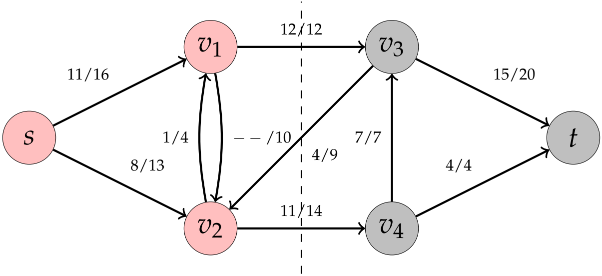

- Flow through a cut: $f(\{s, v_1, v_2\}, \{v_3, v_4, t\}) = f(v_1, v_3) + f(v_2, v_3) + f(v_2, v_4) = 12 + (-4) + 11 = 19.$
- Cut capacity: $c(\{s, v_1, v_2\}, \{v_3, v_4, t\}) = c(v_1, v_3) + c(v_2, v_4) = 12 + 14 = 26.$

Properties

Lemma 26.
Let f be a flow in G with source s and sink t and let (S,T) be a cut of G. Then, $|f| = f(S,T)$.

Proof.

- $f(S,T) = f(S,V) - f(S,S)$

□

Properties

Lemma 26.
Let f be a flow in G with source s and sink t and let (S,T) be a cut of G. Then, $|f| = f(S,T)$.

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Proof.

- $$\begin{aligned} f(S,T) &= f(S,V) - f(S,S) \\ &= f(S,V) \\ &= f(s,V) + f(S - \{s\}, V) \end{aligned}$$

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□

Properties

Corollary 27.

The value of any flow f in G is bounded from above by the capacity of any cut of G.

Proof.

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Proof.

- $$\begin{aligned} |f| &= f(S,T) \\ &= \sum_{u \in S} \sum_{v \in T} f(u,v) \\ &\leq \sum_{u \in S} \sum_{v \in T} c(u,v) \end{aligned}$$

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□

The value of a maximum flow is equal or less than the capacity of a minimum cut.

Max-flow min-cut Theorem

Let f be a flow in G with source s and sink t. Then, the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting path.
3. $|f| = c(S,T)$ for some cut (S,T) of G.

Proof.

- $(1) \Rightarrow (2)$:

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Proof.

- $(1) \Rightarrow (2)$:
  - Let f is maximum flow and p is an augmenting path in G_f.
  - Then, $f + f_p$ is a flow in G and $|f + f_p| > |f|$. Contradition.

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  - Since $s \in S$ and $t \in T$, (S,T) is a cut of G.

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  - For $u \in S$ and $v \in T$, we have $f(u, v) = c(u, v)$, otherwise $(u, v) \in E_f$, so $v \in S$.

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  - $|f| = f(S,T) = c(S,T)$.

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Proof.

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Proof.

- $(3) \Rightarrow (1)$:
  - $|f| \leq c(S, T)$ for any cut (S, T).
  - From $|f| = c(S, T)$, it follows that f is maximum.

□

The basic Ford-Fulkerson algorithm

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```
FORD-FULKERSON(G, s, t)
1  for each edge (u, v) ∈ E
2       do f[u, v] ← 0
3          f[v, u] ← 0
4  while there exists a path p from s to t in the residual network G_f
5          do c_f(p) ← min{c_f(u, v) : (u, v) is in p}
6             for each edge (u, v) in p
7                 do f[u, v] ← f[u, v] + c_f(p)
8                    f[v, u] ← −f[u, v]
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- Time complexity depends on line 4.

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- Time complexity depends on line 4.
- Using BFS gives total complexity $O(nm^2)$ – so called Edmonds-Karp algorithm.

The basic Ford-Fulkerson algorithm – Example

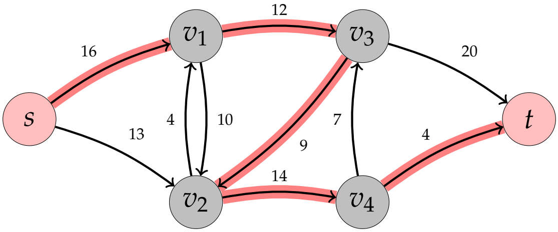

Figure: Residual network with an augmenting path from s to t.

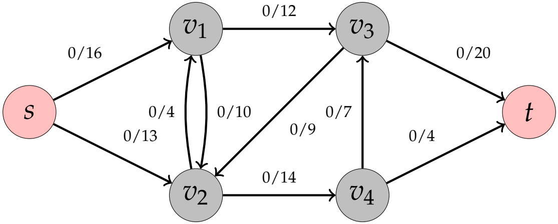

Figure: Network flow augmented along the path.

