Filomat 32:14 (2018), 5011–5021
https://doi.org/10.2298/FIL1814011S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Closure Operators on Graphs for Modeling Connectedness in Digital Spaces

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Abstract. For undirected simple graphs, we introduce closure operators on their vertex sets induced by sets of walks of the same lengths. Some basic properties of these closure operators are studied, with greater attention paid to connectedness. We focus on the closure operators induced by certain sets of walks in the 2-adjacency graph on the digital line $\mathbb{Z}$, which generalize the Khalimsky topology. For the closure operators on $\mathbb{Z}^2$ obtained as particularly defined products of pairs of the induced closure operators on $\mathbb{Z}$, we formulate and prove a digital form of the Jordan curve theorem.

1. Introduction

In our increasingly digital and, thus, discrete world, graph theory plays an important role in many applications we meet in our everyday life. In particular, the theory supplies effective methods for solving problems of discrete geometry. And discrete geometry, especially its digital topology branch ([9, 13]), is used in computer science for solving problems of digital image processing.

The concept of a digital space usually refers to a digital image background such as a computer screen. Since, in this paper, the digital cells (pixels, voxels, etc.) are represented by their midpoints, the m-dimensional digital space ($m > 0$ an integer) is understood to be the set $\mathbb{Z}^m$. The classical approach to digital topology is based on using adjacency relations to define connectedness in the digital space $\mathbb{Z}^m$. For instance, the 4- and 8-adjacencies are used on the digital plane $\mathbb{Z}^2$ while the 6-, 18-, and 26-adjacencies are used on the digital space $\mathbb{Z}^3$ (cf. [8]). It is a drawback of the classical approach that the connectedness given by an adjacency relation on $\mathbb{Z}^m$ does not have properties analogous to the connectedness in the Euclidean space $\mathbb{R}^m$. In particular, on the digital plane $\mathbb{Z}^2$, neither 4-adjacency nor 8-adjacency itself allows for an analogue of the Jordan curve theorem. (Recall that the classical Jordan curve theorem states that a simple closed curve separates the Euclidean plane into precisely two connected components). Such an analogue is obtained only when employing a combination of the two adjacencies - see [12]. Despite this inconvenience, the classical approach building on the combination of the 4- and 8-adjacencies was used for resolving numerous problems of digital image processing (see, e.g., [1]) and creating the bulk of the graphical software.

2010 *Mathematics Subject Classification*. Primary 54A05; Secondary 05C40, 52C99
Keywords. Simple grap, walk, closure operator, digital space, Khalimsky topology, Jordan curve theorem
Received: 05 March 2018; Revised: 22 May 2018; Accepted: 25 May 2018
Communicated by Ljubiša D.R. Kočinac
This work was supported by the Ministry of Education, Youth and Sports of the Czech Republic from the National Programme of Sustainability (NPU II) project "IT4Innovations excellence in science - LQ1602".
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In 1990, Khalimsky, Kopperman and Meyer [7] proposed a new approach to digital topology based on using a topology on the digital space $\mathbb{Z}^m$ to obtain a connectedness in the space. They employed the topology called Khalimsky and showed that, for $m = 2$, it allows for a digital analogue of the Jordan curve theorem thus providing a convenient model of the Euclidean topology on the real plane for the study of digital images. For $m = 3$, the Khalimsky topology was shown to satisfy a digital form of the Jordan-Brouwer separation theorem (i.e., a natural extension of the Jordan curve theorem to the three-dimensional space) in [11]. The topological approach was and still is developed by many authors - see, for instance, [5] and [16].

In [15], it was shown that even closure operators, i.e., structures more general that topologies, may be used to advantage for structuring the digital spaces. Closure operators occur in many branches of mathematics and are utilized in numerous applications, particularly in computer science. In the present paper, we also deal with certain closure operators and will show that they may be used as background connectedness structures on the digital spaces for studying digital images. The closure operators employed are obtained from topologies, i.e., Kuratowski closure operators, by omitting the axioms of idempotency and additivity (but retaining the axiom of monotony).

Each of the two approaches, classical and topological, has its advantages and it may, therefore, be beneficiary to use a combination of them. Being motivated by this idea, we deal with closure operators on graphs. More precisely, we discuss closure operators on (the vertex sets of) graphs that are induced by sets of walks of the same lengths. These closure operators are studied from the viewpoint of applications of the results obtained in digital topology for structuring the digital spaces. We will, therefore, focus on the connectedness provided by the closure operators. It will be shown that a particularly defined product of the operators preserves the basic properties of the connectedness. Thus, having a walk-set induced closure operator on the digital line $\mathbb{Z}$, the connectedness-related behavior of the closure operator is also preserved in the corresponding product closure operator on the digital space $\mathbb{Z}^m$. We will build on this fact. For any integers $m, n > 0$, we consider the graph on the digital space $\mathbb{Z}^m$ obtained as the strong product of m copies of the 2-adjacency graph on $\mathbb{Z}$ and study the closure operator on $\mathbb{Z}^m$ induced by a certain set of walks of length n. If $n = 1$, we obtain the Khalimsky m-dimensional space. For $m = 2$ and $n > 1$, we prove analogues of the Jordan curve theorem for these closure operators by determining the discrete circles that separate the digital plane into precisely two (connected) components. The discrete circles (simple closed curves) employed modify the digital curves in the sense of Rosenfeld [12].

The idea of employing graphs with sets of paths for modeling connectedness in digital spaces was firstly used in [17] where special sets of paths, the path partitions, were introduced and studied. In [19], closure operators on (the vertex sets of) graphs induced by path partitions were dealt with and, as applications of the results attained, closure operators providing convenient background connectedness structures on the digital spaces were discussed. Path partitions from [17] and [19] were generalized and studied in [18] where closure operators on graphs induced by sets of walks of the same lengths were studied and some applications in digital topology were indicated. In the present paper, we focus on the study of connectedness with respect to the closure operators on graphs introduced in [18]. Considering such a connectedness on digital spaces, we prove a digital analogue of the Jordan curve theorem for the digital plane $\mathbb{Z}^2$.

