

Research Article

Josef Šlapal*

A digital Jordan surface theorem with respect to a graph connectedness

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Abstract: After introducing a graph connectedness induced by a given set of paths of the same length, we focus on the 2-adjacency graph on the digital line $\mathbb{Z}$ with a certain set of paths of length n for every positive integer n . The connectedness in the strong product of three copies of the graph is used to define digital Jordan surfaces. These are obtained as polyhedral surfaces bounding the polyhedra that can be face-to-face tiled with digital tetrahedra.

Keywords: simple graph, strong product, path, connectedness, digital space, Jordan surface

MSC 2020: 52C22, 68R10

1 Introduction

Digital Jordan surfaces play an important role in digital imagery because they represent the borders of objects in 3D digital images. It is a classical problem to propose a convenient definition of such surfaces. Since the surfaces are required to be connected and satisfy the digital Jordan surface theorem (i.e., separating the digital space $\mathbb{Z}^3$ into exactly two connected components), to solve the problem, we have to start with choosing a suitable connectedness structure on the digital space $\mathbb{Z}^3$. In the classical approach (see [1,2]), adjacency relations are used to obtain such structures, namely, the well-known 6-, 18-, and 26-adjacencies. A disadvantage of this approach is that the connectedness given by an adjacency does not allow for a digital Jordan surface theorem, i.e., does not behave as the connectedness with respect to the Euclidean topology on $\mathbb{R}^3$. This disadvantage is eliminated by employing two adjacencies simultaneously, one, say k_1 , for the surface and another, say k_2 , for its complement – we then speak about (k_1, k_2) -connectivity (see [3,4]).

In 1990, a new purely topological approach to the problem of providing the digital space with a convenient connectedness structure was proposed in a study by Khalimsky et al. [5] (see also [6]). They showed that there is a topology on $\mathbb{Z}^3$ allowing for a convenient definition of digital Jordan surfaces, thus providing a digital model of the Euclidean 3D space. Therefore, this topology, called the Khalimsky topology, can be used for the study of 3D digital images. Digital Jordan surfaces with respect to the Khalimsky topology were discussed in a study by Kopperman et al. [7].

Another, graph-theoretic approach was proposed in a study by Šlapal [8] based on a connectedness given by a set of paths of the same length in an undirected simple graph with the vertex set $\mathbb{Z}^3$. This approach was developed in [9] and then used in [10], where closure operators associated with sets of paths (of the same length) in a simple undirected graph were used to obtain a convenient connectedness in the digital space $\mathbb{Z}^3$. Such a connectedness was then used for defining digital Jordan surfaces obtained as boundary surfaces of the digital polyhedra that can be face-to-face tiled with (finitely many) digital triangular prisms. In the present article, we will substantially improve the result in [10] by showing that digital Jordan surfaces may be defined

* **Corresponding author: Josef Šlapal**, Institute of Mathematics, Brno University of Technology, 602 00 Brno, Czech Republic, e-mail: slapal@fme.vutbr.cz

to be boundary surfaces of digital polyhedra formed by face-to-face tiling with certain digital tetrahedra that partition the digital prisms used in [10]. Therefore, such digital Jordan surfaces comprise a larger variety than those defined in [10]. We will employ a connectedness given directly by sets of paths of a given length $n > 1$ in an undirected simple graph. This connectedness coincides with the connectedness employed in [10], which is given by closure operators associated with the sets of paths (while only paths of length 2 are considered in [10]).

In the literature, digital surfaces were studied by a number of authors. For example, some homotopy-theoretic properties of digital Jordan surfaces with respect to the (6, 26)- and (6, 18)-connectivities are studied in [11], while in [12] also the (26, 6)- and (18, 6)-connectivities are considered, and a strong homotopy property of the surfaces is discussed. In [13], the authors continue the study performed in [12], and in [14], further homotopy properties of Jordan surfaces are investigated based on combinations of 6- and 18-adjacencies. Digital Jordan surfaces have also been studied by Han [15–18]. In [15], the classical approach to Jordan surfaces using combination of two adjacencies was generalized by employing a single general adjacency. Some properties of the generalized Jordan surfaces were studied there, including the behavior of the connected sum of the surfaces. In [16], the author deals with digital maps preserving the topological properties of digital surfaces and discusses certain minimal simple closed surfaces. A fixed point property for digital surfaces was discussed in [17] including its relationships to Euler characteristics of the surfaces. Finally, a Jordan surface theorem for simple closed surfaces in $\mathbb{R}^3$ with respect to certain special topological structure (called a space set topological structure) is proved in [18]. The theorem may be applied in various topological and geometric fields, including discrete geometry.