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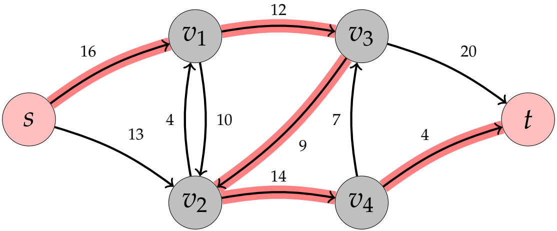

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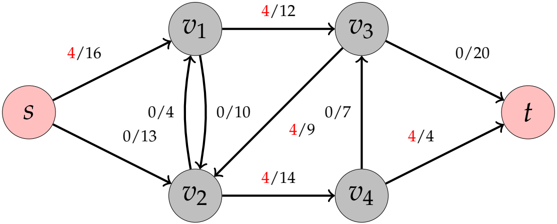

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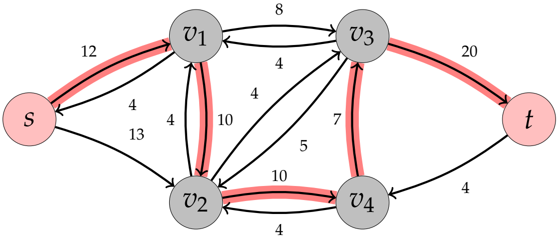

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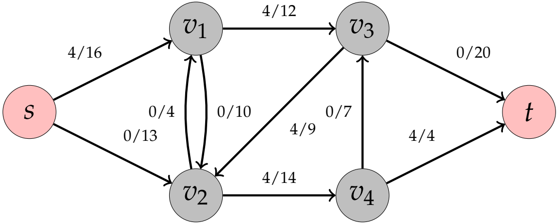

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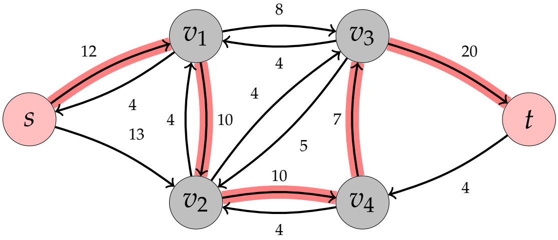

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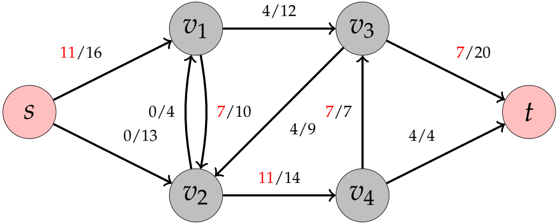

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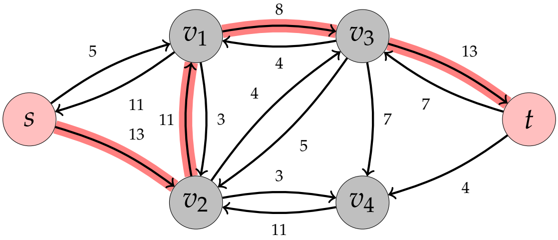

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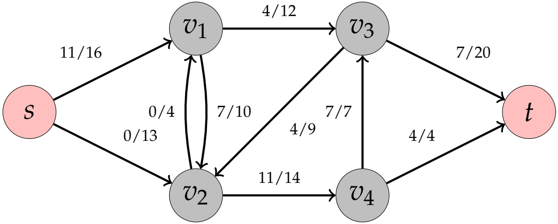

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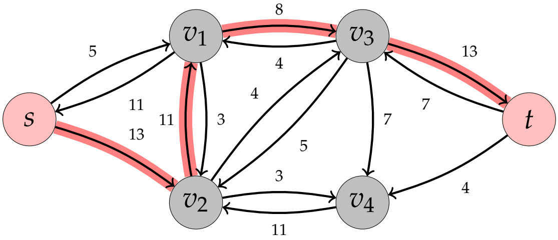

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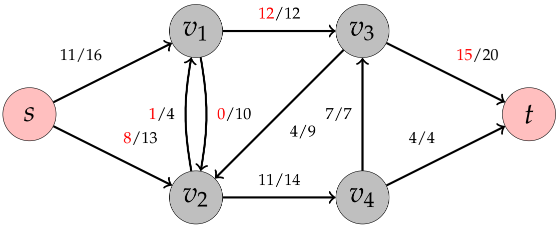

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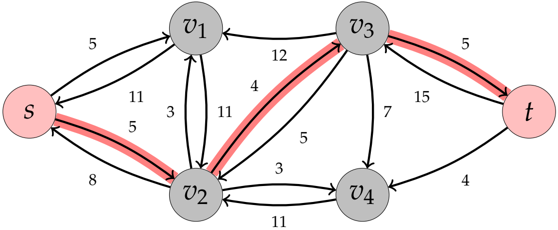
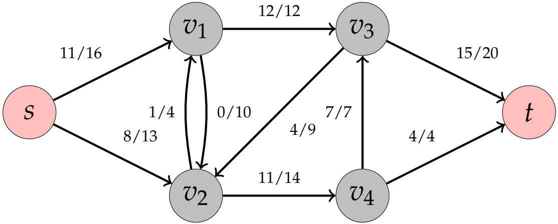


Figure: Residual network with an augmenting path from s to t.