2. Preliminaries

We will work with finite sequences $(x_0, x_1, ..., x_n)$, n a natural number (i.e., a finite ordinal), and these will be written as $(x_i \mid i \leq n)$ or $(x_i \mid i < n + 1)$. For the graph-theoretic terminology used, we refer to [6]. By a *graph* $G = (V, E)$, we understand an (undirected simple) graph (without loops) with $V \neq \emptyset$ the *vertex* set and $E \subseteq \{\{x, y\};\ x, y \in V,\ x \neq y\}$ the set of *edges*. We will say that G is a graph *on* V. Two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$. Recall that a *walk* in G is a (finite) sequence $(x_n \mid i \leq n)$ of vertices of V such that x_i is adjacent to x_{i+1} whenever $i < n$ (note that walks are called paths in [18]) . The natural number n is called the *length* of the walk $(x_n \mid i \leq n)$. A walk $(x_n \mid i \leq n)$ in G is called a *path* if $x_i \neq x_j$ whenever $i, j \leq n, i \neq j$, and it is called a *circle* if $x_i \neq x_j$ whenever $i, j < n, i \neq j$, and $x_0 = x_n$. We will often apply set-theoretic operations to walks regarded as sets (a walk $(x_n \mid i \leq n)$ will be regarded as the set $\{x_n;\ i \leq n\}$).

Recall that, given graphs $G_1 = (V_1, E_1,)$ and $G_2 = (V_2, E_2)$, we say that G_1 is a *subgraph* of G_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. If, moreover, $V_1 = V_2$, then G_1 is called a *factor* of G_2.

Given graphs $G_j = (V_j, E_j,)$, $j = 1, 2, ..., m$ ($m > 0$ a natural number), we define their *strong product* to be the graph $\prod_{j=1}^m G_j = (\prod_{j=1}^m V_j, E)$ with the set of edges $E = \{\{(x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m)\}$; there exists a nonempty subset $J \subseteq \{1, 2, ..., m\}$ such that $\{x_j, y_j\} \in E_j$ for every $j \in J$ and $x_j = y_j$ for every $j \in \{1, 2, ..., m\} - J\}$. Note that the strong product differs from the cartesian product of G_j, $j = 1, 2, ..., m$, i.e., from the graph $(\prod_{j=1}^m V_j, F)$ where $F = \{\{(x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m)\};\ \{x_j, y_j\} \in E_j$ for every $j \in \{1, 2, ..., m\}\}$. Indeed, the cartesian product is a factor of the strong product and we have $F \subsetneq E$ whenever $m > 1$. The strong product of a pair of graphs coincides with that introduced in [14].

By a *closure operator* u on a set X, we mean a map u: $\exp X \to \exp X$ (where $\exp X$ denotes the power set of X) which is

(i) grounded (i.e., $u\emptyset = \emptyset$),

(ii) extensive (i.e., $A \subseteq X \Rightarrow A \subseteq uA$), and

(iii) monotone (i.e., $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$).

The pair (X, u) is then called a *closure space*. Closure spaces were studied by Čech in [2] (where they are called topological spaces).

A closure operator u on X that is

(iv) additive (i.e., $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq X$) and

(v) idempotent (i.e., $uuA = uA$ whenever $A \subseteq X$)

is called a *Kuratowski closure operator* or a *topology* and the pair (X, u) is called a *topological space*.

Given a cardinal $\alpha > 1$, a closure operator u on a set X and the closure space (X, u) are called an *S_α-closure operator* and an *S_α-closure space* (briefly, an *S_α-space*), respectively if the following condition is satisfied:

$A \subseteq X \Rightarrow uA = \bigcup\{uB; B \subseteq A, \text{card } B < \alpha\}$.

For instance, the well-known algebraic closure operators are obtained from the idempotent $S_{\aleph_0}$-closure operators by omitting the requirement of groundedness. In [3], S_2-closure operators and S_2-spaces are called *quasi-discrete*. S_2-topologies (S_2-topological spaces) are usually called *Alexandroff topologies* (*Alexandroff spaces*) - see [10]. Of course, any S_2-closure operator is additive and any S_α-closure operator is an S_β-closure operator whenever $\alpha < \beta$. Evidently, if $\alpha \leq \aleph_0$, then any additive S_α-closure operator is an S_2-closure operator.

Many concepts known for topological spaces (see, e.g., [4]) may naturally be extended to closure spaces. Let us mention some of them. Given a closure space (X, u), a subset $A \subseteq X$ is called *closed* if $uA = A$, and it is called *open* if $X - A$ is closed. A closure space (X, u) is said to be a *subspace* of a closure space (Y, v) if $uA = vA \cap X$ for each subset $A \subseteq X$. We will briefly speak about a subspace X of (Y, v). A closure space (X, u) is said to be *connected* if $\emptyset$ and X are the only subsets of X to be both closed and open. A subset $X \subseteq Y$ is connected in a closure space (Y, v) if the subspace X of (Y, v) is connected. A maximal connected subset of a closure space is called a *component* of this space. All the basic properties of connected sets and components in topological spaces are preserved also in closure spaces. We will specifically employ the fact that the union of a system of connected subsets with nonempty intersection is connected and, consequently, the union of a sequence of connected subsets is connected if every member of the sequence, excluding the first one, has a nonempty intersection with the union of all previous members.

If (X, u) is a closure space and $A \subseteq X$ a subset such that the subspace $X - A$ consists of precisely two components (say B and C), then A is said to *separate* (X, u) into precisely two components (namely, B and C).

Given closure spaces (X, u) and (Y, v), a map $\varphi : X \to Y$ is said to be a *continuous map* of (X, u) into (Y, v) if $f(uA) \subseteq vf(A)$ for each subset $A \subseteq X$. Clearly, continuous maps preserve connectedness of subsets.

For closure operators u, v on a set X, we put $u \leq v$ if $uA \subseteq vA$ for every subset $A \subseteq X$ (clearly, $\leq$ is a partial order on the set of all closure operators on X).

Given a system of closure spaces (X_j, u_j), $j \in J$, the *cartesian product* of this system is the closure space $(\prod_{j \in J} X_j, v)$ where v is the closure operator on $\prod_{j \in J} X_j$ generated by the projections $pr_j : \prod_{j \in J} X_j \to X_j$, $j \in J$ (i.e., the greatest - with respect to $\leq$ - of the closure operators u on $\prod_{j \in J} X_j$ such that all projections pr_j : $(\prod_{j \in J} X_j, u) \to (X_j, u_j)$, $j \in J$, are continuous). Clearly, we have $vA = \prod_{j \in J} u_j pr_j(A)$ whenever $A \subseteq \prod_{j \in J} X_j$.

3. Graphs with Walk Sets and Induced Closure Operators

Since this paper is a continuation of the study of closure operators on graphs induced by sets of walks initiated in [18], we use some notations and concepts introduced there. To make our paper self-contained, we repeat the definitions of all of them.

