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On $k\#\$$-rewriting systems

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Abstract. This paper introduces $k\#\$$-rewriting systems based on early defined #-rewriting systems but with additional pushdown memory. It demonstrates that these systems characterize an infinite hierarchy of language families resulting from the limited number of rewriting positions in every configuration during the generation of a sentence.

Keywords: $k\#\$$-rewriting systems, pushdown, #-rewriting systems, infinite hierarchy, finite index, n-limited state grammars

1 Introduction

The most used classes of formal models in the formal language theory are grammars and automata. Grammars work as generative devices, while automata work as accepting devices. Given a grammar, it uses its rules to derive the string belonging to the language it describes from some initial string. Given an automaton, it uses its rules to decide which actions should be performed, based on its state, first symbol of its input string, and possibly on other additional information. Every string that drives the given automaton to its accepting configuration belongs to the language characterized by that automaton.

In a modern formal language theory, some formal models that share properties both from the grammars and automata has been introduced. Such an example are *state grammars* (see [1]), which were developed from context-free grammars by adding finite-state control. Another example are *rewriting systems* (see Chapter 2 in [2]), which are a generalization of grammars and automata and hence, depending on their rules, they are able to simulate both of them.

In 2006, Meduna, Křivka, and Schönecker introduced a new modification of rewriting systems, called *#-rewriting systems* (see [3]). While ordinary rewriting systems rewrite just one substring to another during one computation step, #-rewriting systems rewrite in fact two substrings, where the first substring is always one symbol long and acts like *state*. Moreover, the success of one computation step in #-rewriting systems depends also on the number of occurrences of # in their sentential forms. If k is an upper bound limit of the number of occurrences of #, #-rewriting systems are said to be of *index* k. Such a restriction has an important influence to their descriptive power. While ordinary rewriting systems characterize the Chomsky hierarchy of languages, the power of #-rewriting systems of index k coincide with the power of programmed grammars of the same index (see [3]).

In this paper, we extend #-rewriting systems with additional storage that can contain both terminals and nonterminals and we call them *$k\#\$$-rewriting systems*. More precisely, every configuration consists of three parts: (1) the current state, (2) a string of terminals (including #), and, newly, (3) a pushdown string of terminals and nonterminals (excluding #). Below, in the string representation of such configuration, part (2) and (3) are separated by \$ symbol.

After giving some preliminaries in Section 2 and introducing the formal definition with an example in Section 3, in Section 4, we show that for some positive integer, k, $k\#\$$-rewriting systems and k-limited state grammars have the same expressive power. Finally, the concluding section outlines an open problem for further investigation.

2 Preliminaries

This paper assumes that the reader is familiar with the fundamental notions of formal language theory (see [4, 5]). For a set X, $\mathrm{card}(X)$ denotes its cardinality and 2^X denotes its power set. By $\mathbb{I}$, we denote a set of all positive integers. Let Σ be an alphabet. Then, Σ^* represents the free monoid generated by Σ under the operation of concatenation with ε as its identity element. Set $\Sigma^+ = \Sigma^* - \{\varepsilon\}$. For $w \in \Sigma^*$, $|w|$ denotes the length of w, $\mathrm{alph}(w) = \{x \mid w = uxv, x \in \Sigma, u, v \in \Sigma^*\}$ denotes the minimal subset of Σ such that $w \in \mathrm{alph}(w)^*$. For $a \in \Sigma$, $\mathrm{occur}(w, a)$ denotes the number of occurrences of a in w; mathematically, $\mathrm{occur}(w, a) = \mathrm{card}(\{u \mid w = uav, u, v \in \Sigma^*\})$. For $W \subseteq \Sigma$, $\mathrm{occur}(w, W) = \sum_{a \in W} \mathrm{occur}(w, a)$. For $k \geq 0$, if w can be expressed as $w = xy$ such that $k = |x|$ and $x, y \in \Sigma^*$, then $\mathrm{prefix}(w, k) = x$; otherwise, $\mathrm{prefix}(w, k) = w$.

Let A be a set and let σ be a (binary) relation over A. The k-fold product of σ, where $k \geq 0$, the transitive closure of σ, and the reflexive and transitive closure of σ are denoted as σ^k, σ^+, and σ^*, respectively. Instead of $(x, y) \in \sigma$, we write $x \; \sigma \; y$.

By $p\colon e$, we express that e has p as its *label*, i.e. p is a unique symbol that is associated with e and that can be used as an alternative name of e. By $p\colon e \in D$, we express that $p\colon e$ and $e \in D$.

A *context-free grammar* is a quadruple, $G = (V, T, P, S)$, where V is a total alphabet, $T \subset V$ is an alphabet of terminals, $P \subseteq (V - T) \times V^*$ is a finite set of rules, and $S \in (V - T)$ is the start symbol. Instead of $(A, x) \in P$, we write $A \to x \in P$. Let $\Rightarrow$ be a relation of direct derivation on V^* defined as follows: $uAv \Rightarrow uxv$ iff $A \to x \in P$, where $A \in (V - T)$ and $u, x, v \in V^*$. By $uAv \Rightarrow uxv \, [A \to x]$, we express that uAv directly derives uxv according to $A \to x$. By $\Rightarrow_G$, we express that a relation of direct derivation, $\Rightarrow$, is associated with a grammar G. The language generated by G, $L(G)$, is defined as $L(G) = \{w \mid S \Rightarrow^* w, w \in T^*\}$. The family of context-free languages is denoted as $\mathscr{L}(\mathrm{CF})$.

Let $k \geq 1$ and $\Sigma_n = \{a_1, b_1, a_2, b_2, \dots, a_n, b_n\}$. The *Dyck language* $\mathscr{D}_n$ over Σ_n is generated by the grammar

$$(\{S\} \cup \Sigma_n, \Sigma_n, \{S \to SS, S \to \varepsilon, S \to a_1 S b_1, \dots, S \to a_n S b_n\}, S).$$

Let G be a grammar of arbitrary type, and let V, T, and S be its total alphabet, terminal alphabet, and start symbol, respectively. For a derivation $D\colon w_1 \Rightarrow w_2 \Rightarrow \ldots \Rightarrow w_r$, $S = w_1$, $w_r \in T^*$, according to G, we set $\mathrm{Ind}(D, G) = \max\{\mathrm{occur}(w_i, V - T) \mid 1 \leq i \leq r\}$, and for $w \in T^*$, we define $\mathrm{Ind}(w, G) = \min\{\mathrm{Ind}(D, G) \mid D$ is a derivation for w in $G\}$. The *index of grammar G* (see page 151 in [6]) is defined as $\mathrm{Ind}(G) = \sup\{\mathrm{Ind}(w, G) \mid w \in L(G)\}$. For a language L in the family $\mathscr{L}(X)$ of languages generated by grammars of some type X, we define $\mathrm{Ind}_X(L) = \inf\{\mathrm{Ind}(G) \mid L(G) = L, G$ is of type $X\}$. For a family $\mathscr{L}(X)$, we set $\mathscr{L}_n(X) = \{L \mid L \in \mathscr{L}(X)$ and $\mathrm{Ind}_X(L) \leq n\}$, $n \geq 1$.

A *state grammar* (see [1]) is a sixtuple $G = (V, T, K, P, S, s)$, where V is a total alphabet, $T \subset V$ is an alphabet of terminals, K is a finite set of states, $V \cap K = \emptyset$, $P \subseteq (V - T) \times K \times V^* \times K$ is a finite set of rules, $S \in (V - T)$ is the start symbol, and $s \in K$ is the start state. Instead of $(A, p, x, q) \in P$, we write $(A, p) \rightarrow (x, q) \in P$. Let $\Rightarrow$ be a relation of direct derivation on $V^* \times K$ defined as follows: $(uAv, p) \Rightarrow (uxv, q)$ iff $(A, p) \rightarrow (x, q) \in P$ and for every $(B, p) \rightarrow (y, t) \in P$, $B \notin \mathrm{alph}(u)$, where $p, q, t \in K$, $A, B \in (V - T)$, and $u, v, x, y \in V^*$. For some $k \geq 1$ satisfying $\mathrm{occur}(uA, V - T) \leq k$, $\Rightarrow$ is said to be k-limited, denoted as ${}_k\!\Rightarrow$. By $(uAv, p) \Rightarrow (uxv, q)\,[(A, p) \rightarrow (x, q)]$, we express that (uAv, p) directly derives (uxv, q) according to $(A, p) \rightarrow (x, q)$. Similarly for ${}_k\!\Rightarrow$. The language generated by G, $L(G)$, is defined as $L(G) = \{w \mid (S, s) \Rightarrow^* (w, q), q \in K, w \in T^*\}$. Let $k \geq 1$. The language generated by G in k-limited way, $L(G, k)$, is defined as $L(G, k) = \{w \mid (S, s)\ {}_k\!\Rightarrow^* (w, q), q \in K, w \in T^*\}$. The families of languages generated by state grammars and by state grammars in k-limited way are denoted as $\mathscr{L}(\mathrm{ST})$ and $\mathscr{L}(\mathrm{ST}, k)$, respectively.

A *#-rewriting system* (see [3]) is a quadruple $M = (Q, \Sigma, s, R)$, where Q is a finite set of states, Σ is an alphabet containing special symbol # called *bounder*, $Q \cap \Sigma = \emptyset$, $s \in Q$ is the start state and $R \subseteq Q \times \mathbb{I} \times \{\#\} \times Q \times \Sigma^*$ is a finite set of rules. Instead of $(p, n, \#, q, x) \in R$, we write $p\,{}_n\# \rightarrow qx$. Let $\Rightarrow$ be a relation of direct rewriting step on $Q\Sigma^*$ defined as follows: $pu\#v \Rightarrow quxv$ iff $p\,{}_n\# \rightarrow qx \in R$ and $\mathrm{occur}(u, \#) = n - 1$, where $p, q \in Q$, $u, v, x \in \Sigma^*$, and $n \in \mathbb{I}$. By $pu\#v \Rightarrow quxv\,[p\,{}_n\# \rightarrow qx]$, we express that $pu\#v$ directly rewrites $quxv$ according to $p\,{}_n\# \rightarrow qx$. The language generated by M, $L(M)$, is defined as $L(M) = \{w \mid s\# \Rightarrow^* qw, q \in Q, w \in (\Sigma - \{\#\})^*\}$. Let $k \in \mathbb{I}$. A #-rewriting system M is said to be of index k if and only if $s\# \Rightarrow^* qy$ implies $\mathrm{occur}(y, \#) \leq k$, where $q \in Q$ and $y \in \Sigma^*$. Let $k \in \mathbb{I}$. The family of languages generated by #-rewriting systems and by #-rewriting systems of index k are denoted as $\mathscr{L}(\#\mathrm{RS})$ and $\mathscr{L}_k(\#\mathrm{RS})$, respectively.

3 Definitions

We are now ready to define $k\#\$$-rewriting systems.