We will use only some basic graph-theoretic concepts [19]. A graph will mean an undirected simple graph (without loops), hence a pair $G = (V, E)$ with $V \neq \emptyset$ the set of *vertices* and E the set of *edges* of G (i.e., $E \subseteq \{\{x, y\}; x, y \in V, x \neq y\}$) where vertices $x, y \in V$ are said to be *adjacent* if $\{x, y\} \in E$. The graph G will be said to be a graph *on* V . Recall that a *walk* in G is a (finite) sequence $(x_i | i \leq n) = (x_0, x_1, \dots, x_n)$ of vertices such that every two consecutive terms of the sequence are adjacent. The nonnegative integer n is called the *length* of the walk. A walk $(x_i | i \leq n)$ in G is called a *path* if $x_i \neq x_j$ for all $i, j \leq n, i \neq j$. A subset $A \subseteq V$ is *connected* if any two different vertices $x, y \in A$ can be joined by a walk (or, equivalently, a path) $(x_i | i \leq n)$ in G contained in A , i.e., such that $x_0 = x, x_n = y$, and $x_i \in A$ for every $i \leq n$.

Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we say that G_1 is a *subgraph* of G_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. If, moreover, $V_1 = V_2$, then G_1 is called a *factor* of G_2 . A graph (V_1, E_1) is said to be an *induced subgraph* of a graph (V_2, E_2) if it is a subgraph of (V_2, E_2) such that $E_1 = E_2 \cap \{\{x, y\}; x, y \in V_1\}$. In short, we speak about the induced subgraph V_1 of (V_2, E_2) in this case.

In accordance with [20], we propose the following definition:

Definition 1.1. Given graphs $G_j = (V_j, E_j)$, $j = 1, 2, \dots, m$ ($m > 0$ an integer), we define their *strong product* to be the graph $\prod_{j=1}^m G_j = (\prod_{j=1}^m V_j, E)$ with the set of edges $E = \{\{(x_1, x_2, \dots, x_m), (y_1, y_2, \dots, y_m)\}; \text{there exists a nonempty subset } J \subseteq \{1, 2, \dots, m\} \text{ such that } \{x_j, y_j\} \in E_j \text{ for every } j \in J \text{ and } x_j = y_j \text{ for every } j \in \{1, 2, \dots, m\} - J\}$.

Thus, the direct (i.e., cartesian) product of a family of graphs $G_j = (V_j, E_j)$, $j = 1, 2, \dots, m$, which is the graph $(\prod_{j=1}^m V_j, E)$ where $E = \{\{(x_1, x_2, \dots, x_m), (y_1, y_2, \dots, y_m)\}; \{x_j, y_j\} \in E_j \text{ for every } j = 1, 2, \dots, m\}$, is a factor of the strong product of the family.

We will use the concept of a *tiling* (i.e., *tessellation* – see [21]) applied to subsets of $\mathbb{Z}^3$. More precisely, we will work with a face-to-face tiling of a (digital) polyhedral subset of $\mathbb{Z}^3$ with certain digital tetrahedra.

2 Path-set induced connectedness in a graph

In order to make our article self-contained, we reproduce some relevant material from [9] and [10].

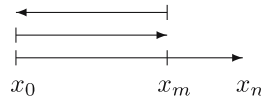
In the sequel, n will denote a positive integer.

Let $G = (V, E)$ be a graph. Then, we denote by $\mathcal{P}_n(G)$ the set of all paths of length n in G . If $\mathcal{B} \subseteq \mathcal{P}_n(G)$ and $V_1 \subseteq V$ is an induced subgraph of G , then the set $\mathcal{B} \cap V_1^{n+1} \subseteq \mathcal{P}_n(G_1)$ will be denoted by $\mathcal{B}|V_1$.

For every set of paths (path set for short) $\mathcal{B} \subseteq \mathcal{P}_n(G)$, we put

$\mathcal{B}^* = \{(x_i | i \leq m) \in \mathcal{P}_m(G); 0 < m \leq n \text{ and there exists } (y_i | i \leq n) \in \mathcal{B} \text{ such that } x_i = y_i \text{ for every } i \leq m \text{ or } x_i = y_{m-i} \text{ for every } i \leq m\}$.

The elements of $\mathcal{B}^*$ will be called $\mathcal{B}$ -initial segments in G . Thus, a $\mathcal{B}$ -initial segment $(x_i | i \leq m)$ in G is a sequence consisting of the first $m + 1$ terms of a path belonging to $\mathcal{B}$ ordered in agreement with or oppositely to the path – see the following figure (with sequences represented by arrows oriented from the first to the last terms):



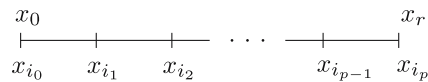
Clearly, we have $\mathcal{B} \subseteq \mathcal{B}^*$.