In the sequel, n will denote a natural number with $n > 0$. Given a graph G, we denote by $\mathcal{W}_n(G)$ the set of all walks of length n in G. For every set of walks (walk set for short) $\mathcal{T} \subseteq \mathcal{W}_n(G)$, we put

$\mathcal{T}^{-1} = \{(x_i \mid i \leq n) \in \mathcal{W}_n(G);\ (x_{n-i} \mid i \leq n) \in \mathcal{T}\}$,

$\hat{\mathcal{T}} = \{(x_i \mid i \leq m) \in \mathcal{W}_m(G);\ 0 < m \leq n$ and there exists $(y_i \mid i \leq n) \in \mathcal{T}$ such that $x_i = y_i$ for every $i \leq m\}$ (so that $\mathcal{T} \subseteq \hat{\mathcal{T}}$), and

$\mathcal{T}^* = \hat{\mathcal{T}} \cup \hat{\mathcal{T}}^{-1}$.

The elements of $\mathcal{T}^*$ will be called *$\mathcal{T}$-initial segments* in G.

Let G be a graph and $\mathcal{T} \subseteq \mathcal{W}_n(G)$. For any subset $X \subseteq V$, we put

$u_{\mathcal{T}}X = X \cup \{x \in V;$ there exists $(x_i \mid i \leq m) \in \hat{\mathcal{T}}$ with $\{x_i;\ i < m\} \subseteq X$ and $x_m = x\}$.

Clearly, $u_{\mathcal{T}}$ is an S_{n+1}-closure operator on V. It will be said to be *induced* by $\mathcal{T}$.

Let G_j be a graph and $\mathcal{T}_j \subseteq \mathcal{W}_n(G_j)$ for every $j = 1, 2, ..., m$ ($m > 0$ a natural number). Then, we put $\prod_{j=1}^{m} \mathcal{T}_j = \{((x_i^1, x_i^2, ..., x_i^m) \mid i \leq n)$; there is a nonempty subset $J \subseteq \{1, 2, ..., m\}$ such that $(x_i^j \mid i \leq n) \in \mathcal{T}_j$ for every $j \in J$ and $(x_i^j \mid i < j)$ is a constant sequence for every $j \in \{1, 2, ..., m\} - J\}$. It is evident that $\prod_{j=1}^{m} \mathcal{T}_j \subseteq \mathcal{W}_n(\prod_{j=1}^{m} G_j)$. $\prod_{j=1}^{m} \mathcal{T}_j$ will be called the *strong product* of $\mathcal{T}_j$, $j = 1, 2, ..., m$ (it will always be clear whether a strong product discussed relates to graphs or walk sets).

Let $G_j = (V_j, E_j)$ be a graph and $\mathcal{T}_j$ a walk set in G_j for every $j = 1, 2, ..., m$ ($m > 0$ a natural number). Then we put $\prod_{j=1}^{m}(V_j, u_{\mathcal{T}_j}) = (\prod_{j=1}^{m} V_j, u_{\prod_{j=1}^{m} \mathcal{T}_j})$. We will need the following, quite obvious observation:

Remark 1. Let $(\prod_{j=1}^{m} V_j, v)$ be the cartesian product of the system of closure spaces $(V_j, u_{\mathcal{T}_j})$, $j = 1, 2, ..., m$. Then, $u_{\prod_{j=1}^{m} \mathcal{T}_j} \leq v$.

Although $u_{\mathcal{T}}$ is not additive in general, it is shown in [18] that the union of a system of closed subsets of $(V, u_{\mathcal{T}})$ is a closed subset of $(V, u_{\mathcal{T}})$. It is also shown in [18] that $u_{\mathcal{T}}$ is idempotent if and only if $(X, u_{\mathcal{T}})$ is an Alexandroff space.

Recall [18] that, given a graph $G = (V, E)$ and a walk set $\mathcal{T} \subseteq \mathcal{W}_n(G)$, a sequence $C = (x_i \mid i \leq m)$, $m > 0$, of vertices of V is called a *$\mathcal{T}$-walk* in G if there is an increasing sequence $(i_k \mid k \leq p)$ of natural numbers with $i_0 = 0$ and $i_p = m$ such that $i_k - i_{k-1} \leq n$ and $(x_i \mid i_{k-1} \leq i \leq i_k) \in \mathcal{T}^*$ for every k with $0 < k \leq p$. The sequence $(i_k \mid k \leq p)$ is said to be a *binding sequence* of C.

If the members of C are pairwise different, then C is called a *$\mathcal{T}$-path* in G.

A $\mathcal{T}$-walk C is said to be a *$\mathcal{T}$-circle* if, for every pair i_0, i_1 of different natural numbers with $i_0, i_1 \leq m$, $x_{i_0} = x_{i_1}$ is equivalent to $\{i_0, i_1\} = \{0, m\}$.

Clearly, every $\mathcal{T}$-walk ($\mathcal{T}$-path, $\mathcal{T}$-circle) in a graph $G = (V, E)$ is a walk (path, circle) in G and both concepts coincide if $\mathcal{T} = \{(x, y);\ \{x, y\} \in E\} \subseteq \mathcal{W}_1(G)$.

Observe that, if $(x_0, x_1, ..., x_m)$ is a $\mathcal{T}$-walk in G, then $(x_m, x_{m-1}, ..., x_0)$ is a $\mathcal{T}$-walk in G, too. Further, if $C_1 = (x_i \mid i \leq m)$ and $C_2 = (y_i \mid i \leq q)$ are $\mathcal{T}$-walks in G such that $x_{m-1} = y_0$, then, putting $z_i = x_i$ for all $i \leq m$ and $z_i = y_{i-m}$ for all i with $m < i \leq m + q$, we get a $\mathcal{T}$-walk $(z_i \mid i \leq m + q)$ in G. We denote the $\mathcal{T}$-walk $(z_i \mid i \leq m + q)$ by $C_1 \oplus C_2$.

It is evident that every $\mathcal{T}$-initial segment of G, and thus every $\mathcal{T}$-walk in G, is connected in $(V, u_{\mathcal{T}})$.

We will need the following statement proved in [18]:

Proposition 1. *Let $G = (V, E)$ be a graph and $\mathcal{T} \subseteq \mathcal{W}_n(G)$. A subset $A \subseteq V$ is connected in the closure space $(V, u_{\mathcal{T}})$ if and only if any two different vertices of G belonging to A can be joined by a $\mathcal{T}$-walk in G contained in A.*

Note that, if two vertices of a graph G may be joined by a $\mathcal{T}$-walk in G, then there may be no $\mathcal{T}$-path in G joining them.

The following statement is evident.

