Linear System Control Using High Order Method

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Abstract. The article deals with a control of the linear system using the high order integration method. The chosen system (inverted pendulum on a cart) is analyzed and a proper controller for such system is designed. The controller that is using the high order method is compared with the controller that uses the state of the art numerical methods. All experiments are performed using MATLAB software.

INTRODUCTION

The design and implementation of the control systems represents the crucial area for todays industry. All modern machinery (cars, refrigerators, airplanes, . . .) uses some sort of control system. The article deals with the numerical integration that is performed during control task. The presented Modern Taylor Series Method (MTSM) is used due to the established favorable properties.

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

The numerical solution of ordinary differential equations (ODEs) can utilize many different methods and approaches. The commonly used explicit methods include Euler and Runge-Kutta fixed order methods [1]. However, these methods need to keep the integration step quite small to maintain accuracy and stability. This can be overcome by using variable-step, variable-order (VSVO) scheme of such numerical method. The best-known and the most accurate method of calculating a new value of the numerical solution of ordinary differential equation [2]

$$y' = f(t, y) \qquad y(t_0) = y_0, \tag{1}$$

is to construct the Taylor series in the form

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2!} f'(t_i, y_i) + \ldots + \frac{h^n}{n!} f^{[n-1]}(t_i, y_i), \tag{2}$$

where h is the integration step, $y_i \doteq y(t_i)$ is the previous approximate value and $y_{i+1} \doteq y(t_i + h)$ is the next approximate value of the function $y(t)$.

Taylor series is not widely used to solve initial value problems (IVPs) due to the generally unknown higher derivatives of a arbitrary function. However, the derivatives of many technical problems can be calculated recursively [3]. Linear systems of ODEs could be written in the form

$$\boldsymbol{y}' = \mathbf{A}\boldsymbol{y} + \mathbf{b} \tag{3}$$

and then the Taylor series expansion follows

$$\boldsymbol{y}_{i+1} = \boldsymbol{y}_i + h(\mathbf{A}\boldsymbol{y}_i + \mathbf{b}) + \frac{h^2}{2!}\mathbf{A}(\mathbf{A}\boldsymbol{y}_i + \mathbf{b}) + \cdots + \frac{h^n}{n!}\mathbf{A}^{(n-1)}(\mathbf{A}\boldsymbol{y}_i + \mathbf{b}) \tag{4}$$

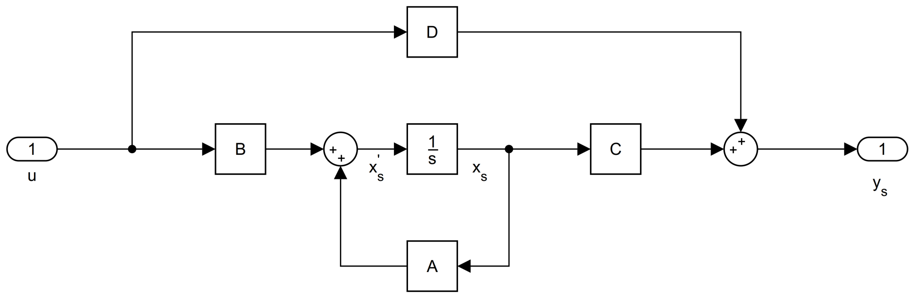

FIGURE 1. State-space representation diagram

where **A** is the constant Jacobian matrix and **b** is the constant right-hand side. Equation (4) can be rewritten

$$\boldsymbol{y}_{i+1} = \boldsymbol{DY}_0 + \boldsymbol{DY}_1 + \boldsymbol{DY}_2 + \cdots + \boldsymbol{DY}_n, \tag{5}$$

where Taylor series terms could be computed recurrently

$$\boldsymbol{DY}_0 = \boldsymbol{y}_i\,, \quad \boldsymbol{DY}_1 = h(\mathbf{A}\boldsymbol{y}_i + \mathbf{b}), \qquad \boldsymbol{DY}_l = \frac{h}{l}\mathbf{A}\boldsymbol{DY}_{l-1}\,, \quad l = 2, \ldots, n. \tag{6}$$

STATE-SPACE REPRESENTATION OF THE CONTROL SYSTEM

In this section, the state-space representation of the control system is discussed. This representation was chosen, because it describes the system accurately using matrices and vectors, that can be used directly in MATLAB. Different representations are also possible (for example, the PI regulator can be represented directly as a differential equation [4], that can be solved using MTSM), the state space representation is more complex look at a system and the desired control. Furthermore, the controllability and observability of the system [5] can be determined. Generally, the state space representation of the system is depicted in the Figure 1 where the vector $\boldsymbol{x}_s$ is the state of the system, **A** is the constant system matrix, **b** is the constant input vector, **C** is the constant output matrix, and **D** is a constant matrix [5]. Matrix **A** and the vector **b** correspond to (3). The system represented in state space can be expressed using

$$\begin{aligned} \boldsymbol{x}'_s &= \mathbf{A}\boldsymbol{x}_s + \mathbf{bu}\,, \\ \boldsymbol{y}_s &= \mathbf{C}\boldsymbol{x}_s + \mathbf{Du}\,, \end{aligned} \tag{7}$$

where $\mathbf{x}_s$ is a state vector and **u** is an input (or control) vector. To effectively control the system defined by equations (7), the derivative has to be amended with:

- constant vector **k**, that describes the required behavior of the system,
- constant vector **r**, that defines the control objective (for example the position or speed that we want to achieve).

The derivative of $\boldsymbol{x}_s$ can therefore be rewritten as

$$\boldsymbol{x}'_s = (\mathbf{A} - \mathbf{bk}^T)\boldsymbol{x}_s + \mathbf{bk}^T\mathbf{r}\,. \tag{8}$$

Matrix **A** and vector **b** are defined below (13).