Definition 2.1. Let G_j be a graph and $\mathcal{B}_j \subseteq \mathcal{P}_n(G_j)$ for every $j = 1, 2, \dots, m$ ($m > 0$ an integer). Then, we define the *strong product* of the paths $\mathcal{B}_j$, $j = 1, 2, \dots, m$, to be the set $\prod_{j=1}^m \mathcal{B}_j = \{((x_i^1, x_i^2, \dots, x_i^m) | i \leq n); \text{ there is a nonempty subset } J \subseteq \{1, 2, \dots, m\} \text{ such that } (x_i^j | i \leq n) \in \mathcal{B}_j \text{ for every } j \in J \text{ and } (x_i^j | i \leq n) \text{ is a constant sequence for every } j \in \{1, 2, \dots, m\} - J\}$.

Clearly, $\prod_{j=1}^m \mathcal{B}_j \subseteq \mathcal{P}_n(\prod_{j=1}^m G_j)$.

Given a graph G and $\mathcal{B} \subseteq \mathcal{P}_n(G)$, we will employ the walks in G that are formed by subsequent $\mathcal{B}$ -initial segments in G to define a connectedness in G .

Definition 2.2. Let $G = (V, E)$ be a graph and let $\mathcal{B} \subseteq \mathcal{P}_n(G)$ be a path set. A $\mathcal{B}$ -walk in G is any sequence $C = (x_i | i \leq r)$, $r > 0$ an integer, of vertices of V having the property that there is an increasing sequence $(i_k | k \leq p)$ of non-negative integers with $i_0 = 0$ and $i_p = r$ such that $i_k - i_{k-1} \leq n$ and $(x_i | i_{k-1} \leq i \leq i_k) \in \mathcal{B}^*$ for all k with $0 < k \leq p$ (see the figure below).



Definition 2.3. Let $G = (V, E)$ be a graph and $\mathcal{B} \subseteq \mathcal{P}_n(G)$ be a path set. A set $A \subseteq V$ is said to be $\mathcal{B}$ -connected in G if any two different vertices of G belonging to A can be joined by a $\mathcal{B}$ -walk in G contained in A . A maximal (with respect to set inclusion) $\mathcal{B}$ -connected set in G is called a $\mathcal{B}$ -component of G .

By [10], Proposition 2.1, the $\mathcal{B}$ -connectedness introduced in Definition 2.3 coincides with the connectedness employed in [10], which is given by certain closure operator associated with $\mathcal{B}$.

Let $G = (V, E)$ be a graph and $\mathcal{B} \subseteq \mathcal{P}_n(G)$. Then, every $\mathcal{B}$ -walk in G is clearly $\mathcal{B}$ -connected in G , but, if $n > 1$ and $A \subseteq V$ is a $\mathcal{B}$ -connected set in G , then A need not be connected in G .

We will need the following, quite obvious property of the $\mathcal{B}$ -connectedness:

Lemma 2.4. Let G be a graph on a vertex set V and let $\mathcal{B} \subseteq \mathcal{P}_n(G)$. Let $\{A_i, i \in I\}$ be a finite or countable set of $\mathcal{B}$ -connected subsets of V . If $\{A_i, i \in I\}$ can be ordered into a sequence such that every term of the sequence (excluding the first one) has a nonempty meet with some of its predecessors, then $\bigcup_{i \in I} A_i$ is $\mathcal{B}$ -connected.

Proposition 2.5. [9] Let $G_j = (V_j, E_j)$ be a graph, $\mathcal{B}_j \subseteq \mathcal{P}_n(G_j)$, and $Y_j \subseteq V_j$ be a subset for every $j = 1, 2, \dots, m$. If Y_j is a $\mathcal{B}_j$ -connected set in G_j for every $j = 1, 2, \dots, n$, then $\prod_{j=1}^m Y_j$ is a $\prod_{j=1}^m \mathcal{B}_j$ -connected set in $\prod_{j=1}^m G_j$.

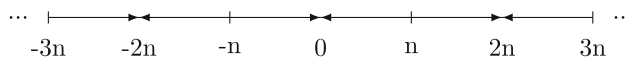
In our investigations, we will employ the following graph on $\mathbb{Z}$:

By the *2-adjacency graph* on $\mathbb{Z}$, we understand the graph $H = (\mathbb{Z}, A)$, where $A = \{\{p, q\}; p, q \in \mathbb{Z}, |p - q| = 1\}$.