Lemma 1. *Let $G = (V, E)$ be a graph and $\mathcal{T} \subseteq \mathcal{W}_n(G)$. Let $(y_i \mid i \leq p)$ be a $\mathcal{T}$-initial segment in G and let $(x_i \mid i \leq n) \in \mathcal{T}$ be a walk with $y_i = x_i$ for all $i \leq p$ or $y_i = x_{p-1-i}$ for all $i \leq p$. If $i_0 \leq p$ is a natural number, then either $(y_{i_0-i} \mid i \leq i_0)$ or $(y_i \mid i_0 \leq i \leq p)$ is a $\mathcal{T}$-initial segment in G with the first member y_{i_0} and the last one x_0.*

Lemma 2. *Let $G_j = (V_j, E_j)$ be a graph, $\mathcal{T}_j \subseteq \mathcal{W}_n(G_j)$, and $(y_i^j \mid i \leq p_j)$ be a $\mathcal{T}_j$-initial segment in G_j for every $j = 1, 2, ..., m$. Then, $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$ is a connected set in $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$.*

Proof. If $m = 1$, then the statement is trivial. Let $m > 1$. For each $j = 1, 2, ..., m$, there is a walk $(x_i^j \mid i \leq n) \in \mathcal{T}$ such that $y_i^j = x_i^j$ for all $i \leq p_j$ or $y_i^j = x_{p_j-i}^j$ for all $i \leq p_j$ (because $(y_i^j \mid i \leq p_j)$ is a $\mathcal{T}_j$-initial segment in G_j). Let $y \in \prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$ be an arbitrary element. Then, for each $j = 1, 2, ..., m$, there is a natural number q_j, $q_j < p_j$, such that $y = (y_{q_1}^1, y_{q_2}^2, ..., y_{q_m}^m)$. By Lemma 1, either $(y_{q_1-i}^1 \mid i \leq q_1)$ or $(y_i^1 \mid q_1 \leq i \leq p_1)$ is a $\mathcal{T}_1$-initial segment in G_1 with the first member $y_{q_1}^1$ and the last one x_0^1. Denote this $\mathcal{T}_1$-initial segment by $(z_i^1 \mid i \leq r_1)$ and put $C_1 = ((z_i^1, y_{q_2}^2, y_{q_3}^3, ..., y_{q_m}^m) \mid i \leq r_1)$. Clearly, C_1 is a $\prod_{j=1}^m \mathcal{T}_j$-initial segment in $\prod_{j=1}^m G_j$ with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$, with the first member y, and with $z_{r_1}^1 = x_0^1$. Further, by Lemma 1, either $(y_{q_2-i}^2 \mid i \leq q_2)$ or $(y_i^2 \mid q_2 \leq i \leq p_2)$ is a $\mathcal{T}_2$-initial segment in G_2 with the first member $y_{q_2}^2$ and the last one x_0^2. Denote this $\mathcal{T}_2$-initial segment by $(z_i^2 \mid i \leq r_2)$ and put $C_2 = ((x_0^1, z_i^2, y_{q_3}^3, y_{q_4}^4, ..., y_{q_m}^m) \mid i \leq r_2)$. Clearly, C_2 is a $\prod_{j=1}^m \mathcal{T}_j$-initial segment in $\prod_{j=1}^m G_j$ with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$ such that $z_0^2 = y_{q_2}^2$ and $z_{r_2}^2 = x_0^2$. Thus, $C_1 \oplus C_2$ is a $\prod_{j=1}^m \mathcal{T}_j$-walk in $\prod_{j=1}^m G_j$ with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$, with the first member y, and with the last one $(x_0^1, x_0^2, y_{q_3}^3, y_{q_4}^4, ..., y_{q_m}^m)$. Repeating this construction m-times, we get $\prod_{j=1}^m \mathcal{T}_j$-initial segments C_1, C_2,...,C_m in $\prod_{j=1}^m G_j$ with the members of each of them belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$ such that $C_1 \oplus C_2 \oplus ... \oplus C_m$ is a $\prod_{j=1}^m \mathcal{T}_j$-walk in $\prod_{j=1}^m G_j$ with the first member y and the last one $(x_0^1, x_0^2, ..., x_0^m)$. We have shown that any point of $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$ can be joined with the point $(x_0^1, x_0^2, ..., x_0^m)$ by a $\prod_{j=1}^m \mathcal{T}_j$-walk in $\prod_{j=1}^m G_j$ contained in $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$. By Proposition 1, $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$ is a connected subset of $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$. □

Lemma 3. *Let $G_j = (V_j, E_j)$ be a graph, $\mathcal{T}_j \subseteq \mathcal{W}_n(G_j)$, and $(x_i^j \mid i \leq p_j)$ be a $\mathcal{T}_j$-walk in G_j for every $j = 1, 2, ..., m$. Then $\prod_{j=1}^m \{x_i^j;\ i \leq p_j\}$ is a connected set in $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$.*

Proof. If $m = 1$, then the statement is trivial. Let $m > 1$. For each $j = 1, 2, ..., m$, let $(i_k^j \mid k \leq q_j)$ be the binding sequence of $(x_i^j \mid i \leq p_j)$, i.e., a sequence of natural numbers with $i_0^j = 0$ and $i_{q_j-1}^j = p_j - 1$ such that $(x_i^j \mid i_k^j \leq i \leq i_{k+1}^j)$ is a $\mathcal{T}_j$-initial segment in G_j whenever $k \leq q_j$. For every $j = 1, 2, ..., m$, putting $C_k^j = \{x_i^j;\ i_k^j \leq i \leq i_{k+1}^j\}$, we get $\{x_i^j;\ i \leq p_j\} = \bigcup_{k<q_j} C_k^j$. Therefore, $\prod_{j=1}^m \{x_i^j;\ i \leq p_j\} = \bigcup_{k_1<q_1} \bigcup_{k_2<q_2} ... \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$ where $\prod_{j=1}^m C_{k_j}^j$ is connected in $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$ whenever $k_j < q_j$, $j = 1, 2, ..., m$, by Lemma 2. Thus, for any $k_j < q_j$, $j = 1, 2, ..., m-1$, $(\prod_{j=1}^m C_{k_j}^j \mid k_m < q_m)$ is a finite sequence of connected sets with nonempty intersection of every consecutive pair of them. Hence, the set $\bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$ is connected in $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$. Consequently, for every k_j with $k_j < q_j$, $j = 1, 2, ..., m-2$, $(\bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j \mid k_{m-1} < q_{m-1})$ is a finite sequence of connected sets with nonempty intersection of any consecutive pair of them. Therefore, the set $\bigcup_{k_{m-1}<q_{m-1}} \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$ is connected in $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$. After repeating this considerations m-times, we arrive at the conclusion that the set $\bigcup_{k_1<q_1} \bigcup_{k_2<q_2} ... \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j = \prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$ is connected in $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$. □

