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$7x \pm 1$: Close Relative of the Collatz Problem

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Abstract: We show an iterated function of which iterates oscillate wildly and grow at a dizzying pace. We conjecture that the orbit of arbitrary positive integer always returns to 1, as in the case of the Collatz function. The conjecture is supported by a heuristic argument and computational results.

Key words: number theory, Collatz problem

I. Introduction

It is conjectured that, for arbitrary positive integer n, a sequence defined by repeatedly applying the function

$$f(n) = \begin{cases} 3n+1 & : \quad \text{if } n \equiv 1 \pmod 2, \\ n/2 & : \quad \text{if } n \equiv 0 \pmod 2 \end{cases} \tag{1}$$

will always converge to the cycle passing through 1. The odd terms of such sequence typically rise and fall repeatedly. The conjecture has never been proven. The problem is known under several different names, including the Collatz problem, $3x + 1$ problem, Syracuse problem, and many others. There is an extensive literature, [1, 2], on this question.

Its close relative is

$$f(n) = \begin{cases} 7n+1 & : \quad \text{if } n \equiv +1 \pmod 4, \\ 7n-1 & : \quad \text{if } n \equiv -1 \pmod 4, \\ n/2 & : \quad \text{if } n \equiv 0 \pmod 2, \end{cases} \tag{2}$$

which also always converges to the cycle passing through 1 when iteratively applied on arbitrary positive integer n. Also here, the odd terms typically rise and fall repeatedly. It is one of many possible generalizations of the $3x+1$ problem. However, unlike others, this one shares incredibly many similarities with the original conjecture.

The main goal of the paper is to present a new problem similar to the Collatz problem and a new conjecture similar to the Collatz conjecture.

II. Heuristic Argument

To prove that such sequences always return to 1, it should be shown that these sequences could never repeat the same number twice and they cannot grow indefinitely. Although the $3x + 1$ conjecture has not been proven, there is a heuristic argument, [3–5], that suggests the sequence should decrease over time. A similar heuristic argument can be used for $7x \pm 1$ problem. The argument is as follows. If n is odd, then $f(n) = 7n \pm 1$ is divisible by 4; thus two iterations of $f(n) = n/2$ must follow. Conversely, when n is even, then $f(n) = n/2$ follows. Furthermore, one can verify that if the input n is uniformly distributed modulo 2^{l+2}, then the output of the two branches above is uniformly distributed modulo 2^l, for an integer $l \geq 0$. All branches of the subsequent iteration therefore occur with equal probability. Now, if the input n is odd, the output of the former branch should be roughly $7/4$ times as large as the input n. Similarly, if the input n is even, the output of the latter branch is $1/2$ times as large as n. If we express the magnitude of n logarithmically, we get expected growth from the input n to the output of the branches above

$$\frac{1}{2}\log\frac{7}{4}+\frac{1}{2}\log\frac{1}{2}<0.$$

Since the growth is negative, the heuristic argument suggests that the magnitude tend to decrease over a long time period.

III. Known Cycles

On positive integers, sequences defined by both the $3x+1$ and the $7x\pm 1$ functions eventually enter a repeating cycle $1\to\cdots\to 1$. When zero is included, there is another cycle $0\to 0$ which, however, cannot be entered from outside. When the $3x+1$ is extended to negative integers, the sequence enters one of a total of three known negative cycles. These are $-1\to\cdots\to -1$, $-5\to\cdots\to -5$, and $-17\to\cdots\to -17$. Nevertheless, when the $7x\pm 1$ is extended to negative integers, the sequence will always converge to the cycle passing through -1. These cycles are listed in Tabs. 1 and 2. In contrast to the $3x+1$ problem, every progression in $7x\pm 1$ on negative numbers corresponds to negated progression on positive numbers, and vice versa.

Tab. 1. $3x+1$ problem. Known cycles. Only odd terms due to limited space

cycle	length
$-17\to -25\to -37\to -55\to -41\to -61\to$ $\to -91\to -17$	18
$-5\to -7\to -5$	5
$-1\to -1$	2
$+1\to +1$	3

Tab. 2. $7x\pm 1$ problem. Known cycles. Only odd terms due to limited space

cycle	length
$-1\to -1$	4
$+1\to +1$	4

IV. Experimental Evidence

For instance, the $7x\pm 1$ sequence for starting value $n=235$ is listed in Tab. 3. It takes 244 steps to reach the number 1 from 235. This is also known as the total stopping time. The highest value reached during the progression is 428 688. For a better mental picture of this sequence, the progression is also graphed in Fig. 1. The odd terms can be recognized as local minima, whereas the even terms as either local maxima or descending lines. One can easily see that the odd terms rise and fall repeatedly. Such behavior is also common to $3x+1$ sequences.