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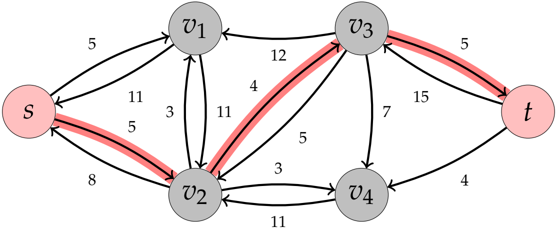

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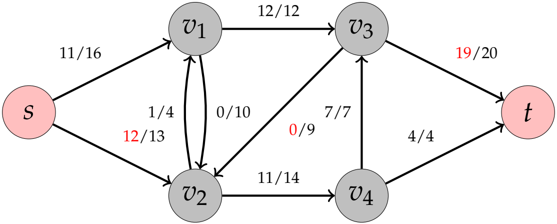

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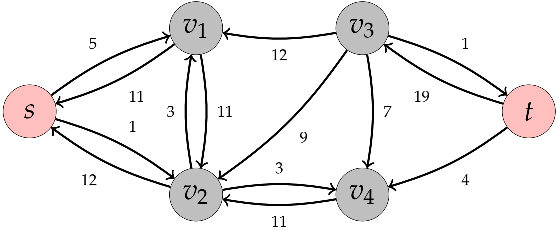

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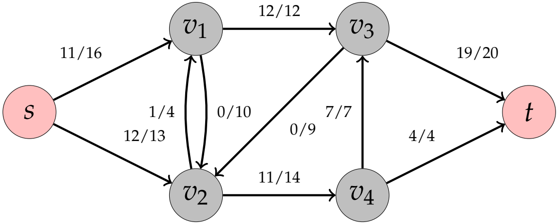

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Maximum bipartite matching

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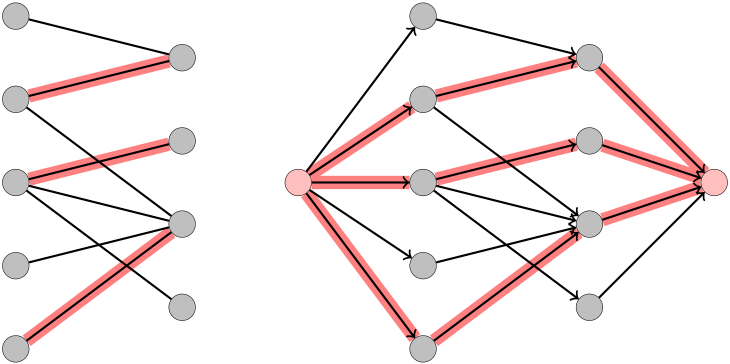
- We consider only connected bipartite graphs. That is, V can be partitioned into $V = L \cup R$, $R \cap L = \emptyset$ and $E \subseteq L \times R$.
- We use the Ford-Fulkerson method to find maximum matching in a connected undirected bipartite graph.

Transformation to Maximum network flow problem



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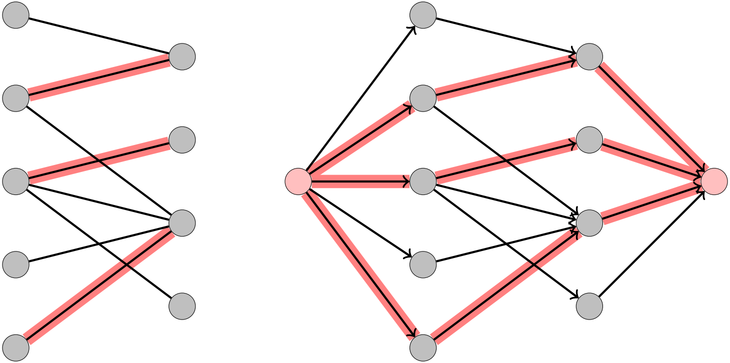

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- Time complexity: $O(nm)$.

Graph Coloring

Notation

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- Δ denotes the maximal degree of G.
- Graph-coloring problem: Determine $\psi_X(G)$ for a given graph, $X \in \{v, e\}$.

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- $\Delta \leq \psi_e(G)$.

Edge Coloring of Bipartite Graph

Theorem 28.
If G is bipartite, then $\psi_e(G) = \Delta$.

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- If they differ, we label these colors by C_1 and C_2.

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- Then, we can paint (u, v) by C_2. □

Edge Coloring of Complete Graph

Theorem 29.

If G is complete with n vertices, then $\psi_e(G) = \begin{cases} \Delta & n \text{ even} \\ \Delta + 1 & n \text{ odd} \end{cases}$

Proof

- Case 1: If n is odd, draw a graph as regular polygon (see below).

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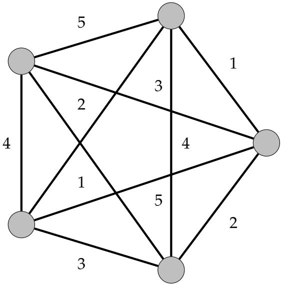

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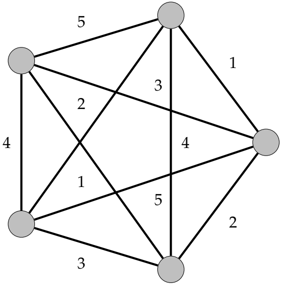

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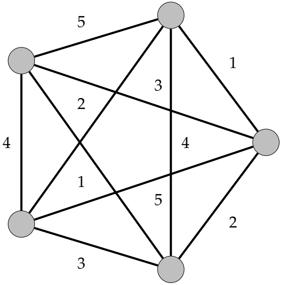

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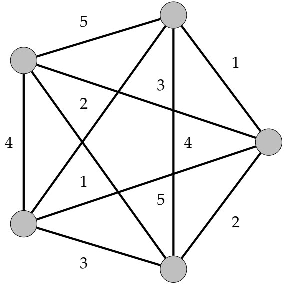

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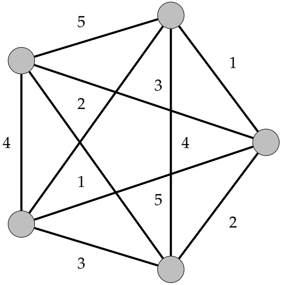
- Therefore, $|M| \leq \frac{1}{2}(n-1)$ – (prove as a homework).