Theorem 1. *Let $G_j = (V_j, E_j)$ be a graph, $\mathcal{T}_j \subseteq \mathcal{W}_n(G_j)$, and $Y_j \subseteq V_j$ be a subset for every $j = 1, 2, ..., m$. Then Y_j is a connected subset of $(V_j, u_{\mathcal{T}_j})$ for every $j = 1, 2, ..., m$ if and only if $\prod_{j=1}^m Y_j$ is a connected subset of $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$.*

Proof. Let Y_j be a connected subset of $(X_j, u_{\mathcal{T}_j})$ for every $j \in \{1, 2, ..., m\}$ and let $(x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m) \in \prod_{j=1}^m G_j$ be arbitrary points. By Proposition 1, for every $j \in \{1, 2, ..., m\}$, there is a $\mathcal{T}_j$-walk $(z_i^j \mid i \le p_j)$ in G_j joining the points x_j and y_j which is contained in Y_j. Then, the set $\prod_{j=1}^m \{z_i^j \mid i \le p_j\}$ contains the points $(x_1, x_2, ..., x_m)$ and $(y_1, y_2, ..., y_m)$ and is a connected subset of $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$ by Lemma 3. Thus, by Proposition 1, there is a $\mathcal{T}_j$-walk C in $\prod_{j=1}^m G_j$ joining the points $(x_1, x_2, ..., x_m)$ and $(y_1, y_2, ..., y_m)$ which is contained in $\prod_{j=1}^m \{z_i^j \mid i \le p_j\}$. Since $\prod_{j=1}^m \{z_i^j \mid i \le p_j\} \subseteq \prod_{j=1}^m Y_j$, C is contained in $\prod_{j=1}^m Y_j$, too, and so $\prod_{j=1}^m Y_j$ is a connected subset of $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$ by Proposition 1.

Conversely, let $\prod_{j=1}^m Y_j$ be a connected subset of $\prod_{j=1}^m (V_j, u_{\mathcal{T}_j})$ and $(\prod_{j=1}^m V_j, v)$ be the cartesian product of the closure spaces $(V_j, u_{\mathcal{T}_j})$, $j = 1, 2, ..., m$. By Remark 1, $u_{\prod_{j=1}^m \mathcal{T}_j} \le v$. Thus, as the projections $pr_j : (\prod_{j=1}^m V_j, v) \to (X_j, u_{\mathcal{T}_j})$, $j = 1, 2, ..., m$, are continuous, the projections $pr_j : (\prod_{j=1}^m V_j, u_{\prod_{j=1}^m \mathcal{T}_j}) \to (V_j, u_{\mathcal{T}_j})$, $j = 1, 2, .., m$, are continuous as well. Consequently, $Y_j = pr_j \prod_{j=1}^m Y_j$ is a connected subset of $(V_j, u_{\mathcal{T}_j})$ for every $j \in \{1, 2, ..., m\}$. □

4. Structuring $\mathbb{Z}^2$ by Products of the Closure Operators Induced by Certain Walk Sets in the 2-Adjacency Graph on $\mathbb{Z}$

As usual in digital geometry, we represent pixels by their center points. The center points are assumed to have integer coordinates, so that we may study adjacency of points in $\mathbb{Z}^2$ rather than pixels.

Recall that the 2-adjacency graph on $\mathbb{Z}$ is the graph $\mathbb{Z}_2 = (\mathbb{Z}, A_2)$ where $A_2 = \{\{p, q\};\ p, q \in \mathbb{Z},\ |p - q| = 1\}$.

Throughout this section, $\mathcal{T} \subseteq \mathcal{W}_n(\mathbb{Z}_2)$ will denote the walk set given as follows:

$\mathcal{T} = \{(x_i \mid i \le n) \in \mathcal{W}_n(\mathbb{Z}_2)$; there exists an odd number $l \in \mathbb{Z}$ such that $x_i = ln + i$ for all $i \le n$ or $x_i = ln - i$ for all $i \le n\}$.

Thus, all walks belonging to $\mathcal{T}$ are paths being just the arithmetic sequences $(x_i \mid i \le n)$ of integers with the difference equal to 1 or -1 and with $x_0 = ln$ where $l \in \mathbb{Z}$ is an odd number. Note that each point of $z \in \mathbb{Z}$ belongs to at least one and at most two paths in $\mathcal{T}$. It belongs to two (different) paths in $\mathcal{T}$ if and only if there is $l \in \mathbb{Z}$ with $z = ln$ (then, z is the first (last) member of each of the two paths if l is odd (even)).

Clearly, $u_{\mathcal{T}}$ is additive if and only if $n = 1$. The closure operator $u_{\mathcal{T}}$ is then Alexandroff and coincides with the Khalimsky topology on $\mathbb{Z}$ generated by the subbase $\{\{2k-1, 2k, 2k+1\};\ k \in \mathbb{Z}\}$ - cf. [10].

Let m be (same as n) a natural number with $m > 0$. Using the results of the previous section, we may propose new structures on the digital spaces convenient for the study of digital images. Such a structure on $\mathbb{Z}^m$ is obtained as the strong product of m copies of the 2-adjacency graph on $\mathbb{Z}$ with the walk set given by the strong product of m-copies of the walk set $\mathcal{T}$. More formally, we may consider the graph $G^m = \prod_{j=1}^m G_j$ on $\mathbb{Z}^m$, where G_j is the 2-adjacency graph on $\mathbb{Z}$ for every $j \in \{1, 2, ..., m\}$, with the walk set $\mathcal{T}^m \subseteq \mathcal{W}_n(\mathbb{G}^m)$ given by $\mathcal{T}^m = \prod_{j=1}^m \mathcal{T}_j$ where $\mathcal{T}_j = \mathcal{T}$ for every $j \in \{1, 2, ..., m\}$ (note that every walk from $\mathcal{T}^m$ is a path). Clearly, G^1 is the 2-adjacency graph on $\mathbb{Z}$ and G^2 and G^3 coincide with the well known 8-adjacency graph on $\mathbb{Z}^2$ and 26-adjacency graph on $\mathbb{Z}^3$, i.e., the graphs $(\mathbb{Z}^2, A_8)$ where $A_8 = \{\{(x_1, y_1), (x_2, y_2)\};\ (x_1, y_1), (x_2, y_2) \in \mathbb{Z}^2,\ \max\{|x_1 - x_2|, |y_1 - y_2|\} = 1\}$ and $(\mathbb{Z}^3, A_{26})$ where $A_{26} = \{\{(x_1, y_1, z_1), (x_2, y_2, z_2)\};\ (x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{Z}^3,\ \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} = 1\}$, respectively.