From now on, $\mathcal{B}_n$ will denote the path set $\mathcal{B}_n \subseteq \mathcal{P}_n(H)$ defined as follows:

$\mathcal{B}_n = \{(x_i | i \leq n) \in \mathcal{P}_n(H); \text{ there is an odd number } l \in \mathbb{Z} \text{ such that } x_i = ln + i \text{ for all } i \leq n \text{ or } x_i = ln - i \text{ for all } i \leq n\}$.

In other words, the paths belonging to $\mathcal{B}_n$ are nothing but the arithmetic sequences $(x_i | i \leq n)$ of integers with $x_0 = ln$, where $l \in \mathbb{Z}$ is an odd number and with the difference 1 or -1 – see the following figure where the paths belonging to $\mathcal{B}_n$ are represented as arrows (oriented from the first to the last terms of the sequences):



It may easily be seen that $\mathbb{Z}$ is a $\mathcal{B}_n$ -connected set in H . The $\mathcal{B}_1$ -connectedness coincides with the connectedness given by the Khalimsky topology on $\mathbb{Z}$ generated by the subbase $\{\{2k - 1, 2k, 2k + 1\}; k \in \mathbb{Z}\}$ – cf. [5].

By Proposition 2.5, we may consider a new connectedness structure on the digital space $\mathbb{Z}^m$ for every positive integer m . Namely, we may consider the strong product of m copies of the 2-adjacency graph on $\mathbb{Z}$ with the path set given by the strong product of m -copies of the path set $\mathcal{B}_n$. More precisely, for every positive integer m , we may consider the graph $H^m = \prod_{j=1}^m H_j$ on $\mathbb{Z}^m$, where $H_j = H$ for every $j \in \{1, \dots, m\}$, with the path set $\mathcal{B}_n^m \subseteq \mathcal{P}_n(H^m)$ given by $\mathcal{B}_n^m = \prod_{j=1}^m \mathcal{B}_j$, where $\mathcal{B}_j = \mathcal{B}_n$ for every $j \in \{1, \dots, m\}$. It immediately follows from Proposition 2.5 that $\mathbb{Z}^m$ is a $\mathcal{B}_n^m$ -connected set in H^m for every positive integer m .

The $\mathcal{B}_1^m$ -connectedness in the graph H^m (on $\mathbb{Z}^m$) coincides with the connectedness in the Khalimsky topology on $\mathbb{Z}^m$ (i.e., the topological product of m copies of the Khalimsky topology on $\mathbb{Z}$) – cf. [5]. The behavior of the Khalimsky topology is well known, and therefore, in the sequel, we will assume that $n > 1$. Of course, with respect to possible applications in digital imagery, the most important cases are $m = 2$ and $m = 3$. Observe that the graphs H^2 and H^3 coincide with the well-known 8-adjacency graph on $\mathbb{Z}^2$ and 26-adjacency graph on $\mathbb{Z}^3$, i.e., the graphs $(\mathbb{Z}^2, A_8)$, where $A_8 = \{\{(x_1, y_1), (x_2, y_2)\}; (x_1, y_1), (x_2, y_2) \in \mathbb{Z}^2, \max\{|x_1 - x_2|, |y_1 - y_2|\} = 1\}$, and $(\mathbb{Z}^3, A_{26})$ where $A_{26} = \{\{(x_1, y_1, z_1), (x_2, y_2, z_2)\}; (x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{Z}^3, \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} = 1\}$, respectively. The path set $\mathcal{B}_n^2$ is shown in Figure 1, where the paths belonging to $\mathcal{B}_n^2$ are represented by line segments directed from the first to the last terms of the paths. Note that between any pair of neighboring parallel line segments (with the same orientation), there are $n - 1$ more line segments parallel to them (and having the same orientation), which are not displayed.

Since the case $m = 2$ is discussed in [9] (and some other articles – see the references in [9]), in this note, we will focus on the graph H^3 with the path set $\mathcal{B}_n^3$.

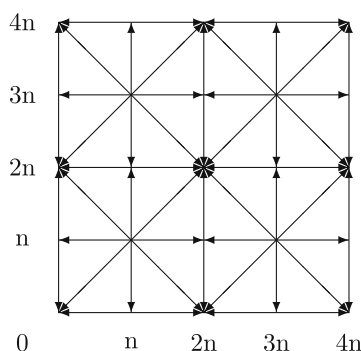


Figure 1: The path set $\mathcal{B}_n^2$.

3 Digital Jordan surfaces with respect to the connectedness in H^3 induced by $\mathcal{B}_n^3$

Every digital cube $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2ln + 2n, 2mn \leq z \leq 2mn + 2n\}$, $k, l, m \in \mathbb{Z}$, will be called an n -fundamental cube. It is evident that every n -fundamental cube is $\mathcal{B}_n^3$ -connected in H^3 and so is every subset of $\mathbb{Z}^3$ obtained from an n -fundamental cube by removing some of its faces.