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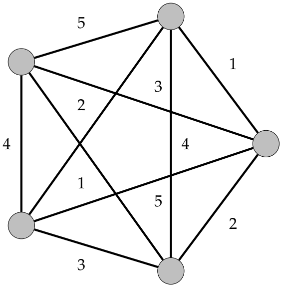

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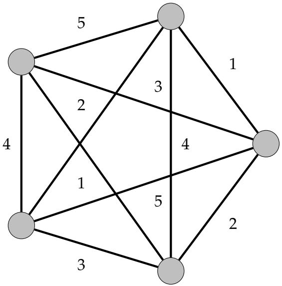

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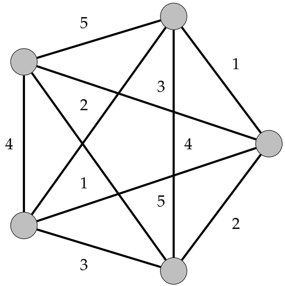

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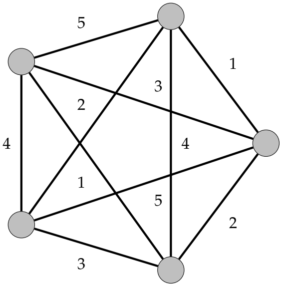

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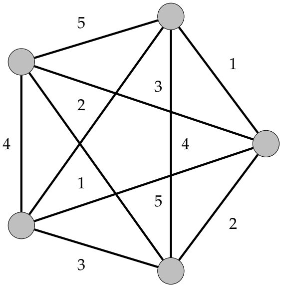

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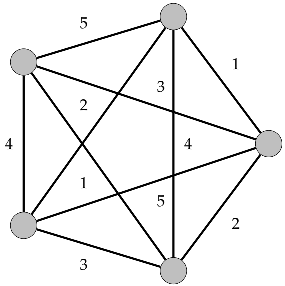
- In the end, we used at most $\Delta = n - 1$ colors. □



Edge Coloring of Undirected Graph

Theorem 30.
Let G is simple graph. Then, $\Delta \leq \psi_e(G) \leq \Delta + 1$.

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- See Chapter 7 in [Gibbons, 1985].

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  - If there is such v and $v \notin \{v_1, v_2, \ldots, v_i\}$, then add (v_0, v_{i+1}) to the sequence, where $v_{i+1} = v$ and C_{i+1} is missing in v_{i+1}.

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  - Otherwise, the sequence is finished.
- Such sequence has always at most Δ edges.

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- Let $(v_0, v_1), (v_0, v_2), (v_0, v_3), \ldots, (v_0, v_j)$ be the built sequence and $C_1, C_2, C_3, \ldots, C_j$, for some $j \geq 0$.

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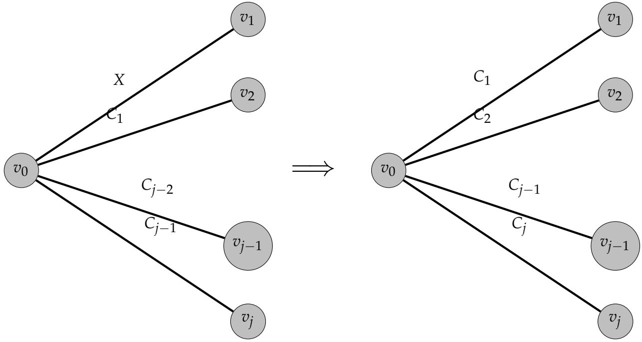
  - i) If there is no (v_0, v) colored by C_j, so we do the recoloring $(X \neq C_j)$:



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  - Then, for $i < k$, we recolor edges (see above), so (v_0, v_i) is colored by C_i.

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- At least one of C_0, C_j is not in v_0, v_k, v_j.
- So not all can be in a single component of $H(C_0, C_j)$:
  $v_0 \overset{C_j}{\to} x \overset{X}{\to} y \ldots \overset{C_0}{\to} v_k$ and we do not reach v_j.

Edge Coloring of Undirected Graph

a) $v_0 \notin H_{v_k}(C_0, C_j)$ – component of $H(C_0, C_j)$ contains v_k – then $C_0 \leftrightarrow C_j$ in $H_{v_k}(C_0, C_j)$, therefore C_0 is missing in v_k.

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- So color (v_0, v_j) by C_0. □

Edge Coloring of Undirected Graph

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Edge Coloring of Undirected Graph

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- But problem whether $\psi_e(G) = \Delta$ is NP-complete.

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- Time complexity: $O(n^2)$

(Vertex) Graph Coloring

Graph Coloring

- NP-Complete problem: Can we find a proper k-coloring of G?

Graph Coloring

Theorem 32.
Any (simple) graph G has $\Delta + 1$-coloring.

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Any (simple) graph G has $\Delta + 1$-coloring.