Having defined the graphs G^m with the walk sets $\mathcal{T}^m$, we may study connectedness with respect to the closure operators $u_{\mathcal{T}^m}$, i.e., in the closure spaces $\prod_{j=1}^m (\mathbb{Z}, u_{\mathcal{T}}) = (\mathbb{Z}^m, u_{\mathcal{T}^m})$ ($m > 0$ an integer). Such a study may be based on the results of the previous section related to the products of closure operators on graphs induced by walk sets. In particular, as an immediate consequence of Theorems 1 and 2, we get:

Theorem 2. *$(\mathbb{Z}^m, u_{\mathcal{T}^m})$ is a connected closure space for every integer $m > 0$.*

Proof. Put $D_l = \{ln + i;\ i \le n\}$ for each $l \in \mathbb{Z}$. Obviously, D_l is connected in $(\mathbb{Z}, u_{\mathcal{T}})$ for every $l \in \mathbb{Z}$ (because $(ln + i;\ i \le n)$ is a $\mathcal{T}$-initial segment in $\mathbb{Z}_2$). It is also evident that $D_l \cup D_{l+1}$ is closed in $(\mathbb{Z}, u_{\mathcal{T}})$ whenever $l \in \mathbb{Z}$ is even.

Let ω denote the least infinite ordinal and let $(B_i \mid i < \omega)$ be the sequence given by $B_i = D_{\frac{i}{2}}$ whenever i is even and $B_i = D_{-\frac{i+1}{2}}$ whenever i is odd, i.e., $(B_i \mid i < \omega) = (D_0, D_{-1}, D_1, D_{-2}, D_2, ...)$. For each $l \in \mathbb{Z}$, we have $D_l \cap D_{l+1} = \{(l+1)n)\} \neq \emptyset$. Thus, $B_0 \cap B_1 \neq \emptyset$. Let i_0 be a natural number with $i_0 > 1$. Then, $B_{i_0} \cap B_{i_0-2} \neq \emptyset$ because $B_{i_0} = D_{\frac{i_0}{2}}$ and $B_{i_0-2} = D_{\frac{i_0}{2}-1}$ whenever i_0 is even, while $B_{i_0} = D_{-\frac{i_0+1}{2}}$ and $B_{i_0-2} = D_{-\frac{i_0+1}{2}+1}$ whenever i_0 is odd. Hence, $(\bigcup_{i<i_0} B_i) \cap B_{i_0} \neq \emptyset$ for each i_0, $0 < i_0 < \omega$. Therefore, $\bigcup_{i<\omega} B_i$ is connected. But $\bigcup_{i<\omega} B_i = \bigcup_{l\in\mathbb{Z}} D_l = \mathbb{Z}$, which proves that $(\mathbb{Z}^m, u_{\mathcal{T}^m})$ is connected for $m = 1$.

Now, the statement follows from Theorem 1. □

We denote by H_n the factor of the 8-adjacency graph on $\mathbb{Z}^2$ whose edges are those $\{(x_1, y_1), (x_2, y_2)\} \in A_8$ that satisfy one of the following four conditions for some $k \in \mathbb{Z}$:

$x_1 - y_1 = x_2 - y_2 = 2kn$,
$x_1 + y_1 = x_2 + y_2 = 2kn$,
$x_1 = x_2 = 2kn$,
$y_1 = y_2 = 2kn$.

A section of the graph H_n is shown in Figure 1 where only the vertices $(2kn, 2ln)$, $k, l \in \mathbb{Z}$, are displayed (as bold dots) and thus, on every edge drawn between two such vertices, there are $2n-1$ more (non-displayed) vertices, so that the edges represent $2n$ edges in the graph H_n. Clearly, every circle C in H_n is a connected subset of the closure space $(\mathbb{Z}^2, u_{\mathcal{T}^2})$ because it is a $\mathcal{T}^2$-circle in G^2. Indeed, C consists (is the union) of a finite sequence of walks from $\mathcal{T}^2$, hence $\mathcal{T}^2$-initial segments, such that every two consecutive walks have a point in common.

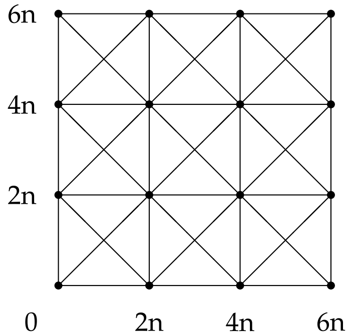

Figure 1: A portion of the graph H_n.

Since, for $n = 1$, the behavior of the closure operator $u_{\mathcal{T}^m}$, i.e., the Khalimsky topology on $\mathbb{Z}^m$, is well known, we will assume that $n > 1$ in the sequel restricting our considerations to $m = 2$ because this case is the most important one with respect to possible applications in digital topology. Thus, we will focus on the closure spaces $(\mathbb{Z}^2, u_{\mathcal{T}^2})$.

Definition 1. For every point $z = ((2k+1)n, (2l+1)n)$, $k, l \in \mathbb{Z}$, each of the following four subsets of $\mathbb{Z}^2$ is called an *n-basic triangle* (given by z):

$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le r + 2ln - 2kn\}$,
$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ (2l+2)n + 2kn - r \le s \le (2l+2)n\}$,
$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ r + 2ln - 2kn \le s \le (2l+2)n\}$,
$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n + 2kn - r\}$.

The points of any n-basic triangle form a segment having the shape of a (digital) rectangular triangle. Obviously, in each of the four n-basic triangles given by z, z is the middle point of the hypotenuse of the triangle. The four n-basic triangles given by a point $z = ((2k+1)n, (2l+1)n)$, $k, l \in \mathbb{Z}$, are demonstrated in Figure 2 as triangles ABC, BCD, CDA, and DAB. Every line segment constituting an edge of any of the four n-basic triangles represents precisely $2n+1$ points forming the corresponding edge of the triangle. Clearly, the edges of any n-basic triangle form a circle in the graph H_n, hence a $\mathcal{T}^2$-circle in G^2.