Definition 3.1. Each of the following subsets of $\mathbb{Z}^3$ will be called an n -fundamental prism:

- (1) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2kn + 2ln + 2n - x, 2mn \leq z \leq 2mn + 2n\}$, $k, l, m \in \mathbb{Z}$,
- (2) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2kn + 2ln + 2n - x \leq y \leq 2ln + 2n, 2mn \leq z \leq 2mn + 2n\}$, $k, l, m \in \mathbb{Z}$,
- (3) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq x - 2kn + 2ln, 2mn \leq z \leq 2mn + 2n\}$, $k, l, m \in \mathbb{Z}$,
- (4) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, x - 2kn + 2ln \leq y \leq 2ln + 2n, 2mn \leq z \leq 2mn + 2n\}$, $k, l, m \in \mathbb{Z}$,
- (5) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2ln + 2n, 2mn \leq z \leq 2kn + 2mn + 2n - x\}$, $k, l, m \in \mathbb{Z}$,
- (6) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2ln + 2n, 2kn + 2mn + 2n - x \leq z \leq 2mn + 2n\}$, $k, l, m \in \mathbb{Z}$,
- (7) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2ln + 2n, 2mn \leq z \leq x - 2kn + 2mn\}$, $k, l, m \in \mathbb{Z}$,
- (8) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2ln + 2n, x - 2kn + 2mn \leq z \leq 2mn + 2n\}$, $k, l, m \in \mathbb{Z}$,
- (9) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2ln + 2n, 2mn \leq z \leq 2ln + 2mn + 2n - y\}$, $k, l, m \in \mathbb{Z}$,
- (10) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2ln + 2n, 2ln + 2mn + 2n - y \leq z \leq 2mn + 2n\}$, $k, l, m \in \mathbb{Z}$,
- (11) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2ln + 2n, 2mn \leq z \leq y - 2ln + 2mn\}$, $k, l, m \in \mathbb{Z}$,
- (12) $\{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq 2kn + 2n, 2ln \leq y \leq 2ln + 2n, y - 2ln + 2mn \leq z \leq 2mn + 2n\}$, $k, l, m \in \mathbb{Z}$,

The concept of a 2-fundamental prism coincides with that of a fundamental prism introduced in [10]. Clearly, every n -fundamental prism has the form of a digital right triangular prism and every n -fundamental cube may be tessellated into two n -fundamental prisms having a face in common. Namely, dividing an n -fundamental cube by a (digital) plane that is perpendicular to a face of the cube and contains one of the two (digital) diagonals of the face, we obtain a pair of n -fundamental prisms. All n -fundamental prisms are obtained in this way, and every n -fundamental cube gives rise to 12 n -fundamental prisms. An n -fundamental prism satisfying equation (1) in Definition 3.1 is shown in Figure 2.

It is proved in [10, Lemma 3.4], that, for $n = 2$, every n -fundamental prism is $\mathcal{B}_n^3$ -connected in H^3 and so is every subset of $\mathbb{Z}^3$ obtained from an n -fundamental prism by removing some of its faces. It may be proved analogously that these properties are satisfied by n -fundamental prisms for all $n > 2$.

Remark 3.2. Every n -fundamental prism is the union of $2n + 1$ digital triangles, namely those lying between the bases of the prism and being parallel to them. We will call these triangles n -fundamental triangles. Clearly, as induced subgraphs of H^3 , these triangles are isomorphic to each other. By [9] (proof of Theorem 3.2), every n -fundamental triangle is $\mathcal{B}_n^3$ -connected and so is every set obtained from an n -fundamental triangle by removing some of its sides.

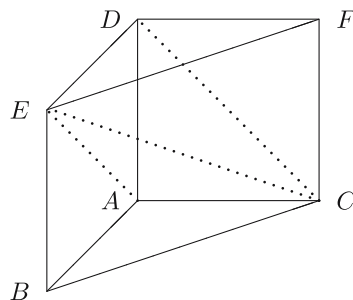


Figure 2: An n -fundamental prism where $A = (2kn, 2ln, 2mn)$, $B = (2kn + 2n, 2ln, 2mn)$, $C = (2kn, 2ln + 2n, 2mn)$, $D = (2kn, 2ln, 2mn + 2n)$, $E = (2kn + 2n, 2ln, 2mn + 2n)$, $F = (2kn, 2ln + 2n, 2mn + 2n)$, and $k, l, m \in \mathbb{Z}$.

Of the three lateral (rectangular) faces of an n -fundamental prism, the one that is not a square will be called the *main face* of the prism.