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- By induction on n.
- $n = 1$, obvious.
- If we add vertex u, then it is connected with at most Δ other vertices.
- Since we have $\Delta + 1$ colors, we have one spare color to paint u.

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- Homework: Design your own algorithm to find some proper coloring of a given graph?

Chromatic polynomial

Chromatic polynomial

- $P_k(G)$ – chromatic polynomial of G;
  determines the number of ways of proper vertex-coloring of G with k colors.

Chromatic polynomial

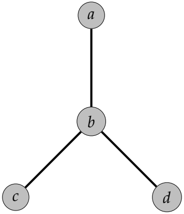
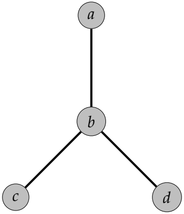


Figure: Graph G_1.

- b ... picks up one of k colors.

Chromatic polynomial



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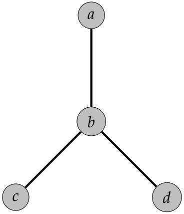

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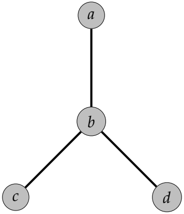

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- $P_k(G_1) = k(k-1)^3$
- In general, let T_n be a tree with n vertices. Then, $P_k(T_n) = k(k-1)^{n-1}$.

Chromatic polynomial

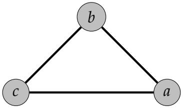

Figure: Graph G_2.

- a ... paint it to any of k colors.

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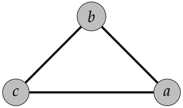

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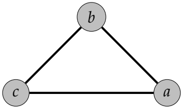

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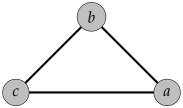

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- $P_k(G_2) = k(k-1)(k-2)$

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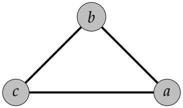

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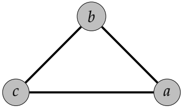

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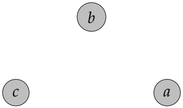
- $P_k(G_2) = k(k-1)(k-2)$
- In general, let K_n be a complete graph with n vertices.
- Then, $P_k(K_n) = \frac{k!}{(k-n)!}$

Chromatic polynomial



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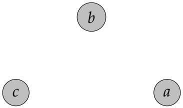

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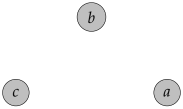

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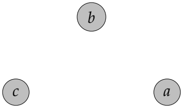

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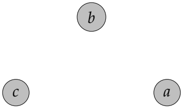

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- In general, let Φ_n be an isolated graph with n vertices; that is, $deg(v) = 0$ for all $v \in V$.

Chromatic polynomial

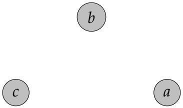

Figure: Graph G_2'.

- a ... gets arbitrary one of k colors.
- b ... gets arbitrary one of k colors.
- c ... gets arbitrary one of k colors.
- $P_k(G_2') = k^3$
- In general, let Φ_n be an isolated graph with n vertices; that is, $deg(v) = 0$ for all $v \in V$.
- Then, $P_k(\Phi_n) = k^n$

Chromatic polynomial

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- $G \circ (u, v)$ ... graph created from G by contracting (u, v).

Chromatic polynomial – Subtracting Recursion Formula

Theorem 33.
Let (u, v) *be an edge in* G*, then*

$$P_k(G) = P_k(G - (u, v)) - P_k(G \circ (u, v))\,.$$

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- So, we subtract them using polynomial $P_k(G \circ (u, v))$.

□

Chromatic polynomial – Example

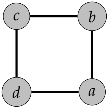

Figure: Graph G_3.

- $P_k(G_3) = P_k(\Phi_4) - 4P_k(\Phi_3) + 6P_k(\Phi_2) - 3P_k(\Phi_1)$

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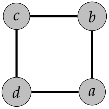

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- If G is dense, there is better variant of the construction:
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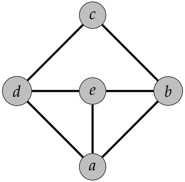
- That is, we add new edges until we reach complete graphs as addends.

Chromatic polynomial – Example



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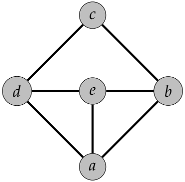

Figure: Graph G_4.

- $$\begin{aligned} P_k(G_4) &= P_k(K_5) + 3P_k(K_4) + 2P_k(K_3) \\ &= k(k-1)(k-2)(k^2-4k+5) \end{aligned}$$

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- What is the time complexity of building chromatic polynomial?
  For $k > 3$, $O(2^n n^r)$ for some constant r.