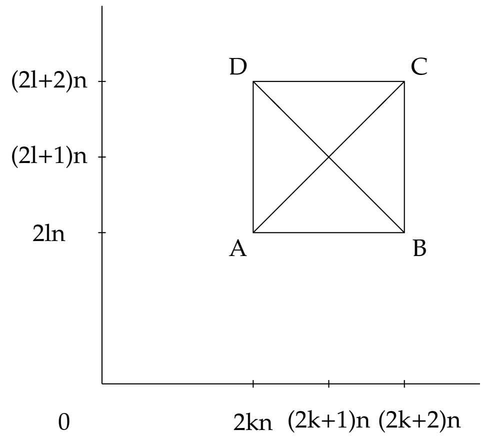

Figure 2: n-basic triangles.

We will need the following

Lemma 4. *Every n-basic triangle is connected in $(\mathbb{Z}^2, u_{\mathcal{T}^2})$ and so is also every set obtained from an n-basic triangle by subtracting some of its edges.*

Proof. Let $z = ((2k+1)n, (2l+1)n), k, l \in \mathbb{Z}$, be a point and consider the n-basic triangle $T = \{(r, s) \in \mathbb{Z}^2;\ 2kn \leq r \leq (2k+2)n,\ 2ln \leq s \leq r + 2ln - 2kn\}$. Then, T is the (digital) triangle ABC with the vertices $A = (2kn, 2ln)$, $B = ((2k+2)n, 2ln)$, $C = ((2k+2)n, (2l+2)n)$. For every $u \in \mathbb{Z}$, $(2k+1)n \leq u \leq (2k+2)n$, the sequence $G_u = ((u, y) \mid 2ln \leq y \leq u + 2(l-k)n)$ is a $\mathcal{T}^2$-path in G^2 (contained in T), so that G_u is a connected set in $(\mathbb{Z}^2, u_{\mathcal{T}^2})$. Similarly, for every $v \in \mathbb{Z}$, $2ln \leq v \leq (2l+1)n$, the sequence $H_v = ((x, v) \mid v + 2(k-l)n \leq x \leq (2k+2)n)$ is a $\mathcal{T}^2$-path in G^2 (contained in T), so that H_v is a connected set in $(\mathbb{Z}^2, u_{\mathcal{T}^2})$. We clearly have $T = \bigcup\{G_u;\ (2k+1)n \leq u \leq (2k+2)n\} \cup \bigcup\{H_v;\ 2ln \leq v \leq (2l+1)n\}$. It may easily be seen that $G_u \cap H_v \neq \emptyset$ whenever $(2k+1)n \leq u \leq (2k+2)n$ and $2ln \leq v \leq (2l+1)n$. For every natural number $i < 2n+2$, we put

$$S_i = \begin{cases} G_{(2k+1)n+\frac{i}{2}} & \text{if } i \text{ is even,} \\ H_{2ln+\frac{i-1}{2}} & \text{if } i \text{ is odd.} \end{cases}$$

Then, $(S_i \mid i < 2n+2)$ is a sequence with the property that its members with even indices form the sequence $(G_u \mid (2k+1)n \leq u \leq (2k+2)n)$ and those with odd indices form the sequence $(H_v \mid 2ln \leq v \leq (2l+1)n)$. Hence, $\bigcup\{S_i \mid i < 2n+2\} = \bigcup\{G_u;\ (2k+1)n \leq u \leq (2k+2)n\} \cup \bigcup\{H_v;\ 2ln \leq v \leq (2l+1)n\}$ and every pair of consecutive members of $(S_i \mid i < 2n+2)$ has a non-empty intersection. Thus, since $T = \bigcup\{S_i \mid i < 2n+2\}$, T is connected in $(\mathbb{Z}^2, u_{\mathcal{T}^2})$. For each of the other three n-basic triangles given by z, the proof is analogous, and the same is true for every set obtained from an n-basic triangle (given by z) by subtracting some of its edges. $\square$

Definition 2. A circle J in the graph H_n is said to be *basic* if, whenever $((2k+1)n, (2l+1)n) \in J$ for some $k, l \in \mathbb{Z}$, one of the following two conditions is true:

$\{((2k+1)n-1, (2l+1)n-1), (2k+1)n+1, (2l+1)n+1))\} \subseteq J$,

$\{((2k+1)n-1, (2l+1)n+1), (2k+1)n+1, (2l+1)n-1))\} \subseteq J$.

The basic circles are exactly the circles in H_n consisting of edges of some n-basic triangles, i.e., the circles in the graph shown in Figure 1.

Theorem 3. *Every basic circle J in the graph H_n separates the closure space $(\mathbb{Z}^2, u_{\mathcal{T}^2})$ into precisely two components, one finite and the other infinite, such that the union of any of them with J is a connected subset of $(\mathbb{Z}^2, u_{\mathcal{T}^2})$.*

Proof. We will say that a (finite or infinite) sequence S of n-basic triangles is a tiling sequence if the members of S are pairwise different and every member of S, except for the first one, has an edge in common with at least one of its predecessors. Given a tiling sequence S of n-basic triangles, we denote by S' the sequence obtained from S by subtracting from every member of the sequence all its edges that are not shared with any other member of the sequence. It immediately follows from Lemma 4 that, for every tiling sequence S of n-basic triangles, the set $\bigcup\{T;\ T \in S\}$ is connected in $(\mathbb{Z}^2, u_{\mathcal{T}^2})$ and the same is true for the set $\bigcup\{T;\ T \in S'\}$.

Let J be a basic circle in the graph H_n. Then, J constitutes the border of a polygon $S_F \subseteq \mathbb{Z}^2$ consisting of n-basic triangles. More precisely, S_F is the union of some n-basic triangles such that any pair of them is disjoint or has just one edge in common. Let U be a tiling sequence of the n-basic triangles contained in S_F. Since S_F is finite, so is U, and we have $S_F = \bigcup\{T;\ T \in U\}$. By Lemma 4, every n-basic triangle $T \in U$ is connected. Thus, S_F is connected, too. Similarly, U' is a finite sequence with $S_F - J = \bigcup\{T;\ T \in U'\}$ and, since every member of U' is connected by Lemma 4, $S_F - J$ is connected, as well.

Further, let V be a tiling sequence of n-basic triangles that are not contained in S_F. Since the complement of S_F in $\mathbb{Z}^2$ is infinite, V is infinite, too. Put $S_I = \bigcup\{T;\ T \in V\}$. By Lemma 4, every n-basic triangle $T \in V$ is connected so that S_I is connected, too. Similarly, V' is a finite sequence with $S_I - J = \bigcup\{T;\ T \in V'\}$ and, since every member of V' is connected by Lemma 4, $S_I - J$ is connected, as well.