Definition 3.3. By an n -fundamental tetrahedron, we understand any of the three digital tetrahedra obtained by dividing an n -fundamental prism by the (digital) planes ACE and CDE , where A, C, D , and E are vertices of the prism such that CE is a diagonal of the main face, AE is a diagonal of a square face, and CD is a diagonal of the other square face of the prism.

Clearly, every n -fundamental prism may be tessellated into three n -fundamental tetrahedra in two different ways (one of them is demonstrated in Figure 2 by dotted line segments), and each of the tetrahedra is inscribed to the n -fundamental prism, i.e., its vertices are (some) vertices of the prism. It may easily be seen that, for every n -fundamental cube, there are 24 n -fundamental tetrahedra inscribed to the cube. Of course, all n -fundamental tetrahedra are congruent to each other (i.e., identical up to translation, rotation, and mirroring).

One may easily see that exactly two faces of an n -fundamental tetrahedron are n -fundamental triangles. The other two faces will be called the *peculiar faces* of the tetrahedron. In the n -fundamental tetrahedron $ABCE$ demonstrated in Figure 2, the faces ABC and ABE are n -fundamental triangles, while the faces ACE and BCE are peculiar. Evidently, each of the peculiar faces of an n -fundamental tetrahedron is isomorphic to any n -fundamental triangle. The following properties of the n -fundamental tetrahedra are crucial for our investigations:

Proposition 3.4. Every n -fundamental tetrahedron is $\mathcal{B}_n^3$ -connected in H^3 and so is every subset of $\mathbb{Z}^3$ obtained from an n -fundamental tetrahedron by removing some of its faces while keeping a peculiar one.

Proof. Let T be the n -fundamental tetrahedron $ABCE$ in Figure 2. Thus, there are $k, l, m \in \mathbb{Z}$ such that $A = (2kn, 2ln, 2mn)$, $B = ((2k+2)n, 2ln, 2mn)$, $C = (2kn, (2l+2)n, 2mn)$, and $E = ((2k+2)n, 2ln, (2m+2)n)$. Hence, we have $T = \{(x, y, z) \in \mathbb{Z}^3; 2kn \leq x \leq (2k+2)n, 2ln \leq y \leq (2k+2l+2)n - x, 2mn \leq z \leq x + (2m-2k)n\}$. The tetrahedron T is demonstrated in Figure 3, and we shall refer to this figure in our further considerations.

Put $U = \{(x, y, z) \in T; x \leq (2k+1)n\}$, $V = \{(x, y, z) \in T; (2k+1)n \leq x, z \leq (2m+1)n\}$, and $W = \{(x, y, z) \in T; (2m+1)n \leq z\}$. Then, U is the union of the triangular prism P with the bases AGH and MKL and the triangular pyramid (tetrahedron) Q with the base MKL and the apex C . Thus, one base (namely MKL)

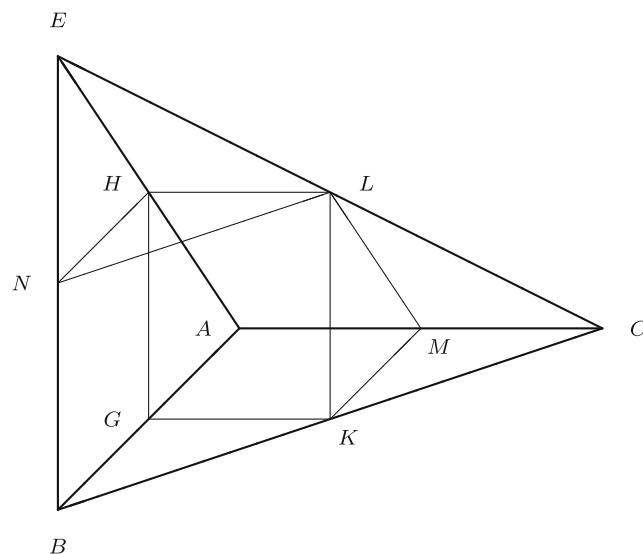


Figure 3: The n -fundamental tetrahedron T .

of P coincides with the base of Q . Clearly, V is the triangular prism with the bases GBK and HNL , and it has the property that one of its faces (namely $GKLH$) is also a face of U . Finally, W is a tetrahedron, namely $HNLE$, and one of its faces, namely HNL , is a face (base) of Q .