Definition 3.1. *Let $k \in \mathbb{I}$. A* $k\#\$$-rewriting system *is a quintuple*

$$M = (Q, V, \Sigma, s, R),$$

where Q is a finite set of states, V is a total alphabet, $V \cap Q = \emptyset$, Σ is an alphabet containing # and \$ called bounders, *$\Sigma \subseteq V$, $s \in Q$ is a start state and*

$$\begin{aligned} R \ \subseteq\ & (Q \times \mathbb{I} \times \{\#\} \times Q \times (\Sigma - \{\$\})^*) \\ \cup\ & (Q \times \{\#\} \times \{\$\} \times Q \times \{\$\} \times (V - \{\#, \$\})^*) \\ \cup\ & (Q \times \{\$\} \times (V - \Sigma) \times Q \times \{\#\} \times \{\$\}) \end{aligned}$$

is a finite set of rules.

Instead of $(p, n, \#, q, x) \in R$, $(p, \#, \$, q, \$, y) \in R$ *and* $(p, \$, A, q, \#, \$) \in R$, *we write* $p\,{}_n\# \to qx \in R$, $p\#\$ \to q\$y \in R$ *and* $p\$A \to q\#\$ \in R$, *respectively.*

Let $\Xi \subseteq Q(\Sigma - \{\$\})^*\{\$\}(V - \{\#, \$\})^*$ *be a set of all* configurations *of* M *such that* $\chi \in \Xi$ *iff* $\mathrm{occur}(\chi, \#) \leq k$.

Let $\Rightarrow$ *be a relation of direct rewriting step on* Ξ *defined as follows:*

- $pu\#v\$\alpha \Rightarrow quxv\$\alpha$ *iff* $p\,{}_n\# \to qx \in R$, $\mathrm{occur}(u, \#) = n - 1$, $p, q \in Q$, $u, v, x \in (\Sigma - \{\$\})^*$, $\alpha \in (V - \{\#, \$\})^*$, *and* $n \in \mathbb{I}$;
- $pu\#\$\alpha \Rightarrow qu\$x\alpha$ *iff* $p\#\$ \to q\$x \in R$, $p, q \in Q$, $u \in (\Sigma - \{\$\})^*$, *and* $x, \alpha \in (V - \{\#, \$\})^*$;
- $pu\$A\alpha \Rightarrow qu\#\$\alpha$ *iff* $p\$A \to q\#\$ \in R$, $p, q \in Q$, $u \in (\Sigma - \{\$\})^*$, $A \in V - \Sigma$, *and* $\alpha \in (V - \{\#, \$\})^*$;
- $pux\$\alpha \Rightarrow pu\$x\alpha$ *iff* $p \in Q$, $u \in (\Sigma - \{\$\})^*$, $x \in (\Sigma - \{\#, \$\})^*$, *and* $\alpha \in (V - \{\#, \$\})^*$;
- $pu\$x\alpha \Rightarrow pux\$\alpha$ *iff* $p \in Q$, $u \in (\Sigma - \{\$\})^*$, $x \in (\Sigma - \{\#, \$\})^*$, *and* $\alpha \in (V - \{\#, \$\})^*$.

By $x \Rightarrow y\,[r]$, *we express that* x *directly rewrites* y *according to* r.

The language generated by M, $L(M)$, *is defined as*

$$L(M) = \{w \mid s\#\$ \Rightarrow^* qw\$, q \in Q, w \in (\Sigma - \{\#, \$\})^*\}.$$

The family of languages generated by $k\#\$$*-rewriting systems is denoted as* $\mathscr{L}_k(\#\$\mathrm{RS})$.

The following example demonstrates a generative capacity of $k\#\$$-rewriting systems.

Example 3.2. *Let* $M = (Q, V, \Sigma, s, R)$ *be a* $2\#\$$*-rewriting system, where*

$$\begin{array}{rcl} Q & = & \{s, p, p', p^{(1)}, p^{(2)}, p^{(X)}, p^{(Y)}, q, f, f^{(A)}, f^{(B)}\} \\ V & = & \{A, B, X, a, b, c, d, 0, 1, \bar{0}, \bar{1}, [_1, [_2,]_1,]_2, \#, \$\} \\ \Sigma & = & \{a, b, c, d, 0, 1, \bar{0}, \bar{1}, [_1, [_2,]_1,]_2, \#, \$\} \end{array}$$

and R *contains rules*

$$\begin{array}{ll} 1\colon s\,{}_1\# \to p\#\# & 9\colon p^{(Y)}\,{}_1\# \to q \\ 2\colon p\,{}_1\# \to p'a\#b & 10\colon q\,{}_1\# \to f \\ 3\colon p'\,{}_2\# \to p^{(1)}c\# & 11\colon f\$A \to f^{(A)}\#\$ \\ 4\colon p'\,{}_2\# \to p^{(2)}d\# & 12\colon f\$B \to f^{(B)}\#\$ \\ 5\colon p^{(1)}\#\$ \to p^{(X)}\$X[_1A]_1 & 13\colon f^{(A)}\,{}_1\# \to f^{(A)}0\#1 \\ 6\colon p^{(2)}\#\$ \to p^{(X)}\$X[_2B]_2 & 14\colon f^{(B)}\,{}_1\# \to f^{(B)}\bar{0}\#\bar{1} \\ 7\colon p^{(X)}\$X \to p\#\$ & 15\colon f^{(A)}\,{}_1\# \to f01 \\ 8\colon p^{(X)}\$X \to p^{(Y)}\#\$ & 16\colon f^{(B)}\,{}_1\# \to f\bar{0}\bar{1} \end{array}$$

First, M *generates two* $\#$ *bounders. Second,* M *uses rules 2 to 7 to generate the following structure*

$$a^m\#b^m z_1 z_2 \ldots z_m\#\$\phi(z_m z_{m-1} \ldots z_1)$$

where $z_i \in \{c,d\}$, $1 \leq i \leq m$, $m \geq 1$ and ϕ is a homomorphism from $\{c,d\}^*$ to $\{A, B, [_1, [_2,]_1,]_2\}^*$ such that $\phi(c) = [_1A]_1$ and $\phi(d) = [_2B]_2$. Finally, M uses rules 8 to 16 to finish the rewriting. Thus, the language generated by M is

$$L(M) = \left\{ w \,\middle|\, \begin{array}{l} w = a^n b^n z_1 z_2 \dots z_n h(z_n, i_1) h(z_{n-1}, i_2) \dots h(z_1, i_n), \\ z_i \in \{c,d\}, 1 \leq i \leq n, i_j \geq 1, 1 \leq j \leq n, n \geq 1 \end{array} \right\}$$

where h is a mapping from $\{c,d\} \times \mathbb{I}$ to $\{0, 1, \bar{0}, \bar{1}, [_1, [_2,]_1,]_2\}^*$ such that $h(c,i) = [_1 0^i 1^i]_1$ and $h(d,i) = [_2 \bar{0}^i \bar{1}^i]_2$.

For instance, M generates $aabbdc[_1 0011]_1 [_2 \bar{0}\bar{1}]_2$ in the following way

$$\begin{array}{rcll}
s\#\$ & \Rightarrow & p\#\#\$ & [1] \\
& \Rightarrow & p'a\#b\#\$ & [2] \\
& \Rightarrow & p^{(2)}a\#bd\#\$ & [4] \\
& \Rightarrow & p^{(X)}a\#bd\$X[_2B]_2 & [6] \\
& \Rightarrow & pa\#bd\#\$[_2B]_2 & [7] \\
& \Rightarrow & p'aa\#bbd\#\$[_2B]_2 & [2] \\
& \Rightarrow & p^{(1)}aa\#bbdc\#\$[_2B]_2 & [3] \\
& \Rightarrow & p^{(X)}aa\#bbdc\$X[_1A]_1[_2B]_2 & [5] \\
& \Rightarrow & p^{(Y)}aa\#bbdc\#\$[_1A]_1[_2B]_2 & [8] \\
& \Rightarrow & qaabbdc\#\$[_1A]_1[_2B]_2 & [9] \\
& \Rightarrow & faabbdc\$[_1A]_1[_2B]_2 & [10] \\
& \Rightarrow & faabbdc[_1\$A]_1[_2B]_2 & \\
& \Rightarrow & f^{(A)}aabbdc[_1\#\$]_1[_2B]_2 & [11] \\
& \Rightarrow & f^{(A)}aabbdc[_1 0\#1\$]_1[_2B]_2 & [13] \\
& \Rightarrow & faabbdc[_1 0011\$]_1[_2B]_2 & [15] \\
& \Rightarrow & faabbdc[_1 0011]_1[_2\$B]_2 & \\
& \Rightarrow & f^{(B)}aabbdc[_1 0011]_1[_2\#\$]_2 & [12] \\
& \Rightarrow & faabbdc[_1 0011]_1[_2\bar{0}\bar{1}\$]_2 & [16] \\
& \Rightarrow & faabbdc[_1 0011]_1[_2\bar{0}\bar{1}]_2\$ &
\end{array}$$

4 Results

First, we prove the identity of $\mathscr{L}(\mathrm{ST}, k)$ and $\mathscr{L}_k(\#\$\mathrm{RS})$ for every $k \geq 1$.

Lemma 4.1. *Let $k \geq 1$. Then, $\mathscr{L}(\mathrm{ST}, k) \subseteq \mathscr{L}_k(\#\$\mathrm{RS})$.*

Proof. Let $G = (V, T, K, P, S, s)$ be a state grammar. Without any loss on generality, suppose that $V \cap \{\#, \$\} = \emptyset$. Now, we construct from G a $k\#\$$-rewriting system

$$M = (Q, V', \Sigma, s', R)$$

such that $L(G, k) = L(M)$. First, we set

$$\begin{array}{rcl}
Q & = & \bigcup_{i=0}^{k} \{\langle q; l; u \rangle \mid q \in K, u \in (V-T)^i, 0 \leq l \leq k\} \\
V' & = & V \cup \{\#, \$\} \\
\Sigma & = & T \cup \{\#, \$\} \\
s' & = & \langle s; 0; S \rangle
\end{array}$$

Every state from Q holds the current G's state and the first k nonterminal symbols from the current G's sentential form. The positions of these k symbols correspond to #'s in the simulation. Additionally, it also holds a number that has a meaning of a type of state—0 is for regular state and 1 to k are for auxiliary states.