Tab. 3. $7x\pm 1$ sequence starting at 235. Steps through odd numbers in bold

235, 1644, 822, **411**, 2876, 1438, **719**, 5032, 2516, 1258, **629**, 4404, 2202, **1101**, 7708, 3854, **1927**, 13488, 6744, 3372, 1686, **843**, 5900, 2950, **1475**, 10324, 5162, **2581**, 18068, 9034, **4517**, 31620, 15810, **7905**, 55336, 27668, 13834, **6917**, 48420, 24210, **12105**, 84736, 42368, 21184, 10592, 5296, 2648, 1324, 662, **331**, 2316, 1158, **579**, 4052, 2026, **1013**, 7092, 3546, **1773**, 12412, 6206, **3103**, 21720, 10860, 5430, **2715**, 19004, 9502, **4751**, 33256, 16628, 8314, **4157**, 29100, 14550, **7275**, 50924, 25462, **12731**, 89116, 44558, **22279**, 155952, 77976, 38988, 19494, **9747**, 68228, 34114, **17057**, 119400, 59700, 29850, **14925**, 104476, 52238, **26119**, 182832, 91416, 45708, 22854, **11427**, 79988, 39994, **19997**, 139980, 69990, **34995**, 244964, 122482, **61241**, 428688, 214344, 107172, 53586, **26793**, 187552, 93776, 46888, 23444, 11722, **5861**, 41028, 20514, **10257**, 71800, 35900, 17950, **8975**, 62824, 31412, 15706, **7853**, 54972, 27486, **13743**, 96200, 48100, 24050, **12025**, 84176, 42088, 21044, 10522, **5261**, 36828, 18414, **9207**, 64448, 32224, 16112, 8056, 4028, 2014, **1007**, 7048, 3524, 1762, **881**, 6168, 3084, 1542, **771**, 5396, 2698, **1349**, 9444, 4722, **2361**, 16528, 8264, 4132, 2066, **1033**, 7232, 3616, 1808, 904, 452, 226, **113**, 792, 396, 198, **99**, 692, 346, **173**, 1212, 606, **303**, 2120, 1060, 530, **265**, 1856, 928, 464, 232, 116, 58, **29**, 204, 102, **51**, 356, 178, **89**, 624, 312, 156, 78, **39**, 272, 136, 68, 34, **17**, 120, 60, 30, **15**, 104, 52, 26, **13**, 92, 46, **23**, 160, 80, 40, 20, 10, **5**, 36, 18, **9**, 64, 32, 16, 8, 4, 2, **1**

The progression lengths for both the $3x+1$ and the $7x\pm 1$ problems are shown in Fig. 2. Regarding the successive n, the behavior of total stopping time is obviously irregular. Despite this, we can see regular patterns in graphs of these times for both of the problems. Consecutive starting values tend to reach the same total stopping time.

In order to compare the behavior of the $3x+1$ and $7x\pm 1$ sequences, consider the following tables. Tabs. 4 and 5 show the longest progression (total stopping time) for any starting number less than the given limit. One can see that the $3x+1$ sequences tend to have recognizably longer progressions. Moreover, Tabs. 6 and 7 show that the maximum value reached during a progression for any starting number below the given limit. This value grows significantly faster in the $7x\pm 1$ problem that in the $3x+1$ case.

A lot of generalizations, e.g., [4–8], of the original Collatz problem can be found in the literature. In [5], the author also mentions the $7x+1$ problem. The definition of such a problem is, however, different from the definition discussed in this paper. To the best of my knowledge, the $7x\pm 1$ function studied in this paper has never appeared before. I have computationally verified the convergence of the $7x\pm 1$ problem for all numbers up to 10^{15}.