It may easily be seen that every point of U can be joined by a $\mathcal{B}_n^3$ -walk (consisting of at most two $\mathcal{B}_n^3$ -initial segments) with a point of the face ABE , namely with the orthogonal projection of the point on the face. Since the face ABE is connected (it is an n -fundamental triangle – see Remark 3.2), every pair of points of ABE can be joined by a $\mathcal{B}_n^3$ -walk in ABE . Therefore, every pair of points of U can be joined by a $\mathcal{B}_n^3$ -walk in T . Furthermore, every point of V may be joined by a $\mathcal{B}_n^3$ -walk (a $\mathcal{B}_n^3$ -initial segment) with a point of the face $GKLH$, namely with the orthogonal projection of the point on the face. Since $GKLH$ is a subset (face) of U , every pair of its points can be joined by a $\mathcal{B}_n^3$ -walk in T . It follows that every pair of points of V can be joined by a $\mathcal{B}_n^3$ -walk in T , and the same is true if one of the points belongs to U and the other one belongs to V . Finally, every point of W may be joined by a $\mathcal{B}_n^3$ -walk (a $\mathcal{B}_n^3$ -initial segment) with a point of the face HNL , namely with the orthogonal projection of the point on the face. Since HNL is a subset (face) of V , every pair of its points can be joined by a $\mathcal{B}_n^3$ -walk in T . Hence, every pair of points of W can be joined by a $\mathcal{B}_n^3$ -walk in T , and the same is true if one of the points belongs to V and the other one belongs to W . Consequently, every pair of points of T can be joined by a $\mathcal{B}_n^3$ -walk in T . Therefore, T is connected.

If T' is the set obtained from the n -fundamental tetrahedron T by removing some of its faces while keeping a peculiar one, then the proof of $\mathcal{B}_n^3$ -connectedness of T' is much the same. The assumption that at least one peculiar face of T must be included in T' is substantial. Indeed, if $n > 2$ and T' is obtained from T by removing all faces of T , then the set $\{(x, y, z) \in T'; z = (2m + 1)n - 1\}$ has at least two points, but none pair of points of the set may be joined by a $\mathcal{B}_n^3$ -walk contained in T' .

For any other n -fundamental tetrahedron, the proof is analogous. □

Proposition 3.4 will be used to prove the main result of this article:

Theorem 3.5. (Digital Jordan surface theorem) *Let $S \subseteq \mathbb{Z}^3$ be a (finite) polyhedral surface such that the polyhedron it bounds can be face-to-face tiled with n -fundamental tetrahedra such that each of them has the property that at most one of its peculiar faces is a subset of S . Then, the induced subgraph $\mathbb{Z}^3 - S$ of H^3 has exactly two $\mathcal{B}_n^3|(\mathbb{Z}^3 - S)$ -components such that one of them is finite, the other is infinite, and the union of any of them with S is a $\mathcal{B}_n^3$ -connected set in H^3 .*

Proof. Let S satisfy the conditions of the statement. Then, S is the union of all faces of a polyhedron $P \subseteq \mathbb{Z}^3$ that may be face-to-face tiled with finitely many n -fundamental tetrahedra such that each of them has the property that at most one of its peculiar faces is a subset of S . Then, the set $Q = (\mathbb{Z}^3 - P) \cup S$ may be face-to-face tiled with countably many n -fundamental tetrahedra such that each of them has the property that at most one of its peculiar faces is a subset of S . By Proposition 3.4, every n -fundamental tetrahedron is $\mathcal{B}_n^3$ -connected and so is the subset of $\mathbb{Z}^3$ obtained from the tetrahedron by removing some of its faces while keeping at least one peculiar face. Thus, any of the sets P , Q , $P - S$, $Q - S$ is the union of a sequence of $\mathcal{B}_n^3$ -connected subsets of $\mathbb{Z}^3$ such that every term (excluding the first one) of the sequence has a nonempty meet with some of its predecessors. In the cases of P and Q , such a sequence is given by n -fundamental tetrahedra, and in the cases of $P - S$ and $Q - S$, such a sequence consists of the subsets of $\mathbb{Z}^3$ obtained from n -fundamental tetrahedra by removing those of their faces that are subsets of S . Therefore, P , $P - S$, Q , and $Q - S$ are $\mathcal{B}_n^3$ -connected by Lemma 2.4..