Next, we construct R. Let

$$\text{rules}(p, u) = \left\{ r \;\middle|\; \begin{array}{l} r\colon (B, p) \to (x, q) \in P, B \in ((V - T) \cap \text{alph}(u)), \\ p, q \in K, x \in V^+, u \in V^* \end{array} \right\}$$

and let g and h be two homomorphisms from V^* to $(\Sigma - \{\$\})^*$ and from V^* to $(V' - \Sigma)^*$, respectively, defined as

$$\begin{aligned} g(x) &= \begin{cases} \# & \text{for every } x \in (V - T) \\ x & \text{for every } x \in T \end{cases} \\ h(x) &= \begin{cases} x & \text{for every } x \in (V - T) \\ \varepsilon & \text{for every } x \in T \end{cases} \end{aligned}$$

Initially, set $R = \emptyset$. For every rule $(A, p) \to (x, q) \in P$ and for every state $\langle p; 0; uAv \rangle \in Q$ such that $\text{rules}(p, u) = \emptyset$ perform the following steps:

(A) If $k - |uv| \geq |h(x)|$, then add $\langle p; 0; uAv \rangle\ _{|uA|}\# \to \langle q; 0; uh(x)v \rangle g(x)$ to R.

(B) If $k - |uv| < |h(x)|$, then express v as $v = X_{m-1}X_{m-2} \dots X_0$, where $X_i \in (V' - \Sigma)$, $0 \leq i \leq m - 1$, $m = |v|$, and

   (i) for every i such that $0 \leq i \leq m - 1$, add $\langle p; i; uAv \rangle \#\$ \to \langle p; i + 1; uAv \rangle \$ X_i$ to R;

   (ii) add $\langle p; m; uAv \rangle \#\$ \to \langle q; 0; u \rangle \$ x$ to R.

Finally, for every state $\langle p; 0; u \rangle \in Q$ such that $|u| \leq k - 1$ and for every $B \in (V' - \Sigma)$ add rule

$$\langle p; 0; u \rangle \$ B \to \langle p; 0; uB \rangle \#\$$$

to R.

Due to the lack of space, we leave the proofs of the following claims to the kind reader. Both of them can be proved by induction on the number of derivation or rewriting steps, respectively.

Claim 4.2. *Let* $(S, p)\ _k{\Rightarrow}^m\ (wy, q)$ *in* G*, where* $p, q \in K$*,* $w \in T^*$*,* $y \in ((V - T)T^*)^*$*, and* $m \geq 0$*. Then,* $\langle p; 0; S \rangle \#\$ \Rightarrow^* \langle q; 0; \text{prefix}(h(y), k) \rangle w g(\alpha) \$ \beta$ *in* M*, where* $y = \alpha\beta$*,* $\alpha \in (T^*(V - T))^{|\text{prefix}(h(y),k)|}$*, and* $\beta \in V^*$*.*

Claim 4.3. *Let* $\langle p; 0; S \rangle \#\$ \Rightarrow^m \langle q; i; h(\alpha\bar{\alpha}) \rangle w g(\alpha) \$ \bar{\alpha} \beta$ *in* M*, where* $p, q \in K$*,* $0 \leq i \leq k$*,* $w \in (\Sigma - \{\#, \$\})^*$*,* $\alpha \in ((V' - \Sigma)(V' - \{\#, \$\})^*)^*$*,* $\text{occur}(\alpha, V' - \Sigma) \leq k$*,* $\bar{\alpha}, \beta \in (V' - \{\#, \$\})^*$*,* $\text{occur}(\bar{\alpha}, V' - \Sigma) = i$*, and* $m \geq 0$*. Then,* $(S, p)\ _k{\Rightarrow}^*\ (w\alpha\bar{\alpha}\beta, q)$ *in* G*.*

If we set $p = s$ and $y = \varepsilon$ in Claim 4.2, then $(S, s)\ _k{\Rightarrow}^*\ (w, q)$ in G implies $\langle s; 0; S \rangle \#\$ \Rightarrow^* \langle q; 0; \varepsilon \rangle w\$$ in M which proves $L(G, k) \subseteq L(M)$. Conversely, for $p = s$, $i = 0$, and $\alpha = \bar{\alpha} = \beta = \varepsilon$ in Claim 4.3, $\langle s; 0; S \rangle \#\$ \Rightarrow^* \langle q; 0; \varepsilon \rangle w\$$ in M implies $(S, s)\ _k{\Rightarrow}^*\ (w, q)$ in G which proves $L(M) \subseteq L(G, k)$. Hence, $L(G, k) = L(M)$ and the lemma holds. □

Lemma 4.4. *Let* $k \geq 1$*. Then,* $\mathscr{L}_k(\#\$\text{RS}) \subseteq \mathscr{L}(\text{ST}, k)$*.*

Proof. Let $M = (Q, V, \Sigma, s, R)$ be a $k\#\$$-rewriting system. Without any loss on generality, suppose that $? \notin V$ and $\#_i \notin V$, for all $1 \leq i \leq k$. From M, we construct a state grammar

$$G = (V', T, K, P, \#_1, s')$$

such that $L(M) = L(G, k)$. First, we set

$$\begin{array}{rcl} V' & = & (V - \{\#, \$\}) \cup \{\#_i \mid 1 \leq i \leq k\} \\ T & = & \Sigma - \{\#, \$\} \\ K & = & \{\langle p; i\rangle \mid p \in Q, 0 \leq i \leq k\} \\ & \cup & \{\langle p; i; [\![r, j]\!]\rangle \mid p \in Q, r \in R, 0 \leq i \leq k, 1 \leq j \leq k\} \\ & \cup & \{\langle p; i; [\![r, X]\!]\rangle \mid p \in Q, r \in R, X \in (V - \Sigma) \cup \{?\}, 0 \leq i \leq k\} \\ & \cup & \{q_{\text{fail}}\} \\ s' & = & \langle s; 1\rangle \end{array}$$

Every state from K holds the M's current state, the number of $\#$'s in the current M's configuration and occasionally the simulated rule together with information either about leftmost non-$\#$ nonterminal symbol or simulation progress. There is also a special state q_{fail} in K that brings G to configuration that makes the next derivation step in G impossible and in this way the simulation of M is abnormally stopped.

Let τ be a mapping from $(\Sigma - \{\$\})^* \times \{1, 2, \ldots, k\}$ to $(T \cup \{\#_i \mid 1 \leq i \leq k\})^*$ defined recursively as follows

- $\tau(\varepsilon, i) = \varepsilon$, for every $1 \leq i \leq k$
- $\tau(ax, i) = a\tau(x, i)$, for every $a \in (\Sigma - \{\#, \$\})$, $x \in (\Sigma - \{\$\})^*$, and $1 \leq i \leq k$
- $\tau(\#x, i) = \#_i\tau(x, i + 1)$, for every $x \in (\Sigma - \{\$\})^*$ and $1 \leq i \leq k - 1$

We are now ready to construct P. Initially, set $P = \emptyset$. For every rule $r\colon p\,{}_n\# \rightarrow qx \in R$ and for every state $\langle p; \kappa\rangle \in K$ such that $n \leq \kappa$ and $\kappa - 1 + \text{occur}(x, \#) \leq k$ perform the following steps:

(A) If $\text{occur}(x, \#) = 0$ and $\kappa - n = 0$, then add

$$(\#_\kappa, \langle p; \kappa\rangle) \rightarrow (x, \langle q; \kappa - 1\rangle)$$

to P.

(B) If $\text{occur}(x, \#) = 0$ and $\kappa - n \geq 1$, then

- add $(\#_n, \langle p; \kappa\rangle) \rightarrow (x, \langle q; \kappa - 1; [\![r, 1]\!]\rangle)$ to P;
- for every $1 \leq i \leq \kappa - n - 1$, add

  $$(\#_{n+i}, \langle q; \kappa - 1; [\![r, i]\!]\rangle) \rightarrow (\#_{n+i-1}, \langle q; \kappa - 1; [\![r, i + 1]\!]\rangle)$$

  to P;
- add $(\#_\kappa, \langle q; \kappa - 1; [\![r, \kappa - n]\!]\rangle) \rightarrow (\#_{\kappa-1}, \langle q; \kappa - 1\rangle)$ to P.

(C) If $\text{occur}(x, \#) = 1$, then add

$$(\#_n, \langle p; \kappa\rangle) \rightarrow (\tau(x, n), \langle q; \kappa\rangle)$$

to P.

(D) If $\mathrm{occur}(x, \#) \geq 2$, then

- add $(\#_n, \langle p; \kappa\rangle) \rightarrow (\#_n, \langle p; \kappa; [\![r, 1]\!]\rangle)$ to P;
- for every $0 \leq i \leq \kappa - n - 1$, add

$$(\#_{\kappa-i}, \langle p; \kappa; [\![r, i+1]\!]\rangle) \rightarrow (\#_{\kappa+\eta-i}, \langle p; \kappa; [\![r, i+2]\!]\rangle)$$

  to P, where $\eta = \mathrm{occur}(x, \#) - 1$;
- add $(\#_n, \langle p; \kappa; [\![r, \kappa - n + 1]\!]\rangle) \rightarrow (\tau(x, n), \langle q; \kappa + \eta\rangle)$ to P, where $\eta = \mathrm{occur}(x, \#) - 1$.

Next, for every rule $p\#\$ \rightarrow q\$x \in R$ and for every state $\langle p; \kappa\rangle \in K$ such that $\kappa \geq 1$, add

$$(\#_\kappa, \langle p; \kappa\rangle) \rightarrow (x, \langle q; \kappa - 1\rangle)$$

to P.

Finally, for every rule $r\colon p\$A \rightarrow q\#\$ \in R$ and for every state $\langle p; \kappa\rangle \in K$ such that $\kappa \leq k - 1$, add

- $(A, \langle p; \kappa\rangle) \rightarrow (A, \langle p; \kappa; [\![r, ?]\!]\rangle)$
- $(X, \langle p; \kappa; [\![r, ?]\!]\rangle) \rightarrow (X, \langle p; \kappa; [\![r, X]\!]\rangle)$, for all $X \in (V - \Sigma)$
- $(Y, \langle p; \kappa; [\![r, Y]\!]\rangle) \rightarrow (Y, q_{\mathrm{false}})$, for all $Y \in (V - \Sigma)$, where $Y \neq A$
- $(A, \langle p; \kappa; [\![r, A]\!]\rangle) \rightarrow (\#_{\kappa+1}, \langle q; \kappa + 1\rangle)$

to P.

As in the proof of Lemma 4.1, both following claims can be proved by induction on the number of rewriting or derivation steps, respectively, and we leave the proofs to the kind reader.

Claim 4.5. *Let $p\#\$ \Rightarrow^m qw\alpha\$\beta$ in M, where $p, q \in Q$, $w \in (\Sigma - \{\#, \$\})^*$, $\alpha \in (\{\#\}(\Sigma - \{\$\})^*)^*$, $\beta \in (V - \{\#, \$\})^*$, and $m \geq 0$. Then,*

$$(\#_1, \langle p; 1\rangle)\ {}_k\!\Rightarrow^* (w\tau(\alpha, 1)\beta, \langle q; \mathrm{occur}(\alpha, \#)\rangle)$$

in G.