It may easily be seen that every $\mathcal{T}^2$-walk $C = (z_i \mid i \leq k)$, $k > 0$ a natural number, in the 8-adjacency graph G^2 on $\mathbb{Z}^2$ connecting a point of $S_F - J$ with a point of $S_I - J$ meets J (i.e., meets an edge of an n-basic triangle contained in J). Therefore, the set $\mathbb{Z}^2 - J = (S_F - J) \cup (S_I - J)$ is not connected in $(\mathbb{Z}^2, u_{\mathcal{T}^2})$. We have shown that $S_F - J$ and $S_I - J$ are components of the subspace $\mathbb{Z}^2 - J$ of $(\mathbb{Z}^2, u_{\mathcal{T}^2})$, $S_F - J$ finite and $S_I - J$ infinite, with S_F and S_I connected. □

Remark 2. Note that Theorem 3 is formulated for the closure spaces $(\mathbb{Z}^2, u_{\mathcal{T}^2})$ with $\mathcal{T} \subseteq \mathcal{W}_n(\mathbb{Z}_2)$ where $n > 1$. For the space $(\mathbb{Z}^2, u_{\mathcal{T}^2})$ with $\mathcal{T} \subseteq \mathcal{W}_1(\mathbb{Z}_2)$, i.e., for the Khalimsky plane, a Jordan curve theorem has been proved in [7] stating that every simple closed curve J in the Khalimsky plane having at least four points separates this plane into precisely two components. Here, simple closed curves in the Khalimsky plane are (in accordance with [12]) the circles in the connectedness graph of the Khalimsky topology on $\mathbb{Z}^2$ that contain, with each of its points, exactly two points adjacent to it. Recall that the connectedness graph of the Khalimsky topology on $\mathbb{Z}^2$ is the graph with the vertex set $\mathbb{Z}^2$ in which two vertices are adjacent if and only if they are different and form a connected set - see Figure 3. Thus, the simple closed curves in the Khalimsky plane are special cases of digital manifolds. Note that every basic circle that does not turn, at any of its points, at the acute angle $\frac{\pi}{4}$ is a simple closed curve in the Khalimsky plane. And, conversely, every simple closed curve in the Khalimsky plane is a circle in the graph $(\mathbb{Z}^2, A \cup A_4)$ where A is the set of edges of the graph H_n and A_4 is the 4-adjacency, i.e., $A_4 = \{\{(x_1, y_1), (x_2, y_2)\} \in \mathbb{Z}^2;\ |x_1 - x_2| + |y_1 - y_2| = 1\}$.

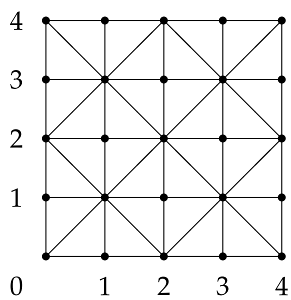

Figure 3: A section of the connectedness graph of the Khalimsky topology on $\mathbb{Z}^2$.

Example 1. It is a disadvantage of the closure operator $u_{\mathcal{T}^2}$ with $n = 1$, i.e., Khalimsky topology, that simple closed curves may never turn at the acute angle $\frac{\pi}{4}$ so that the Jordan curve theorem proved in [7] cannot be applied to such curves. This disadvantage is eliminated if a closure operator $u_{\mathcal{T}^2}$ is used with $\mathcal{T} \subseteq \mathcal{W}_n(\mathbb{Z}_2)$ where $n > 1$ instead and Theorem 3 is applied. For instance, consider the digital curve in Figure 4, which represents the (border of) letter M. This curve is not a simple closed curve in the Khalimsky plane. For it to be a simple closed curve in the Khalimsky plane (separating the plane into precisely two components), we have to delete the eight points that are ringed. But this will lead to a certain deformation (loss of sharpness) of the letter - recall that the points of $\mathbb{Z}^2$ represent pixels so that the eight pixels will belong to the white background of the black image of M. On the other hand, for $n = 2$, the curve is a basic circle in the graph H_n and, therefore, it separates the closure space $(\mathbb{Z}^2, u_{\mathcal{T}^2})$ into precisely two components by Theorem 3.

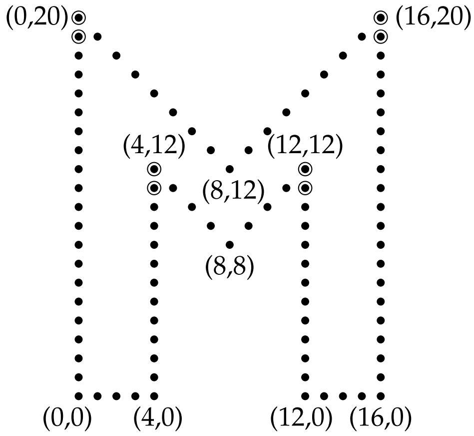

Figure 4: A digital image of M.

5. Conclusion

We introduced and investigated the concept of closure operators on (the vertex sets of) graphs induced by sets of walks of the same lengths in the graphs. We showed that connectedness with respect to these closure operators is preserved by the introduced product of the operators. We discussed a possible application of this result in digital topology to the study of geometric and topological properties of digital images. Since digital images may be regarded as approximations of real ones, to be able to perform such a study, we need to provide the digital plane (and, more generally, digital space) with a structure that would conveniently model the Euclidean plane (Euclidean space). For the digital plane, an important criterion of such a convenience is the validity of a digital analogue of the Jordan curve theorem. In two-dimensional digital images, which are regarded as approximations of real ones, digital versions of simple closed curves are used to represent borders of objects. To avoid undesirable paradoxes, it is necessary that the curves satisfy a digital analogue of the Jordan curve theorem, i.e., separate the digital plane into precisely two components - the inside and the outside of the objects. The above Theorem 3 provides such a digital analogue of the Jordan curve theorem and it, therefore, makes it possible to use the closure operators $u_{\mathcal{T}^2}$ with $\mathcal{T} \subseteq \mathcal{W}_n(\mathbb{Z}_2)$ where $n > 1$ for modeling connectedness in the digital plane $\mathbb{Z}^2$ in order to get convenient backgrounds for studying digital images.

The most natural extension of the classical Jordan curve theorem to the m-dimensional Euclidean spaces $\mathbb{R}^m$, $m > 2$, is known as the Jordan-Brouwer separation theorem. The method we used for proving a digital Jordan curve theorem (Theorem 3) seems to be applicable also to proving a digital analogue of the Jordan-Brouwer separation theorem. Such a proof, however, would be rather laborious and, therefore, we prefer dealing with the problem in a separate paper.

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