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for each vertex u ∈ V
    do for c ← 1 to Δ + 1
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for each vertex u ∈ V
    do c ← 1
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       for each v ∈ Adj[u]
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       color[u] ← c
```

- Time Complexity: $O(n^2)$

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- Time Complexity: $O(n^2)$
- Performance ratio $\text{A-S-V-C}(G)/\psi_v(G)$ is non-constant.

Exercises

1. Consider 3×3 chessboard represented as a graph with 9 vertices where an undirected edge (u, v) represents that a chess piece placed at u dominates v (it can attack the other piece at v) and vice versa. Use graph coloring to determine how many queens we can place on this chessboard so they do not attack each other.
2. Derive chromatic polynomial using subtracting formula for the complete graph with 4 vertices.
3. Derive chromatic polynomial using adding formula for the isolated graph with 4 vertices.
4. Use approximate vertex coloring algorithm for a bipartite graph with $L = \{u_1, u_2, \ldots, u_k\}$, $R = \{v_1, v_2, \ldots, v_k\}$, and $E = \{(u_i, v_j) : i \neq j\}$, $k \geq 2$. First, consider the vertices are colored in the order u_1, u_2, ..., u_k, v_1, v_2, ..., v_k. Second, apply the algorithm in the other order u_1, v_1, u_2, v_2, ..., u_k, v_k. Compare the results.

Eulerian Tours

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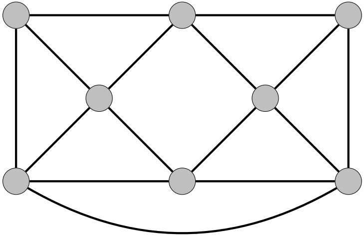
  - Graph exploration that walks through every vertex exactly once.
- Definition note: Tour = path or circuit; Cycle/Circuit = closed path

Eulerian graph

- Eulerian graph is a graph that contains an Eulerian circuit; that is, a closed path that visits all edges exactly once.



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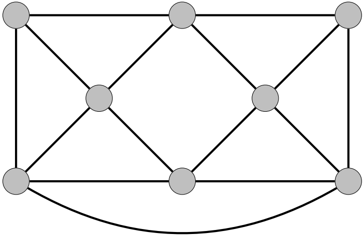
- Eulerian graph is a graph that contains an Eulerian circuit; that is, a closed path that visits all edges exactly once.
- Note that Eulerian path does not have to be closed, but then the graph is not Eulerian.



Theorem: Existence of an Eulerian tour

Theorem 34.

An undirected graph G, has an Eulerian tour if and only if it is connected and the number of odd-degree vertices is 0 or 2.

Proof

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Proof (continued)

- Let $G' = G - T = (V_{G'} = \{u, v | (u, v) \in E_G - E_T\}, E_G - E_T)$. G' can be unconnected, but contains only even-degree vertices.

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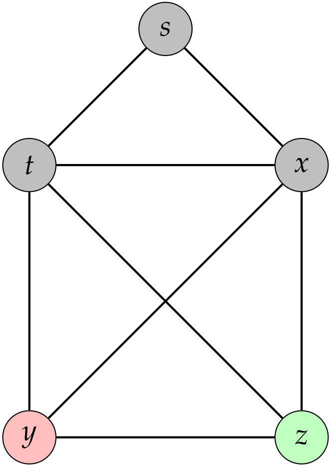
- Since G is connected and if G' is nonempty, then $V_T \cap V_{G'} \neq \emptyset$.
- Now, we inject Eulerian tours from G' into T using any of these common vertices. □

Example: Draw a house by a tour



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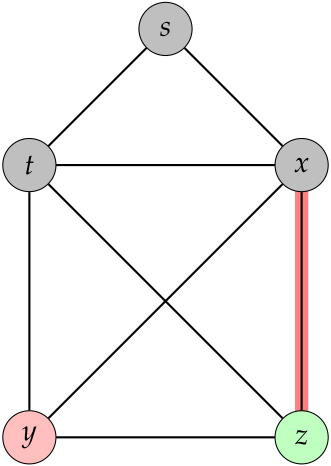

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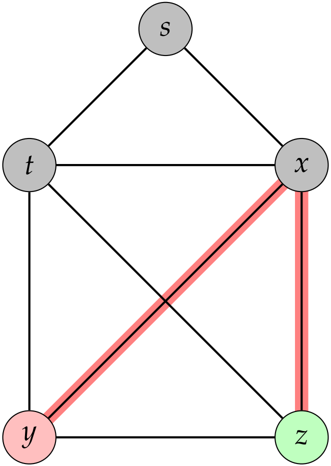