Claim 4.6. *Set $\Omega = \{[\![r, X]\!] \mid r \in R, X \in (\{1, 2, \ldots, k\} \cup (V - \Sigma) \cup \{?\})\}$ and express K as $K = K_Q \cup K_\Omega \cup \{q_{\mathrm{false}}\}$, where*

$$\begin{aligned} K_Q &= \{\langle p; i\rangle \mid p \in Q, 0 \leq i \leq k\} \\ K_\Omega &= \{\langle p; i; Z\rangle \mid p \in Q, 0 \leq i \leq k, Z \in \Omega\} \end{aligned}$$

Define a binary operation $\bullet$ from $K_Q \times (\Omega \cup \{\lambda\})$ to K such that

$$\begin{aligned} \langle p; i\rangle \bullet Z &= \langle p; i; Z\rangle, \quad \text{for all } Z \in \Omega \\ \langle p; i\rangle \bullet \lambda &= \langle p; i\rangle \end{aligned}$$

Furthermore, set $N_\# = \{\#_i \mid 1 \leq i \leq k\}$ and define a homomorphism $\bar{\tau}$ from $(N_\# \cup T)$ to $(\Sigma - \{\$\})$ such that $\bar{\tau}(a) = a$ for every $a \in T$ and $\bar{\tau}(X) = \#$ for every $X \in N_\#$.

Based on a state to which G enters, the following two cases are considered:

(a) *Let* $(\#_1, \langle p, 1\rangle)\ {}_k\!\Rightarrow^m\ (w\alpha\beta, \langle q, \mathrm{occur}(\bar{\tau}(\alpha), \#)\rangle \bullet Z)$ *in* G*, where* $p, q \in Q$*,* $w \in T^*$*,* $\alpha \in (N_\#(N_\# \cup T)^*)^*$*,* $\beta \in (V - \{\#, \$\})^*$*,* $Z \in (\Omega \cup \{\lambda\})$*, and* $m \geq 0$*. Then,* $p\#\$ \Rightarrow^* qw\bar{\tau}(\alpha)\$\beta$ *in* M*.*

(b) *Let* $(\#_1, \langle p, 1\rangle)\ {}_k\!\Rightarrow^m\ (w\alpha\beta, q_{\mathrm{false}})$ *in* G*, where* $p \in Q$*,* $w \in T^*$*,* $\alpha \in (N_\#(N_\# \cup T)^*)^*$*,* $\beta \in (V - \{\#, \$\})^*$*, and* $m \geq 0$*. Then,* $p\#\$ \Rightarrow^* \bar{q}w\bar{\tau}(\alpha)\$\beta$ *in* M*, where* $\bar{q} \in Q$*,* $\beta = z_1Yz_2Az_3$*,* $Y, A \in (V - \Sigma)$*,* $Y \neq A$*,* $z_1 \in (\Sigma - \{\#, \$\})^*$*,* $z_2 \in (V - \{A, \#, \$\})^*$*,* $z_3 \in (V - \{\#, \$\})^*$*, and there is a rule* $\bar{r}\colon \bar{q}\$A \to q'\#\$ \in R$*,* $q' \in Q$*, such that* $\bar{r}$ *is not applicable on* $\bar{q}w\bar{\tau}(\alpha)\$\beta$*.*

If we set $p = s$ and $\alpha = \beta = \varepsilon$ in Claim 4.5, then $s\#\$ \Rightarrow^* qw\$$ in M implies $(\#_1, \langle s; 1\rangle)\ {}_k\!\Rightarrow^*\ (w, \langle q; 0\rangle)$ in G which proves $L(M) \subseteq L(G, k)$. Conversely, for $p = s$, $\alpha = \beta = \varepsilon$, and $Z = \lambda$ in Claim 4.6, $(\#_1, \langle s; 1\rangle)\ {}_k\!\Rightarrow^*\ (w, \langle q; 0\rangle)$ in G implies $s\#\$ \Rightarrow^* qw\$$ in M which proves $L(G, k) \subseteq L(M)$. Hence, $L(M) = L(G, k)$ and the lemma holds. □

Corollary 4.7. *Let* $k \geq 1$*. Then,* $\mathscr{L}_k(\#\$\mathrm{RS}) = \mathscr{L}(\mathrm{ST}, k)$*.*

Proof. It directly follows from Lemma 4.1 and Lemma 4.4. □

Next, we show that $\mathscr{L}_k(\#\mathrm{RS})$ is properly included in $\mathscr{L}_k(\#\$\mathrm{RS})$ for every $k \geq 1$.

Theorem 4.8. *For every* $k \geq 1$*. Then,* $\mathscr{L}_k(\#\mathrm{RS}) \subset \mathscr{L}_k(\#\$\mathrm{RS})$*.*

Proof. The inclusion $\mathscr{L}_k(\#\mathrm{RS}) \subseteq \mathscr{L}_k(\#\$\mathrm{RS})$ follows directly from the definitions of #-rewriting system of index k and $k\#\$$-rewriting system. It remains to find a language from $\mathscr{L}_k(\#\$\mathrm{RS})$ that is not contained in $\mathscr{L}_k(\#\mathrm{RS})$.

For $k = 1$, such a language is $\mathscr{D}_2$. As $\mathscr{L}_1(\#\$\mathrm{RS}) = \mathscr{L}(\mathrm{CF})$ (by [1] and Corollary 4.7), $\mathscr{D}_2 \in \mathscr{L}_1(\#\$\mathrm{RS})$, but $\mathscr{D}_2 \notin \mathscr{L}_1(\#\mathrm{RS})$ (see page 169 in [6]).

For $k \geq 2$, let $\Sigma_k = \{a_1, a_2, \ldots, a_{4k-2}\}$ be an alphabet. Define a language L_k over Σ_k as

$$L_k = \{a_1^i a_2^i \ldots a_{4k-2}^i \mid i \geq 1\}.$$

By Theorem 4 in [1], $L_k \in \mathscr{L}(\mathrm{ST}, k)$ and since $\mathscr{L}_k(\#\$\mathrm{RS}) = \mathscr{L}(\mathrm{ST}, k)$, $L_k \in \mathscr{L}_k(\#\$\mathrm{RS})$ as well.

It is easy to see that matrix grammars of finite index k generates the same language family as $\mathscr{L}_k(\#\mathrm{RS})$ (see [3] and Theorem 3.1.2 on page 155 in [6]). By an application of pumping lemma for matrix grammars of finite index (see Lemma 3.1.6 on page 159 in [6]), we can proof that $L_k \notin \mathscr{L}_k(\#\mathrm{RS})$. Assume that $L_k \in \mathscr{L}_k(\#\mathrm{RS})$. Then there exists $z \in L_k$ such that

$$z = u_1v_1w_1x_1u_2v_2w_2x_2 \ldots u_lv_lw_lx_lu_{l+1}$$

with $l \leq k$, $|v_1x_1v_2x_2 \ldots v_lx_l| > 0$, and

$$u_1v_1^iw_1x_1^iu_2v_2^iw_2x_2^i \ldots u_lv_l^iw_lx_l^iu_{l+1} \in L_k$$

for every $i \geq 1$. Now, consider the following cases:

- There exists $y \in \{v_1, x_1, v_2, x_2, \ldots, v_l, x_l\}$ such that $\mathrm{card}(\mathrm{alph}(y)) \geq 2$. In this case, there exists $i \geq 1$ such that

$$u_1v_1^iw_1x_1^iu_2v_2^iw_2x_2^i \ldots u_lv_l^iw_lx_l^iu_{l+1} \notin L_k.$$

$$\begin{array}{ccccccc}
\mathscr{L}_1(\#\$\text{RS}) & \subset & \mathscr{L}_2(\#\$\text{RS}) & \subset & \cdots & \subset & \mathscr{L}_k(\#\$\text{RS}) \\
\cup & & \cup & & & & \cup \\
\mathscr{L}_1(\#\text{RS}) & \subset & \mathscr{L}_2(\#\text{RS}) & \subset & \cdots & \subset & \mathscr{L}_k(\#\text{RS})
\end{array}$$

Fig. 1: The relations between #-rewriting systems with finite index and #$-rewriting systems.

- All $v_1, x_1, v_2, x_2, \dots, v_l, x_l$ are strings over one-letter alphabet. As for $k \geq 2$ it is always true that $4k - 2 > 2k$, there will be always symbols from $\text{alph}(z)$ that are not contained in $\text{alph}(v_1 x_1 v_2 x_2 \dots v_l x_l)$. Hence there must exist $i \geq 1$ such that

$$u_1 v_1^i w_1 x_1^i u_2 v_2^i w_2 x_2^i \dots u_l v_l^i w_l x_l^i u_{l+1} \notin L_k.$$

Such $z \in L_k$ does not exist and therefore $L_k \notin \mathscr{L}_k(\#\text{RS})$ for every $k \geq 2$. □

The relationship between infinite hierarchies of #-rewriting systems of finite index and $k\#\$$-rewriting systems is summed in Figure 1.

5 Conclusion

Since we have new characterization of $\mathscr{L}(\text{ST}, k)$, for some $k \geq 1$, in the future investigation, we can naturally study the relationship between $k\#\$$-rewriting systems and generalized #-rewriting systems (studied in Sections 4.1.4 and 5.1.3 of [7]).

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Appendix

In this section, we recall the claims from the paper together with their proofs that have to be omitted from the paper due to the page limit. Since we assume that the appendix will not be part of the final version of the paper, we include no references into the appendix from the text of the paper.

Claim 4.2. *Let $(S,p)\ {}_k{\Rightarrow}^m\ (wy,q)$ in G, where $p,q \in K$, $w \in T^*$, $y \in ((V-T)T^*)^*$, and $m \geq 0$. Then, $\langle p;0;S\rangle\#\$ \Rightarrow^* \langle q;0;\mathrm{prefix}(h(y),k)\rangle wg(\alpha)\$\beta$ in M, where $y=\alpha\beta$, $\alpha \in (T^*(V-T))^{|\mathrm{prefix}(h(y),k)|}$, and $\beta \in V^*$.*

Proof. This claim is proved by induction on $m \geq 0$.

Basis. Let $m=0$, so $(S,p)\ {}_k{\Rightarrow}^0\ (S,p)$ in G, $w=\varepsilon$ and $y=S$. Then,

$$\langle p;0;S\rangle\#\$ \Rightarrow^0 \langle p;0;\mathrm{prefix}(h(S),k)\rangle g(\alpha)\$\beta$$

in M. Since $\mathrm{prefix}(h(S),k)=S$, it holds that $\alpha=S$ and $\beta=\varepsilon$, so we have

$$\langle p;0;\mathrm{prefix}(h(S),k)\rangle g(\alpha)\$\beta = \langle p;0;S\rangle\#\$$$

and the basis holds.

Induction Hypothesis. Suppose that the claim holds for all $0 \leq m \leq l$, where l is a non-negative integer.