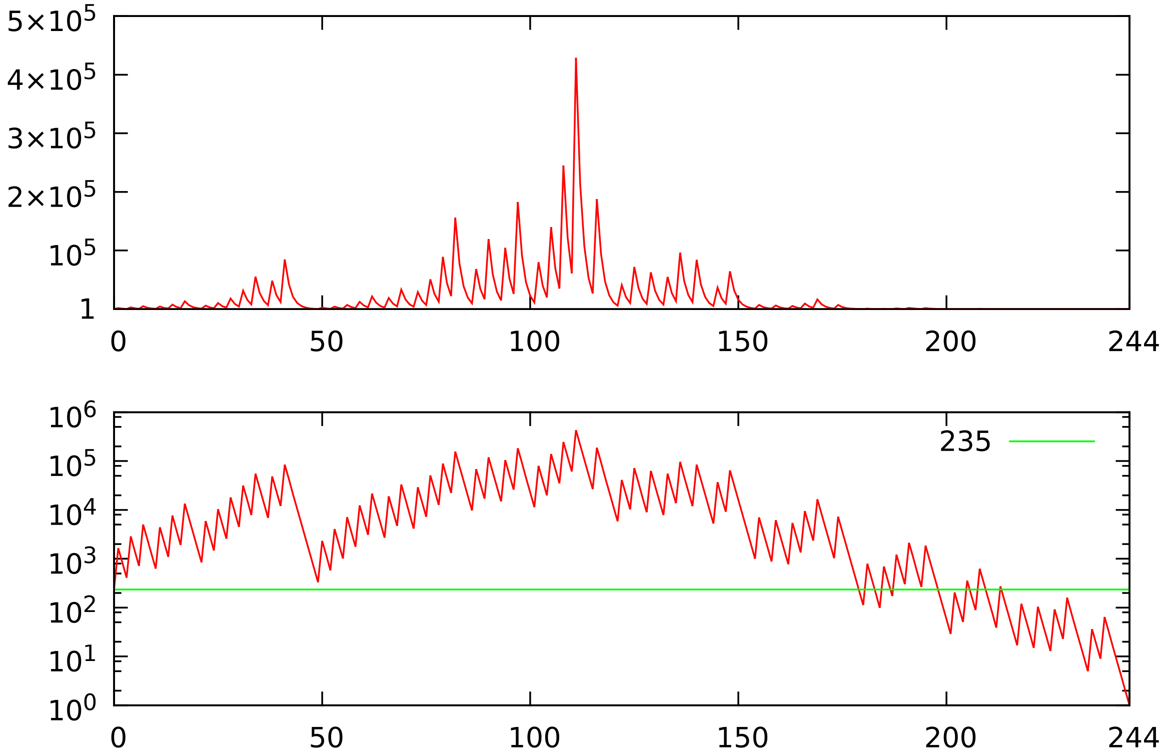

Fig. 1. $7x \pm 1$ sequence starting at 235. Due to a very large number range, the sequence in the linear scale is shown at the top, and in the logarithmic scale at the bottom