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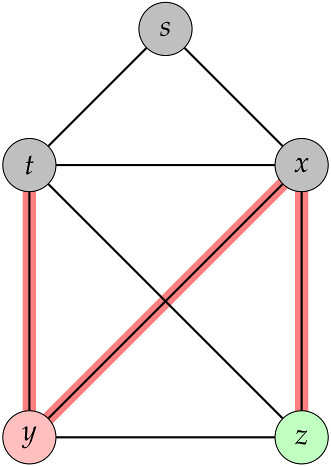

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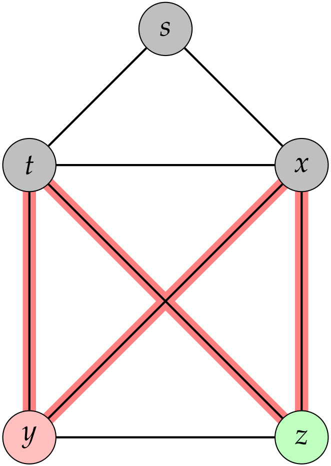

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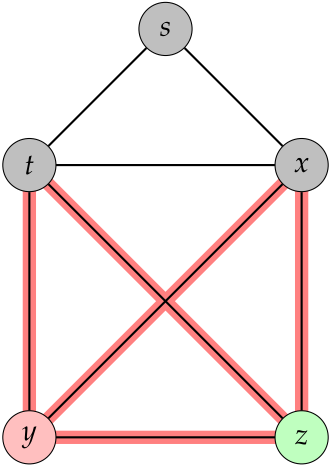

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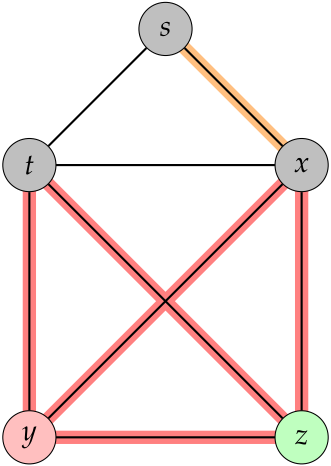

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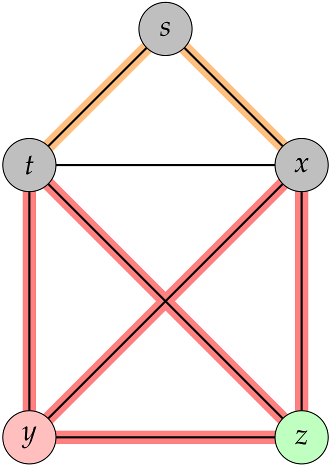

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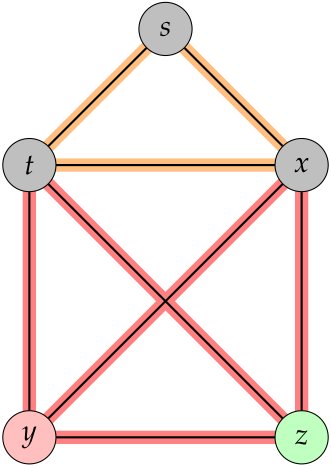

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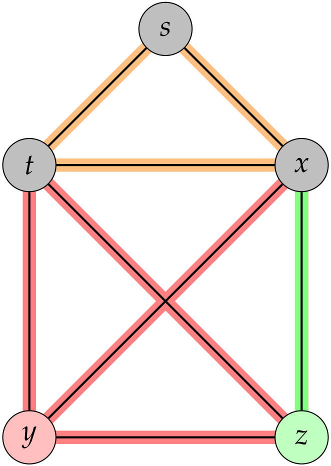

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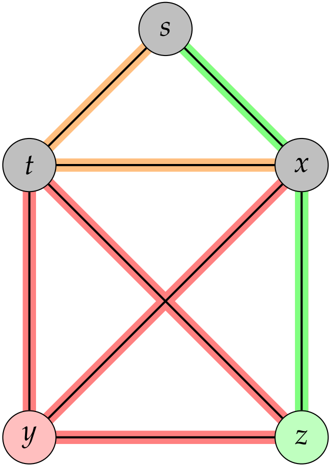

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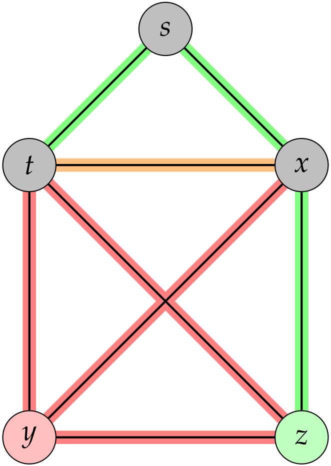

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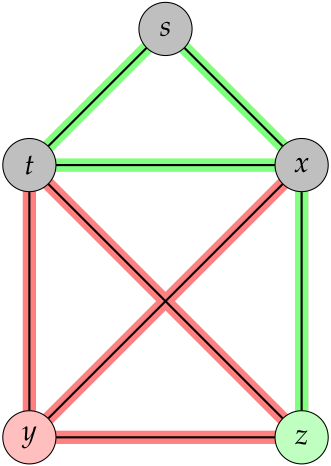

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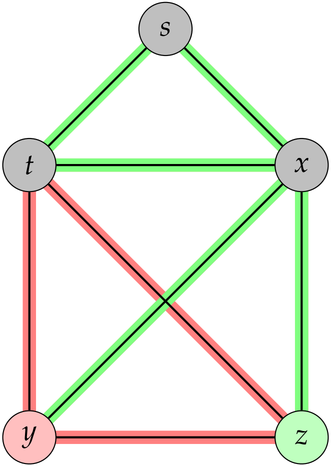

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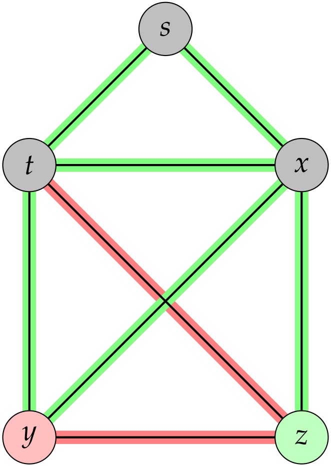

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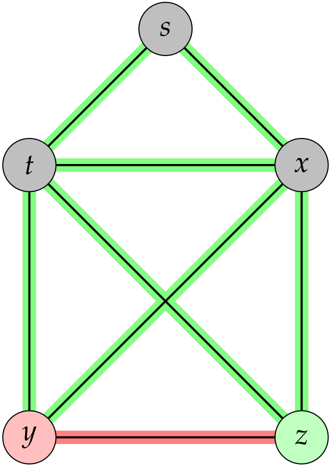

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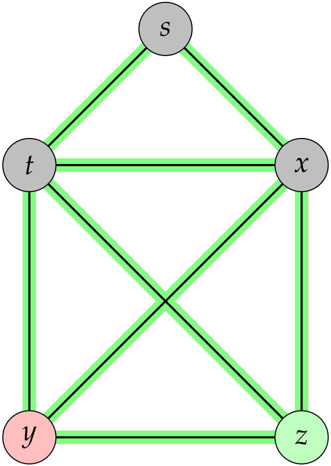

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Eulerian tour in digraphs

Out-tree of a graph $G = (V, E)$ is a directed subgraph (spanning tree) $T = (V, E')$ with root $u \in V$ where $E' \subseteq E$ and $d_+(u) = 0$ and $d_+(v) = 1$ for every $v \in V - \{u\}$.

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Proof. The first part in analogy to undirected Eulerian graph.

Directed Eulerian Tour – Examples

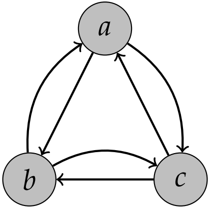

Figure: Eulerian digraph

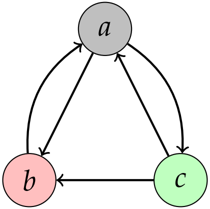

Figure: Eulerian path that is not a circuit

Theorem: Spanning out-tree of Eulerian digraph

Theorem 36.
Let $G = (V, E)$ be an Eulerian digraph and T its subgraph created by Eulerian tour from any vertex u in the following way: for every $v \neq u$, we add the first edge leading to v. Then, T is a spanning out-tree of digraph G rooted at u.

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- Assume that T contains a cycle finished by (v_i, v_j).
- Since $d_+(u) = 0$, $v_j \neq u$.
- Since (v_i, v_j) closes a cycle, so v_j was already processed, which is a contradiction! □

Theorem about directed Eulerian tour

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If G is connected and balanced digraph with a directed spanning tree T rooted at u, then we can find Eulerian circuit in the reverse order in the following way:

(a) *Start with any edge incident to u.*
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- Now, traverse the sequence of edges reversely back to u.
- Since G is balanced, we find unvisited edge that is incident to u, which is a contradiction with step (c). □

Algorithm for searching directed Eulerian path