Induction Step. Let $(S,p)\ {}_k{\Rightarrow}^{l+1}\ (wy,q)$ in G, where $p,q \in K$, $w \in T^*$, and $y \in ((V-T)T^*)^*$. Since $l+1 \geq 1$, express $(S,p)\ {}_k{\Rightarrow}^{l+1}\ (wy,q)$ as $(S,p)\ {}_k{\Rightarrow}^{l}\ (w'uAv,t)\ {}_k{\Rightarrow}\ (w'uxv,q)$, where $t \in K$, $w' \in T^*$, $u \in ((V-T)T^*)^*$, $|h(u)| \leq k-1$, $A \in (V-T)$, $x,v \in V^*$, $(A,t) \to (x,q) \in P$, $w=w'\hat{w}$, and $\hat{w}y=uxv$ with $\hat{w} \in T^*$. By the induction hypothesis, $\langle p;0;S\rangle\#\$ \Rightarrow^* \langle t;0;\mathrm{prefix}(h(uAv),k)\rangle w'g(uAz)\$\bar{z}$ in M, where $v=z\bar{z}$, $z \in (T^*(V-T))^{|\mathrm{prefix}(h(uAv),k)|-|h(uA)|}$, and $\bar{z} \in V^*$. As $(A,t) \to (x,q) \in P$, the following rules were added to R during its construction, based on the relation between $k-|\mathrm{prefix}(h(uAv),k)|+1$ and $|h(x)|$:

(A) $k-|\mathrm{prefix}(h(uAv),k)|+1 \geq |h(x)|$. Then, based on construction of R,

$$\langle t;0;\mathrm{prefix}(h(uAv),k)\rangle\ {}_{|h(uA)|}\# \to \langle q;0;\mathrm{prefix}(h(uxv),k)\rangle g(x) \in R.$$

Now, we must consider the following three cases:

1. $\mathrm{occur}(uAv,V-T) \geq k$ and $\mathrm{occur}(x,V-T)=1$. Then,

   $$\langle t;0;\mathrm{prefix}(h(uAv),k)\rangle w'g(uAz)\$\bar{z} \Rightarrow^* \langle q;0;\mathrm{prefix}(h(uxv),k)\rangle w'g(uxz)\$\bar{z}$$

   in M. Clearly, uxz can be expressed as $\hat{w}\alpha$. As $h(uxv)=h(\hat{w}y)=h(y)$, $g(uxz)=g(\hat{w}\alpha)=\hat{w}g(\alpha)$, $w=w'\hat{w}$, and $\beta=\bar{z}$, we have

   $$\langle q;0;\mathrm{prefix}(h(uxv),k)\rangle w'g(uxz)\$\bar{z} = \langle q;0;\mathrm{prefix}(h(y),k)\rangle wg(\alpha)\$\beta.$$

2. $\mathrm{occur}(uAv,V-T) \geq k$ and $\mathrm{occur}(x,V-T)=0$. Then,

   $$\langle t;0;\mathrm{prefix}(h(uAv),k)\rangle w'g(uAz)\$\bar{z} \Rightarrow^* \langle q;0;h(uxz)\rangle w'g(uxz)\$\bar{z}$$

in M and since for every $B \in (V' - \Sigma)$,

$$\langle q; 0; h(uxz)\rangle \$B \to \langle q; 0; h(uxzB)\rangle \#\$ \in R$$

there exists

$$\langle q; 0; h(uxz)\rangle w' g(uxz)\$\bar{z} \Rightarrow^* \langle q; 0; \text{prefix}(h(uxv), k)\rangle w' g(uxz')\$\bar{z}'$$

in M with $z' \in (T^*(V-T))^{|\text{prefix}(h(uxv),k)|-|h(ux)|}$, $\bar{z}' \in V^*$, and $v = z'\bar{z}'$. Again, we can express uxz' as $\hat{w}\alpha$ and with $\beta = \bar{z}'$, we have

$$\langle q; 0; \text{prefix}(h(uxv), k)\rangle w' g(uxz')\$\bar{z}' = \langle q; 0; \text{prefix}(h(y), k)\rangle w g(\alpha)\$\beta.$$

3. occur$(uAv, V - T) < k$ and occur$(uxv, V - T) \leq k$. Then,

$$\langle t; 0; \text{prefix}(h(uAv), k)\rangle w' g(uAz)\$\bar{z} \Rightarrow^* \langle q; 0; \text{prefix}(h(uxv), k)\rangle w' g(uxz)\$\bar{z}$$

in M. As in previous cases, set $\hat{w}\alpha = uxz$ and $\beta = \bar{z}$. Therefore,

$$\langle q; 0; \text{prefix}(h(uxv), k)\rangle w' g(uxz)\$\bar{z} = \langle q; 0; \text{prefix}(h(y), k)\rangle w g(\alpha)\$\beta.$$

(B) $k - |\text{prefix}(h(uAv), k)| + 1 < |h(x)|$. Express prefix$(h(uAv), k)$ as $h(uA)\delta$, where $\delta = D_1 D_2 \ldots D_{|\delta|}$, $D_i \in (V-T)$, $1 \leq i \leq |\delta|$. Furthermore, express z as $d_1 D_1 d_2 D_2 \ldots d_{|\delta|} D_{|\delta|}$, where $d_i \in T^*$, $1 \leq i \leq |\delta|$. As there are following rules in R introduced by step (B.i) of the construction of R

$$\begin{array}{rcl}
\langle t; 0; h(uA)\delta\rangle \#\$ & \to & \langle t; 1; h(uA)\delta\rangle \$D_{|\delta|} \\
\langle t; 1; h(uA)\delta\rangle \#\$ & \to & \langle t; 2; h(uA)\delta\rangle \$D_{|\delta|-1} \\
 & \vdots & \\
\langle t; |\delta| - 1; h(uA)\delta\rangle \#\$ & \to & \langle t; |\delta|; h(uA)\delta\rangle \$D_1
\end{array}$$

there exists

$$\begin{array}{rl}
 & \langle t; 0; \text{prefix}(h(uAv), k)\rangle w' g(uAd_1 D_1 \ldots d_{|\delta|} D_{|\delta|})\$\bar{z} \\
\Rightarrow^* & \langle t; 1; h(uA)\delta\rangle w' g(uAd_1 D_1 \ldots d_{|\delta|-1} D_{|\delta|-1})\$d_{|\delta|} D_{|\delta|}\bar{z} \\
\vdots & \\
\Rightarrow^* & \langle t; |\delta|; h(uA)\delta\rangle w' g(uA)\$z\bar{z}
\end{array}$$

in M. Since step (B.ii) of the construction of R introduces a rule

$$\langle t; |\delta|; h(uA)\delta\rangle \#\$ \to \langle q; 0; h(u)\rangle \$x$$

to R, there also exists

$$\langle t; |\delta|; h(uA)\delta\rangle w' g(uA)\$z\bar{z} \Rightarrow^* \langle q; 0; h(u)\rangle w' g(u)\$xz\bar{z}$$

in M. Finally, for every state $\langle o; 0; \gamma\rangle \in Q$ such that $|\gamma| \leq k - 1$ and for every $B \in (V' - \Sigma)$, there is a rule

$$\langle o; 0; \gamma\rangle \$B \to \langle o; 0; \gamma B\rangle \#\$$$

in R and hence

$$\langle q; 0; h(u)\rangle w' g(u)\$xz\bar{z} \Rightarrow^* \langle q; 0; \text{prefix}(h(uxv), k)\rangle w' g(uz')\$\bar{z}'$$

in M, where $z' \in (T^*(V-T))^{|\text{prefix}(h(uxv),k)|-|h(u)|}$, $\bar{z}' \in V^*$, and $xv = z'\bar{z}'$. Express uz' as $\hat{w}\alpha$ and set $\bar{z}' = \beta$. Thus,

$$\langle q; 0; \text{prefix}(h(uxv), k)\rangle w' g(uz')\$\bar{z}' = \langle q; 0; \text{prefix}(h(y), k)\rangle w g(\alpha)\$\beta$$

and the claim holds.

□

Claim 4.3. *Let* $\langle p;0;S\rangle\#\$ \Rightarrow^m \langle q;i;h(\alpha\bar{\alpha})\rangle wg(\alpha)\$\bar{\alpha}\beta$ *in* M*, where* $p,q \in K$*,* $0 \le i \le k$*,* $w \in (\Sigma - \{\#,\$\})^*$*,* $\alpha \in ((V' - \Sigma)(V' - \{\#,\$\})^*)^*$*,* $\text{occur}(\alpha, V' - \Sigma) \le k$*,* $\bar{\alpha}, \beta \in (V' - \{\#,\$\})^*$*,* $\text{occur}(\bar{\alpha}, V' - \Sigma) = i$*, and* $m \ge 0$*. Then,* $(S,p)\ {}_k\!\Rightarrow^* (w\alpha\bar{\alpha}\beta, q)$ *in* G*.*

Proof. This claim is proved by induction on $m \ge 0$.

Basis. Let $m = 0$, so $\langle p;0;S\rangle\#\$ \Rightarrow^0 \langle p;0;S\rangle\#\$$ in M, $w = \varepsilon$, $\alpha = S$, $\bar{\alpha} = \varepsilon$ and $\beta = \varepsilon$. Then, $(S,p)\ {}_k\!\Rightarrow^* (S,p)$ in G and the basis holds.

Induction Hypothesis. Suppose that the claim holds for all $0 \le m \le l$, where l is a non-negative integer.

Induction Step. Let $\langle p;0;S\rangle\#\$ \Rightarrow^{l+1} \langle q;\hat{\jmath};h(\alpha\bar{\alpha})\rangle wg(\alpha)\$\bar{\alpha}\beta$ in M, where $p,q \in K$, $0 \le \hat{\jmath} \le k$, $w \in (\Sigma - \{\#,\$\})^*$, $\alpha \in ((V' - \Sigma)(V' - \{\#,\$\})^*)^*$, $\text{occur}(\alpha, V' - \Sigma) \le k$, and $\bar{\alpha}, \beta \in (V' - \{\#,\$\})^*$, $\text{occur}(\bar{\alpha}, V' - \Sigma) = \hat{\jmath}$. Since $l+1 \ge 1$, express $\langle p;0;S\rangle\#\$ \Rightarrow^{l+1} \langle q;\hat{\jmath};h(\alpha\bar{\alpha})\rangle wg(\alpha)\$\bar{\alpha}\beta$ as

$$\langle p;0;S\rangle\#\$ \Rightarrow^l \langle t;\hat{\imath};h(uAv\hat{\alpha})\rangle w'g(uAv)\$\hat{\alpha}z \Rightarrow \langle q;\hat{\jmath};h(uxv'\bar{\alpha})\rangle w'g(uxv')\$\bar{\alpha}\beta$$

where $t \in K$, $0 \le \hat{\imath} \le k$, $w' \in (\Sigma - \{\#,\$\})^*$, $A \in (V' - \Sigma) \cup \{\varepsilon\}$, $u \in ((V' - \Sigma)(V' - \{\#,\$\})^*)^*$, $x, v, v', \hat{\alpha}, z \in (V' - \{\#,\$\})^*$, $\text{occur}(uAv, V' - \Sigma) \le k$, $\text{occur}(\hat{\alpha}, V' - \Sigma) = \hat{\imath}$, $w = w'\hat{w}$, and $\hat{w}\alpha = uxv'$ with $\hat{w} \in (\Sigma - \{\#,\$\})^*$. By the induction hypothesis, $(S,p)\ {}_k\!\Rightarrow^* (w'uAv\hat{\alpha}z, t)$ in G. M can rewrite $\langle t;\hat{\imath};h(uAv\hat{\alpha})\rangle w'g(uAv)\$\hat{\alpha}z$ to $\langle q;\hat{\jmath};h(uxv'\bar{\alpha})\rangle w'g(uxv')\$\bar{\alpha}\beta$ according to the one of following cases:

Case 1. $t = q$, $\hat{\imath} = \hat{\jmath}$, $A = x$, $v = v'\bar{v}$, $\bar{\alpha} = \bar{v}\hat{\alpha}$, $z = \beta$, and $\bar{v} \in (\Sigma - \{\#,\$\})^*$. In this case, we have

$$(w'uAv'\bar{v}\hat{\alpha}z, t)\ {}_k\!\Rightarrow^* (w\alpha\bar{\alpha}\beta, q)$$

in G.