PENDULUM ON A CART

Pendulum on a cart can be visualized using Figure 2, where M is the mass of the cart, m is the mass of the pendulum, F is amplitude of the force that causes the cart-pendulum system to move, l is the length of the pendulum and θ is the deviation angle of the pendulum. The equations of the system can be derived using Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q'_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{9}$$

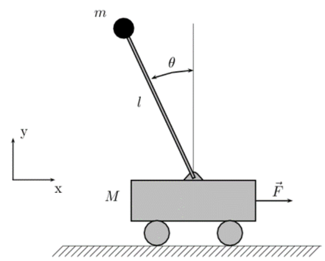

FIGURE 2. Inverted pendulum

where q_i is the unknown function, q_i' is its time derivative and Lagrangian L summarizes the dynamics of the entire system. The Lagrangean can be calculated as a difference between kinetic (K) and potential (P) energy of the system

$$P = mgl\cos\theta \tag{10}$$

$$K = \frac{1}{2}(M+m)x'^2 + \frac{1}{2}m(-2x'l\theta'\cos\theta + l^2\theta'^2(\cos^2\theta + \sin^2\theta)) \tag{11}$$

Substituting L into (9) and solving for $q_i = x$ and $q_i = \theta$ yields the equations of motion, that are used during the calculation

$$\begin{aligned} (M+m)x'' + ml\theta''\cos\theta + ml\theta'^2\sin\theta &= F \\ l\theta'' - x''\cos\theta - g\sin\theta &= 0\,. \end{aligned} \tag{12}$$

To define the state-space representation of the control system, the state variables were defined as $\boldsymbol{x}_s = (\theta, \theta', x, x')^T$. The system is linearized around 0 to obtain the matrix $\mathbf{A}$ that describes the behavior of the linearized system and input vector $\mathbf{b}$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{M+m}{Ml}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M}g & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ -\frac{1}{lM} \\ 0 \\ \frac{1}{M} \end{pmatrix}. \tag{13}$$

The matrix $\mathbf{A}$ and the vector $\mathbf{b}$ can be substituted into (3).

NUMERICAL EXPERIMENTS

This section of the paper contains the numerical experiments performed using (13) in MATLAB [6]. Initial conditions for the differential equations were set

$$x_{s1} = 0.1\,, \qquad x_{s2} = 0\,, \qquad x_{s3} = 0.1\,, \qquad x_{s4} = 0\,, \tag{14}$$

to set the initial position of the pendulum (x_{s1}) and initial position of the cart (x_{s3}). The control objective for this experiment was to keep the pendulum upright ($x_{s1} = 0$) and move the cart into position ($x_{s3} = 1$). To achieve this objective, the vector $\mathbf{r}$ is equal to $\mathbf{r}_{Tmax} = (0, 0, 1, 0)^T$.

Before the experiment, the eigenvalues of the matrix $\mathbf{A}$ were calculated to check, if the system is stable without control. Due to the fact, that one of the eigenvalues was positive, the controller is needed to control the system. The controllability [5] of the system was confirmed by constructing the controllability matrix and checking its rank against the number of state variables.

The controller design was then performed by placing the new eigenvalues for the system so that they are all negative. Vector of eigenvalues $\mathbf{p} = (-0.8, -0.7, -0.6, -0.5)^T$ was used. Using the vector $\mathbf{p}$, the vector $\mathbf{k}$ can be calculated using the `place` command in MATLAB. The results are in the Figure 3, which shows the behavior of the system and the used order of the Modern Taylor Series Method.

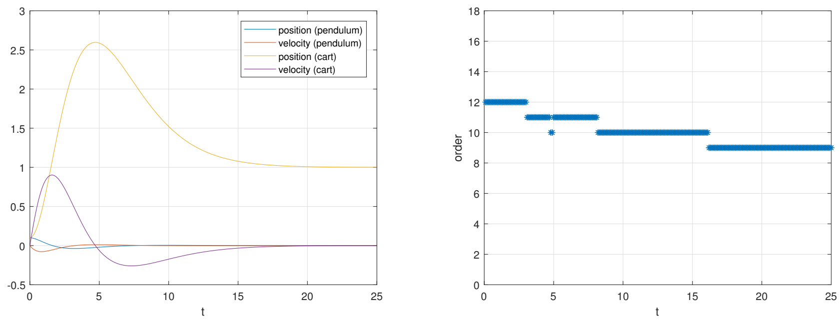

FIGURE 3. The behavior of the controlled system (left), number of used Taylor series terms (right)

The figure confirms that the control system achieved the set objective $\mathbf{r}_{T_{max}}$ at maximum time $T_{max} = 25$s. The more aggressive placement of the poles (vector **p**) would speed up the controller response. The accuracy of all solvers was set to 10^{-13} and the step size $h = 0.1$.

Numerical results are summed up in Table 1. The MTSM performed the control tasks faster than the ode solvers present in MATLAB (mean of calculation time from 100 runs).

TABLE 1. Solvers comparison

Solver	Time [s]	Order	Steps
MTSM	0.00476482	9 – 12	250
ode45	0.04452096	4 – 5	4305
ode23	0.80037904	2 – 3	49432
ode113	0.0065460	1 –12	227

Due to the fact, that the MTSM can use variable order, we can increase the step size and speed up the calculation further with the same accuracy e.g. for the integration step size $h = 1$ the order is increased to 15 – 18, while the speed of the calculation decreased tenfold.

CONCLUSION

This article presented a high order numerical method (MTSM) and used it to solve the set control problem. The method was compared to the state of the art and seems to have favorable properties for control systems. The solution of the non-linear variant of the presented problem is the objective of a future research.

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