```
EULER-CIRCUIT(G)
Find an oriented spanning out-tree T = (V, E_T) of G = (V, E) (root u)
for every vertex v ∈ V
    do  A[v] ← ∅
        I[v] ← 0
for every edge (v_i, v_j) ∈ E
    do if (v_i, v_j) ∈ E_T
          then add v_i to the tail of list A[v_j]
          else add v_i to the head of list A[v_j]
EC ← ∅
CV ← u
while I[CV] ≤ d_+(CV)
    do  add CV to the head of list EC
        I[CV] ← I[CV] + 1
        CV ← A[CV][I[CV]]
Print EC
```

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- Therefore, the total time complexity $O(m)$.

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  - Optimal solution for non-Eulerian graph: $O(m + n^3)$

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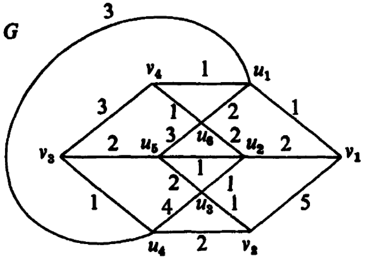

$d(v_1, v_2) = 4$ along (v_1, u_2, u_3, v_2)
$d(v_1, v_3) = 5$ along (v_1, u_2, u_5, v_3)
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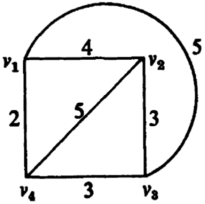

A minimum-weight perfect matching consists of the edges (v_1, v_4) and (v_2, v_3).

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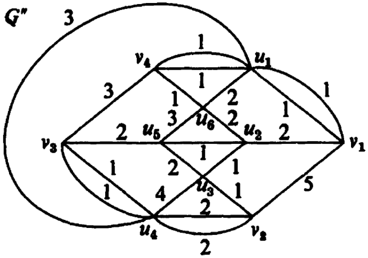

An Eulerian circuit of G'' and a solution to the Chinese postman problem for G is (v_1, u_1, v_4, v_3, u_4, v_2, v_1, u_2, u_3, v_2, u_4, u_3, u_5, v_3, u_4, u_1, v_4, u_6, u_5, u_2, u_6, u_1, v_1).

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Theorem 41.
If $G = (V, E)$ is a graph such that $|V| > 3$ and $\min_{v \in V}(d(v)) > \frac{n}{2}$ then G is Hamiltonian.

Chvátal theorem (1972)

Theorem 42.
Let G be undirected graph with $n \geq 3$ vertices. If $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$ is a non-descending sequence of degrees of vertices and, in addition, the following holds:

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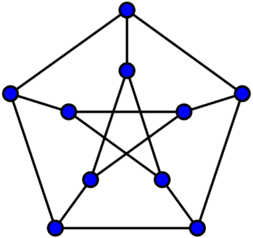

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- Applications: Transportation tasks, Process scheduling, ...

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- Intractable/ineffective since enumeration grows with $n!$.