Case 2. $t = q$, $\hat{\imath} = \hat{\jmath}$, $A = x$, $v' = v\bar{v}$, $\hat{\alpha} = \bar{v}\bar{\alpha}$, $z = \beta$, and $\bar{v} \in (\Sigma - \{\#,\$\})^*$. In this case, we have

$$(w'uAv\bar{v}\bar{\alpha}z, t)\ {}_k\!\Rightarrow^* (w\alpha\bar{\alpha}\beta, q)$$

in G.

Case 3. $t = q$, $\hat{\imath} = \hat{\jmath} = 0$, $A = x$, $\hat{\alpha} = \bar{\alpha} = \varepsilon$, $z = B\beta$, $v' = vB$, and $B \in (V' - \Sigma)$. In this case, the rewriting step was performed by rule $\langle q;0;h(uv)\rangle\$B \to \langle q;0;h(uvB)\rangle\#\$$ which was introduced to R for every state $\langle q;0;h(uv)\rangle \in Q$, $|h(uv)| \le k-1$, and for every $B \in (V' - \Sigma)$. Since this rule only changes symbol B to $\#$ and updates M's state to remember $\#$'s meaning, we have

$$(w'uAv\hat{\alpha}B\beta, t)\ {}_k\!\Rightarrow^* (w'uxv'\bar{\alpha}\beta, q)\ {}_k\!\Rightarrow^* (w\alpha\bar{\alpha}\beta, q)$$

in G.

Case 4. $\hat{\imath} = \hat{\jmath} = 0$, $v = v'$, $\hat{\alpha} = \bar{\alpha}$ and $z = \beta$. In this case, $\langle t; 0; h(uAv)\rangle_{|h(uA)|}\# \rightarrow \langle q; 0; h(uxv)\rangle g(x) \in R$ was used, so there exists a rule $(A, t) \rightarrow (x, q)$ in P such that rules$(t, u) = \emptyset$ and hence

$$(w'uAv\hat{\alpha}z, t)\ {}_k\!\Rightarrow (w'uxv\hat{\alpha}z, q)\ {}_k\!\Rightarrow^* (w\alpha\bar{\alpha}\beta, q)$$

in G.

Case 5. $t = q$, $\hat{\imath} = \hat{\jmath} - 1$, $A = x$, $v = v'B$, $\bar{\alpha} = B\hat{\alpha}$, $z = \beta$, and $B \in (V' - \Sigma)$. In this case, $\langle t; \hat{\imath}; h(uAv\hat{\alpha})\rangle\#\$ \rightarrow \langle t; \hat{\imath}+1; h(uAv'\bar{\alpha})\rangle\$B \in R$ introduced in step (B.i) was used and hence

$$(w'uAv'B\hat{\alpha}z, t)\ {}_k\!\Rightarrow^* (w\alpha\bar{\alpha}\beta, q)$$

in G.

Case 6. $\hat{\jmath} = 0$, $v = x = v' = \bar{\alpha} = \varepsilon$, $\beta = y\hat{\alpha}z$, and $y \in (V' - \{\#, \$\})^*$. In this case, $\langle t; \hat{\imath}; h(uA\hat{\alpha})\rangle\#\$ \rightarrow \langle q; 0; h(u)\rangle\$y \in R$ introduced in step (B.ii) was used which means that there is a rule $(A, t) \rightarrow (y, q)$ in P such that rules$(t, u) = \emptyset$. Therefore,

$$(w'uAv\hat{\alpha}z, t)\ {}_k\!\Rightarrow (w'uyv\hat{\alpha}z, q)\ {}_k\!\Rightarrow^* (w\alpha\bar{\alpha}\beta, q)$$

in G, which completes the induction step.

□

Claim 4.5. *Let* $p\#\$ \Rightarrow^m qw\alpha\$\beta$ *in* M*, where* $p, q \in Q$*,* $w \in (\Sigma - \{\#, \$\})^*$*,* $\alpha \in (\{\#\}(\Sigma - \{\$\})^*)^*$*,* $\beta \in (V - \{\#, \$\})^*$*, and* $m \geq 0$*. Then,*

$$(\#_1, \langle p; 1\rangle)\ {}_k\!\Rightarrow^* (w\tau(\alpha, 1)\beta, \langle q; \text{occur}(\alpha, \#)\rangle)$$

in G*.*

Proof. This claim is proved by induction on $m \geq 0$.

Basis. Let $m = 0$, $p\#\$ \Rightarrow^0 p\#\$$ in M, $w = \varepsilon$, $\alpha = \#$, and $\beta = \varepsilon$. Then, $(\#_1, \langle p; 1\rangle)\ {}_k\!\Rightarrow^* (\#_1, \langle p; 1\rangle)$ in G and the basis holds.

Induction Hypothesis. Suppose that the claim holds for all $0 \leq m \leq l$, where l is a non-negative integer.

Induction Step. Let $p\#\$ \Rightarrow^{l+1} qw\alpha\$\beta$ in M, where $p, q \in Q$, $w \in (\Sigma - \{\#, \$\})^*$, $\alpha \in (\{\#\}(\Sigma - \{\$\})^*)^*$, and $\beta \in (V - \{\#, \$\})^*$. Since $l + 1 \geq 1$, express $p\#\$ \Rightarrow^{l+1} qw\alpha\$\beta$ as

$$p\#\$ \Rightarrow^l tw'uA_\#v\$z \Rightarrow qw'uxv'\$\beta$$

where $t \in Q$, $w' \in (\Sigma - \{\#, \$\})^*$, $u \in (\{\#\}(\Sigma - \{\$\})^*)^*$, $A_\# \in \{\#, \varepsilon\}$, $x, v, v' \in (\Sigma - \{\$\})^*$, $z \in (V - \{\#, \$\})^*$, $w = w'\hat{w}$, and $\hat{w}\alpha = uxv'$ with $\hat{w} \in (\Sigma - \{\#, \$\})^*$. By the induction hypothesis, $(\#_1, \langle p; 1\rangle)\ {}_k\!\Rightarrow^* (w'\tau(uA_\#v, 1)z, \langle t; \text{occur}(uA_\#v, \#)\rangle)$ in G. M can perform $tw'uA_\#v\$z \Rightarrow qw'uxv'\$\beta$ in the following ways:

(I) $tw'uA_{\#}v\$z \Rightarrow qw'uxv'\$\beta$, where $t = q$, $A_{\#} = x$, $v = v'\bar{v}$, $\beta = \bar{v}z$, and $\bar{v} \in (\Sigma - \{\#, \$\})^*$. This rewriting step only transfers terminal symbols from the left to the right, relatively to \$. Clearly,

$$\begin{aligned} & (w'\tau(uA_{\#}v, 1)z, \langle t; \mathrm{occur}(uA_{\#}v, \#)\rangle) \\ {}_k\Rightarrow^* \ & (w'\tau(uxv', 1)\beta, \langle q; \mathrm{occur}(uxv', \#)\rangle) \\ {}_k\Rightarrow^* \ & (w\tau(\alpha, 1)\beta, \langle q; \mathrm{occur}(\alpha, \#)\rangle) \end{aligned}$$

in G.

(II) $tw'uA_{\#}v\$z \Rightarrow qw'uxv'\$\beta$, where $t = q$, $A_{\#} = x$, $v' = v\bar{v}$, $\bar{v} = \bar{v}\beta$, and $\bar{v} \in (\Sigma - \{\#, \$\})^*$. This rewriting step only transfers terminal symbols from the right to the left, relatively to \$. Thus,

$$\begin{aligned} & (w'\tau(uA_{\#}v, 1)z, \langle t; \mathrm{occur}(uA_{\#}v, \#)\rangle) \\ {}_k\Rightarrow^* \ & (w'\tau(uxv', 1)\beta, \langle q; \mathrm{occur}(uxv', \#)\rangle) \\ {}_k\Rightarrow^* \ & (w\tau(\alpha, 1)\beta, \langle q; \mathrm{occur}(\alpha, \#)\rangle) \end{aligned}$$

in G.

(III) $tw'uA_{\#}v\$z \Rightarrow qw'uxv'\$\beta$, where $A_{\#} = \#$, $v = v'$, $z = \beta$, $\mathrm{occur}(u, \#) = n - 1$, and $1 \leq n \leq k$. This rewriting step was performed by applying a rule $r\colon t\,{}_n\# \to qx \in R$. Set $\kappa = \mathrm{occur}(uA_{\#}v, \#)$. G simulates application of r in the following ways:

(1) $\kappa - n = 0$ and $\mathrm{occur}(x, \#) = 0$. Then, $(\#_\kappa, \langle t; \kappa\rangle) \to (x, \langle q; \kappa - 1\rangle) \in P$, so

$$\begin{aligned} & (w'\tau(uA_{\#}v, 1)z, \langle t; \mathrm{occur}(uA_{\#}v, \#)\rangle) \\ {}_k\Rightarrow \ & (w'\tau(uxv', 1)\beta, \langle q; \mathrm{occur}(uxv', \#)\rangle) \\ {}_k\Rightarrow^* \ & (w\tau(\alpha, 1)\beta, \langle q; \mathrm{occur}(\alpha, \#)\rangle) \end{aligned}$$

in G, where $\mathrm{occur}(uA_{\#}v, \#) - 1 = \kappa - 1 = \mathrm{occur}(uxv', \#)$.