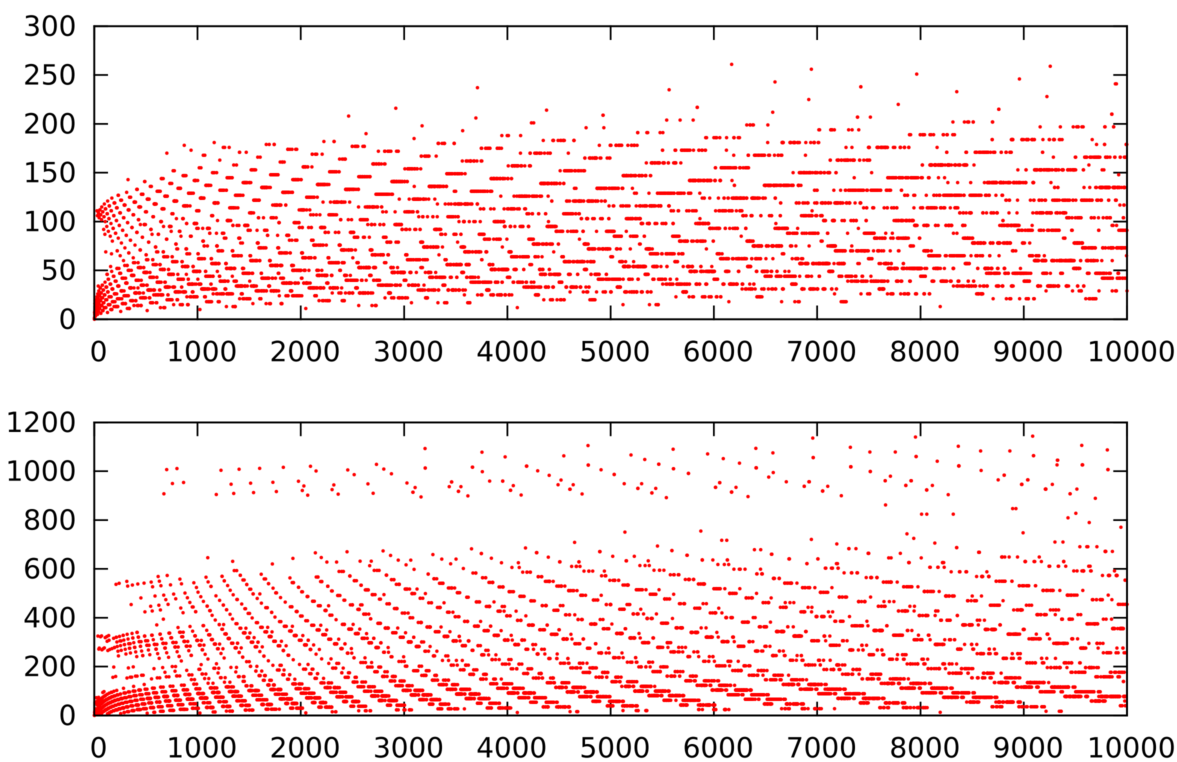

Fig. 2. Numbers 1 to 10 000 and their total stopping time. The $3x + 1$ at the top, the $7x \pm 1$ at the bottom

Tab. 4. $3x + 1$ problem. Longest progression for values less than the given value

below	peak steps	start value
10^1	19	9
10^2	118	97
10^3	178	871
10^4	261	6 171
10^5	350	77 031
10^6	524	837 799
10^7	685	8 400 511
10^8	949	63 728 127
10^9	986	670 617 279
10^{10}	1 132	9 780 657 630

Tab. 5. $7x \pm 1$ problem. Longest progression for values less than the given value

below	peak steps	start value
10^1	18	7
10^2	326	70
10^3	1 011	801
10^4	1 144	9 087
10^5	1 551	98 003
10^6	2 799	775 533
10^7	3 480	7 632 037
10^8	5 025	61 475 411
10^9	5 444	983 358 845
10^{10}	5 717	6 346 893 259

Tab. 6. $3x + 1$ problem. Maximum value reached in progressions

below	peak value	start value
10^1	52	7
10^2	9 232	27
10^3	250 504	703
10^4	27 114 424	9 663
10^5	1 570 824 736	77 671
10^6	56 991 483 520	704 511
10^7	60 342 610 919 632	6 631 675
10^8	2 185 143 829 170 100	80 049 391
10^9	1 414 236 446 719 942 480	319 804 831
10^{10}	18 144 594 937 356 598 024	8 528 817 511

Tab. 7. $7x \pm 1$ problem. Maximum value reached in progressions

below	peak value	start value
10^1	64	3
10^2	428 688	35
10^3	20 492 891 264	701
10^4	34 462 899 848	8 317
10^5	965 557 666 410 854 560	56 925
10^6	16 785 854 261 378 324 480	199 093
10^7	387 911 901 837 284 812 874 137 728	4 351 011
10^8	432 862 432 624 267 939 703 128 640 368	98 600 229
10^9	1 278 593 034 093 037 189 798 609 704 765 568	662 844 973
10^{10}	421 614 662 439 923 712 249 655 593 962 998 304	9 725 365 821

V. Final Remarks

- The paper presented a conjecture that the orbit of arbitrary positive integer always returns to 1 under the $7x \pm 1$ function.
- Although the conjecture has not been proven, there is a heuristic argument that suggests the sequence should decrease over time.

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David Barina received the PhD degree at the Faculty of Information Technology, Brno University of Technology, Czech Republic. He is currently a member of the Graph@FIT group at the Department of Computer Graphics and Multimedia at FIT, Brno University of Technology. His research interests include wavelets and fast algorithms in signal and image processing.