The rest of the proof is similar to that of the proof of Theorem 3.5 in [10]: Since every $\mathcal{B}_n^3$ -walk $C = (z_i | i \leq k)$, $k > 0$ an integer, joining a point of $P - S$ with a point of $Q - S$ clearly meets S (i.e., meets an n -fundamental tetrahedral face contained in S), the set $\mathbb{Z}^3 - S = (P - S) \cup (Q - S)$ is not $\mathcal{B}_n^3$ -connected. Therefore, $P - S$ and $Q - S$ are $\mathcal{B}_n^3$ -components of the induced subgraph $\mathbb{Z}^3 - S$ of H^3 , $P - S$ finite and $Q - S$ infinite, and P and Q are $\mathcal{B}_n^3$ -connected. □

Remark 3.6. In [10], a digital Jordan surface theorem is proved for the polyhedral surfaces bounding a polyhedron that can be face-to-face tiled with 2-fundamental prisms. Thus, Theorem 3.5 provides

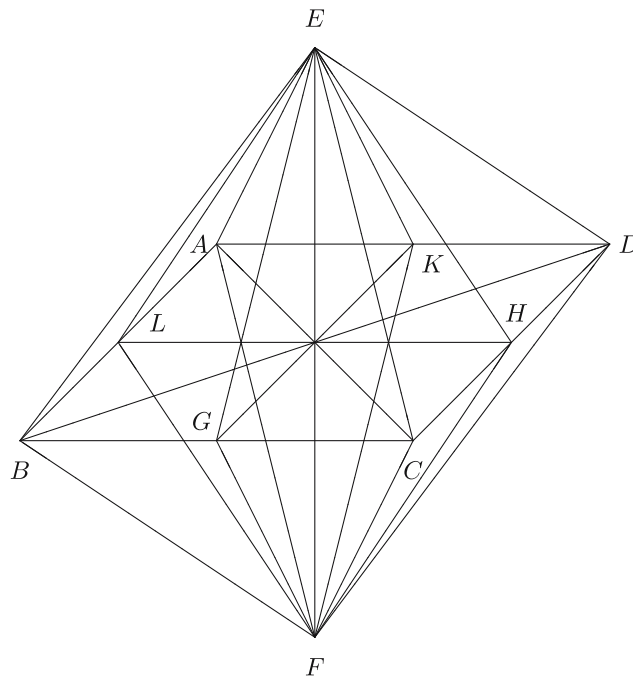


Figure 4: A right square bipyramid (octahedron) in $\mathbb{Z}^3$ with the base $ABCD$ and the apexes E and F tiled with 16-fundamental tetrahedra, where $A = (2kn, 2ln, 2mn)$, $B = (2kn + 4n, 2ln, 2mn)$, $C = (2kn + 4n, 2ln + 4n, 2mn)$, $D = (2kn, 2ln + 4n, 2mn)$, $E = (2kn + 2n, 2ln + 2n, 2mn + 2n)$, and $F = (2kn + 2n, 2ln + 2n, 2mn - 2n)$ $k, l, m \in \mathbb{Z}$.

a substantial generalization (improvement) of the digital Jordan surface theorem in [10] because employing n -fundamental tetrahedra results in a variety of Jordan surfaces that is much larger than the one resulting from using 2-fundamental prisms only. The following example demonstrates a digital Jordan surface in the sense of Theorem 3.4, which is not a Jordan surface in the sense of [10].

Example 3.7. In Figure 4, a digital surface in $\mathbb{Z}^3$ is displayed, which is a boundary of a digital right square bipyramid. The surface satisfies the conditions of Theorem 3.5, thus it is a digital Jordan surface with respect to $\mathcal{B}_n^3$ -connectedness. The same is true for the boundary surface of each of the two pyramids $ABCDE$ and $ABCDF$, as well as the boundary surface of each of the four heptahedra obtained by cutting the bipyramid by the digital plane $GFKE$ or $LFHE$, but none of these six surfaces is a Jordan surface in the sense of [10].

4 Conclusion

We have proposed (countably many) connectedness structures for the digital space $\mathbb{Z}^3$, namely the path sets $\mathcal{B}_n^3$ ($n > 1$ an integer), such that each of them allows for a definition of a digital Jordan surface (Theorem 3.5). The digital Jordan surfaces consist (i.e., are the union) of digital triangles such that any two of them are disjoint or only share one vertex or one full edge. The digital Jordan surfaces may be used to digitize borders of objects in 3D digital images, hence connected surfaces in $\mathbb{R}^3$ (satisfying the 3D Jordan-Brouwer separation theorem – cf. [22]). The advantage of these digital Jordan surfaces over those with respect to the Khalimsky topology is that the former may possess acute dihedral angles $\frac{\pi}{4}$, while the latter may never possess a dihedral angle less than $\frac{\pi}{2}$. Thus, the path sets $\mathcal{B}_n^3$ ($n > 1$ an integer) equip the digital space $\mathbb{Z}^3$ with connectedness structures convenient for the study of 3D digital images because they give a larger variety of digital Jordan surfaces than the Khalimsky topology. The digital prisms used to define Jordan surfaces in [10] can be face-to-face tiled with

the tetrahedra employed in Theorem 3.5. Therefore, Theorem 3.5 provides digital Jordan surfaces that are “finer,” hence more diversified, than those proposed in [10].

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