(2) $\kappa - n \geq 1$ and $\mathrm{occur}(x, \#) = 0$. Then, $(\#_n, \langle t; \kappa\rangle) \to (x, \langle q; \kappa - 1; [\![r, 1]\!]\rangle) \in P$, so

$$\begin{aligned} & (w'\tau(uA_{\#}v, 1)z, \langle t; \mathrm{occur}(uA_{\#}v, \#)\rangle) \\ {}_k\Rightarrow \ & (w'\tau(ux, 1)\tau(v', n + 1)\beta, \langle q; \kappa - 1; [\![r, 1]\!]\rangle) \end{aligned}$$

in G. If $\kappa - n \geq 2$, then there are rules

$$\begin{aligned} (\#_{n+1}, \langle q; \kappa - 1; [\![r, 1]\!]\rangle) &\to (\#_n, \langle q; \kappa - 1; [\![r, 2]\!]\rangle) \\ (\#_{n+2}, \langle q; \kappa - 1; [\![r, 2]\!]\rangle) &\to (\#_{n+1}, \langle q; \kappa - 1; [\![r, 3]\!]\rangle) \\ &\vdots \\ (\#_{\kappa-1}, \langle q; \kappa - 1; [\![r, \kappa - n - 1]\!]\rangle) &\to (\#_{\kappa-2}, \langle q; \kappa - 1; [\![r, \kappa - n]\!]\rangle) \end{aligned}$$

in P. Set $\bar{\eta} = \kappa - n = \mathrm{occur}(v', \#)$. Since $\kappa - n \geq 2$, it follows that also $\bar{\eta} \geq 2$, so v' can be expressed as $v' = \delta_1\delta_2 \dots \delta_{\bar{\eta}}$, where $\delta_i \in (\Sigma - \{\#, \$\})^*\{\#\}(\Sigma - \{\#, \$\})^*$, for all $1 \leq i \leq \bar{\eta}$, and G can perform the following sequence of derivation steps:

$$\begin{aligned} & (w'\tau(ux, 1)\tau(\delta_1\delta_2 \dots \delta_{\bar{\eta}}, n + 1)\beta, \langle q; \kappa - 1; [\![r, 1]\!]\rangle) \\ {}_k\Rightarrow \ & (w'\tau(ux\delta_1, 1)\tau(\delta_2\delta_3 \dots \delta_{\bar{\eta}}, n + 2)\beta, \langle q; \kappa - 1; [\![r, 2]\!]\rangle) \\ {}_k\Rightarrow \ & (w'\tau(ux\delta_1\delta_2, 1)\tau(\delta_3\delta_4 \dots \delta_{\bar{\eta}}, n + 3)\beta, \langle q; \kappa - 1; [\![r, 3]\!]\rangle) \\ & \vdots \\ {}_k\Rightarrow \ & (w'\tau(ux\delta_1\delta_2 \dots \delta_{\bar{\eta}-1}, 1)\tau(\delta_{\bar{\eta}}, n + \bar{\eta})\beta, \langle q; \kappa - 1; [\![r, \bar{\eta}]\!]\rangle) \end{aligned}$$

The simulation of r is finished by the rule

$$(\#_\kappa, \langle q; \kappa-1; [\![r, \kappa-n]\!]\rangle) \rightarrow (\#_{\kappa-1}, \langle q; \kappa-1\rangle) \in P$$

If $\kappa - n \geq 2$, then

$$\begin{array}{ll} & (w'\tau(ux\delta_1\delta_2 \ldots \delta_{\bar{\eta}-1}, 1)\tau(\delta_{\bar{\eta}}, n+\bar{\eta})\beta, \langle q; \kappa-1; [\![r, \bar{\eta}]\!]\rangle) \\ {}_k\Rightarrow & (w'\tau(ux\delta_1\delta_2 \ldots \delta_{\bar{\eta}}, 1)\beta, \langle q; \kappa-1\rangle) \\ {}_k\Rightarrow^* & (w\tau(\alpha)\beta, \langle q; \mathrm{occur}(\alpha, \#)\rangle) \end{array}$$

in G. Otherwise, $\kappa - n = 1$, and

$$\begin{array}{ll} & (w'\tau(ux, 1)\tau(v', n+1)\beta, \langle q; \kappa-1; [\![r, 1]\!]\rangle) \\ {}_k\Rightarrow & (w'\tau(uxv', 1)\beta, \langle q; \kappa-1\rangle) \\ {}_k\Rightarrow^* & (w\tau(\alpha)\beta, \langle q; \mathrm{occur}(\alpha, \#)\rangle) \end{array}$$

in G, where $\tau(v', n+1) = v_1\#_\kappa v_2$, $v_1, v_2 \in T^*$, and $\mathrm{occur}(uxv', \#) = \kappa - 1 = \mathrm{occur}(\alpha, \#)$.

(3) $\mathrm{occur}(x, \#) = 1$. Then, $(\#_n, \langle t; \kappa\rangle) \rightarrow (\tau(x, n), \langle q; \kappa\rangle) \in P$, so

$$\begin{array}{ll} & (w'\tau(uA_\# v, 1)z, \langle t; \mathrm{occur}(uA_\# v, \#)\rangle) \\ {}_k\Rightarrow & (w'\tau(uxv, 1)z, \langle q; \mathrm{occur}(uxv, \#)\rangle) \\ {}_k\Rightarrow^* & (w\tau(\alpha, 1)\beta, \langle q; \mathrm{occur}(\alpha, \#)\rangle) \end{array}$$

in G.

(4) $\mathrm{occur}(x, \#) \geq 2$. Set $\eta = \mathrm{occur}(x, \#) - 1$. If $\kappa - n \geq 1$, then the following rules were introduced to P:

$$\begin{array}{rcl} (\#_n, \langle t; \kappa\rangle) & \rightarrow & (\#_n, \langle t; \kappa; [\![r, 1]\!]\rangle) \\ (\#_\kappa, \langle t; \kappa; [\![r, 1]\!]\rangle) & \rightarrow & (\#_{\kappa+\eta}, \langle t; \kappa; [\![r, 2]\!]\rangle) \\ (\#_{\kappa-1}, \langle t; \kappa; [\![r, 2]\!]\rangle) & \rightarrow & (\#_{\kappa+\eta-1}, \langle t; \kappa; [\![r, 3]\!]\rangle) \\ & \vdots & \\ (\#_{n+1}, \langle t; \kappa; [\![r, \kappa-n]\!]\rangle) & \rightarrow & (\#_{\eta+n+1}, \langle t; \kappa; [\![r, \kappa-n+1]\!]\rangle) \\ (\#_n, \langle t; \kappa; [\![r, \kappa-n+1]\!]\rangle) & \rightarrow & (\tau(x, n), \langle q; \kappa+\eta\rangle) \end{array}$$

Set $\bar{\eta} = \kappa - n$ and express v as $v = \delta_1\delta_2 \ldots \delta_{\bar{\eta}}$, where

$$\delta_i \in (\Sigma - \{\#, \$\})^*\{\#\}(\Sigma - \{\#, \$\})^*$$

for all $1 \leq i \leq \bar{\eta}$. Then, G is able to perform the following sequence of derivation steps:

$$\begin{array}{ll} & (w'\tau(uA_\# v, 1)z, \langle t; \kappa\rangle) \\ {}_k\Rightarrow & (w'\tau(uA_\#\delta_1\delta_2 \ldots \delta_{\bar{\eta}}, 1)z, \langle t; \kappa; [\![r, 1]\!]\rangle) \\ {}_k\Rightarrow & (w'\tau(uA_\#\delta_1\delta_2 \ldots \delta_{\bar{\eta}-1}, 1)\tau(\delta_{\bar{\eta}}, \kappa+\eta)z, \langle t; \kappa; [\![r, 2]\!]\rangle) \\ {}_k\Rightarrow & (w'\tau(uA_\#\delta_1\delta_2 \ldots \delta_{\bar{\eta}-2}, 1)\tau(\delta_{\bar{\eta}-1}\delta_{\bar{\eta}}, \kappa+\eta-1)z, \langle t; \kappa; [\![r, 3]\!]\rangle) \\ \vdots_k\Rightarrow & \\ {}_k\Rightarrow & (w'\tau(uA_\#, 1)\tau(\delta_1\delta_2 \ldots \delta_{\bar{\eta}}, \eta+n+1)z, \langle t; \kappa; [\![r, \kappa-n+1]\!]\rangle) \\ {}_k\Rightarrow & (w'\tau(uxv, 1)z, \langle q; \kappa+\eta\rangle) \\ {}_k\Rightarrow^* & (w\tau(\alpha, 1)\beta, \langle q; \mathrm{occur}(\alpha, \#)\rangle) \end{array}$$

Observe that $\kappa+\eta-(\kappa-n-1) = \eta+n+1$ and $\mathrm{occur}(uxv, \#) = \kappa+\eta = \mathrm{occur}(\alpha, \#)$ since r removes one and add $\mathrm{occur}(x, \#)$ $\#$ symbols.

If $\kappa - n = 0$, only the rules

$$\begin{array}{rcl} (\#_n, \langle t;\kappa\rangle) & \rightarrow & (\#_n, \langle t;\kappa;[\![r,1]\!]\rangle) \\ (\#_n, \langle t;\kappa;[\![r,1]\!]\rangle) & \rightarrow & (\tau(x,n), \langle q;\kappa+\eta\rangle) \end{array}$$

from P were used during the simulation of r by G as the following sequence of derivation steps in G demonstrates:

$$\begin{array}{ll} & (w'\tau(uA_\# v,1)z, \langle t;\kappa\rangle) \\ {}_k\Rightarrow & (w'\tau(uA_\# v,1)z, \langle t;\kappa;[\![r,1]\!]\rangle) \\ {}_k\Rightarrow & (w'\tau(uxv,1)z, \langle q;\kappa+\eta\rangle) \\ {}_k\Rightarrow^* & (w\tau(\alpha,1)\beta, \langle q;\mathrm{occur}(\alpha,\#)\rangle) \end{array}$$

(IV) $tw'uA_\# v\$z \Rightarrow qw'uxv'\$\beta$, where $A_\# = \#$, $x = v = v' = \varepsilon$, $\beta = yz$, and $y \in (V - \{\#,\$\})^*$. Then, a rule $t\#\$ \rightarrow q\$y \in R$ was applied and hence $(\#_\kappa, \langle t;\kappa\rangle) \rightarrow (x, \langle q;\kappa-1\rangle) \in P$. This gives

$$\begin{array}{ll} & (w'\tau(uA_\# v,1)z, \langle t;\kappa\rangle) \\ {}_k\Rightarrow & (w'\tau(u,1)yz, \langle q;\kappa-1\rangle) \\ {}_k\Rightarrow^* & (w\tau(\alpha,1)\beta, \langle q;\mathrm{occur}(\alpha,\#)\rangle) \end{array}$$

in G.

(V) $tw'uA_\# v\$z \Rightarrow qw'uxv'\$\beta$, where $A_\# = \varepsilon$, $x = v = \varepsilon$, $v' = \#$, $z = A\beta$, and $A \in (V - \Sigma)$. In this case, a rule $r\colon t\$A \rightarrow q\#\$ \in R$ was applied, so for every $X, Y \in (V - \Sigma)$, where $Y \neq A$, the following rules

$$\begin{array}{rcl} (A, \langle t;\kappa\rangle) & \rightarrow & (A, \langle t;\kappa;[\![r,?]\!]\rangle) \\ (X, \langle t;\kappa;[\![r,?]\!]\rangle) & \rightarrow & (X, \langle t;\kappa;[\![r,X]\!]\rangle) \\ (Y, \langle t;\kappa;[\![r,Y]\!]\rangle) & \rightarrow & (Y, q_{\mathrm{false}}) \\ (A, \langle t;\kappa;[\![r,A]\!]\rangle) & \rightarrow & (\#_{\kappa+1}, \langle q;\kappa+1\rangle) \end{array}$$

were introduced in P. Hence, G simulates the application of r in the following way:

$$\begin{array}{ll} & (w'\tau(uA_\# v,1)A\beta, \langle t;\kappa\rangle) \\ {}_k\Rightarrow & (w'\tau(uA_\# v,1)A\beta, \langle t;\kappa;[\![r,?]\!]\rangle) \\ {}_k\Rightarrow & (w'\tau(uA_\# v,1)A\beta, \langle t;\kappa;[\![r,A]\!]\rangle) \\ {}_k\Rightarrow & (w'\tau(ux\#,1)\beta, \langle q;\kappa+1\rangle) \\ {}_k\Rightarrow^* & (w\tau(\alpha,1)\beta, \langle q;\mathrm{occur}(\alpha,\#)\rangle) \end{array}$$

Observe that A must be the first nonterminal symbol just behind $\#_i$ nonterminals, $1 \leq i \leq k$. When $z = BA\beta$, with $B \in (V - \Sigma)$ and $B \neq A$, then G reach the state q_{false} and the simulation is blocked.

□

Claim 4.6. *Set* $\Omega = \{[\![r,X]\!] \mid r \in R, X \in (\{1,2,\ldots,k\} \cup (V - \Sigma) \cup \{?\})\}$ *and express* K *as* $K = K_Q \cup K_\Omega \cup \{q_{\mathrm{false}}\}$*, where*

$$\begin{array}{rcl} K_Q & = & \{\langle p;i\rangle \mid p \in Q, 0 \leq i \leq k\} \\ K_\Omega & = & \{\langle p;i;Z\rangle \mid p \in Q, 0 \leq i \leq k, Z \in \Omega\} \end{array}$$

Define a binary operation $\bullet$ from $K_Q \times (\Omega \cup \{\lambda\})$ to K such that

$$\begin{array}{lcll} \langle p; i\rangle \bullet Z &=& \langle p; i; Z\rangle, & \textit{for all } Z \in \Omega \\ \langle p; i\rangle \bullet \lambda &=& \langle p; i\rangle & \end{array}$$

Furthermore, set $N_\# = \{\#_i \mid 1 \le i \le k\}$ and define a homomorphism $\bar\tau$ from $(N_\# \cup T)$ to $(\Sigma - \{\$\})$ such that $\bar\tau(a) = a$ for every $a \in T$ and $\bar\tau(X) = \#$ for every $X \in N_\#$.

Based on a state to which G enters, the following two cases are considered:

(a) Let $(\#_1, \langle p, 1\rangle)\ {}_k{\Rightarrow}^m\ (w\alpha\beta, \langle q, \mathrm{occur}(\bar\tau(\alpha), \#)\rangle \bullet Z)$ in G, where $p, q \in Q$, $w \in T^*$, $\alpha \in (N_\#(N_\# \cup T)^*)^*$, $\beta \in (V - \{\#, \$\})^*$, $Z \in (\Omega \cup \{\lambda\})$, and $m \ge 0$. Then, $p\#\$ \Rightarrow^* qw\bar\tau(\alpha)\$\beta$ in M.

(b) Let $(\#_1, \langle p, 1\rangle)\ {}_k{\Rightarrow}^m\ (w\alpha\beta, q_{\mathrm{false}})$ in G, where $p \in Q$, $w \in T^*$, $\alpha \in (N_\#(N_\# \cup T)^*)^*$, $\beta \in (V - \{\#, \$\})^*$, and $m \ge 0$. Then, $p\#\$ \Rightarrow^* \bar q w\bar\tau(\alpha)\$\beta$ in M, where $\bar q \in Q$, $\beta = z_1 Y z_2 A z_3$, $Y, A \in (V - \Sigma)$, $Y \ne A$, $z_1 \in (\Sigma - \{\#, \$\})^*$, $z_2 \in (V - \{A, \#, \$\})^*$, $z_3 \in (V - \{\#, \$\})^*$, and there is a rule $\bar r\colon \bar q\$A \to q'\#\$ \in R$, $q' \in Q$, such that $\bar r$ is not applicable on $\bar q w\bar\tau(\alpha)\$\beta$.

Proof. This claim is proved by induction on $m \ge 0$.

Basis. Let $m = 0$, so $(\#_1, \langle p; 1\rangle)\ {}_k{\Rightarrow}^0\ (\#_1, \langle p; 1\rangle)$ in G, where $w = \varepsilon$, $\alpha = \#_1$, $\beta = \varepsilon$, and $Z = \lambda$. Then, $p\#\$ \Rightarrow^* p\#\$$ in M and the basis holds. Observe that the basis also holds for the case (b) of this claim. Since $\langle p; 1\rangle \ne q_{\mathrm{false}}$, $(\#_1, \langle p; 1\rangle)\ {}_k{\Rightarrow}^0\ (\#_1, q_{\mathrm{false}})$ not in G and the implication is automatically true.

Induction Hypothesis. Suppose that the claim holds for all $0 \le m \le l$, where l is a non-negative integer.

Induction Step. Let $(\#_1, \langle p; 1\rangle)\ {}_k{\Rightarrow}^{l+1}\ (w\alpha\beta, \mathscr{X})$ in G, where $p \in Q$, $w \in T^*$, $\alpha \in (N_\#(N_\# \cup T)^*)^*$, $\beta \in (V - \{\#, \$\})^*$, and either $\mathscr{X} = \langle q; \mathrm{occur}(\bar\tau(\alpha), \#)\rangle \bullet Z$, $q \in Q$, $Z \in (\Omega \cup \{\lambda\})$, or $\mathscr{X} = q_{\mathrm{false}}$. Since $l + 1 \ge 1$, express $(\#_1, \langle p; 1\rangle)\ {}_k{\Rightarrow}^{l+1}$ $(w\alpha\beta, \mathscr{X})$ as $(\#_1, \langle p; 1\rangle)\ {}_k{\Rightarrow}^{l}\ (w'uA_\# vz, \mathscr{X}')\ {}_k{\Rightarrow}\ (w'uxv'\beta, \mathscr{X})$, where $w' \in T^*$, $u \in (N_\#(N_\# \cup T)^*)^*$, $A_\# \in (N_\# \cup \{\varepsilon\})$, $x, v, v' \in (N_\# \cup T)^*$, $z \in (V - \{\#, \$\})^*$, $\hat w\alpha = uxv'$, $w = w'\hat w$, $\hat w \in T^*$, and either $\mathscr{X}' = \langle t; \mathrm{occur}(\bar\tau(uA_\# v), \#)\rangle \bullet Z'$, $t \in Q$, $Z' \in (\Omega \cup \{\lambda\})$, or $\mathscr{X}' = q_{\mathrm{false}}$.

By the induction hypothesis, $p\#\$ \Rightarrow^* \chi$ in M, where $\chi = tw'\bar\tau(uA_\# v)\$z$, if $\mathscr{X}' = \langle t; \mathrm{occur}(\bar\tau(uA_\# v), \#)\rangle \bullet Z'$, or $\chi = \bar q w'\bar\tau(uA_\# v)\$z$, if $\mathscr{X}' = q_{\mathrm{false}}$, and there is a rule $\bar r\colon \bar q\$A \to q'\#\$ \in R$ such that $\bar r$ is not applicable on χ, where $\bar q, q' \in Q$, $z = z_1 Y z_2 A z_3$, $Y, A \in (V - \Sigma)$, $Y \ne A$, $z_1 \in (\Sigma - \{\#, \$\})^*$, $z_2 \in (V - \{A, \#, \$\})^*$, and $z_3 \in (V - \{\#, \$\})^*$.

Set $\kappa = \mathrm{occur}(\bar\tau(uA_\# v), \#)$. Based on a forms of applied rules, G can perform

$$(w'uA_\# vz, \mathscr{X}')\ {}_k{\Rightarrow}\ (w'uxv'\beta, \mathscr{X})$$

according to the following cases:

Case 1 G performs $(w'uA_\# vz, \mathscr{X}')\ {}_k{\Rightarrow}\ (w'uxv'\beta, \mathscr{X})$ using $(\#_\kappa, \langle t; \kappa\rangle) \to (x, \langle q; \kappa - 1\rangle) \in P$, where $x \in T^*$. In this case, $\mathscr{X}' = \langle t; \mathrm{occur}(\bar\tau(uA_\# v), \#)\rangle \bullet Z'$, $\mathscr{X} = \langle q; \mathrm{occur}(\bar\tau(uxv), \#)\rangle \bullet Z$, $Z' = Z = \lambda$, $\mathrm{occur}(\bar\tau(u), \#) = \kappa - 1$, $A_\# = \#_\kappa$, $v = v'$, $\mathrm{occur}(\bar\tau(v), \#) = 0$, and $z = \beta$. Based on a construction of P, there must be $t\ {}_\kappa\# \to qx \in R$ and therefore

$$tw'\bar\tau(uA_\# v)\$z \Rightarrow qw'\bar\tau(uxv)\$z \Rightarrow^* qw\bar\tau(\alpha)\$\beta$$

in M.

Case 2 G performs $(w'uA_{\#}vz, \mathscr{X}')\ {}_k\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using $(\#_n, \langle t;\kappa\rangle) \to (x, \langle q;\kappa-1;[\![r,1]\!]\rangle) \in P$, where $1 \leq n \leq \kappa - 1$, $x \in T^*$, and $r = t\,{}_n\# \to qx$. In this case, $\mathscr{X}' = \langle t;\text{occur}(\bar\tau(uA_{\#}v),\#)\rangle \bullet Z'$, $\mathscr{X} = \langle q;\kappa-1\rangle \bullet Z$, $Z' = \lambda$, $Z = [\![r,1]\!]$, $\text{occur}(\bar\tau(u),\#) = n-1$, $A_{\#} = \#_n$, $v = v'$, $\text{occur}(\bar\tau(v),\#) = \kappa - n$, $z = \beta$, and $r \in R$. Thus,

$$tw'\bar\tau(uA_{\#}v)\$z \Rightarrow qw'\bar\tau(uxv)\$z \Rightarrow^* qw\bar\tau(\alpha)\$\beta$$

in M.

Case 3 G performs $(w'uA_{\#}vz, \mathscr{X}')\ {}_k\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using $(\#_{n+i}, \langle t;\kappa;[\![r,i]\!]\rangle) \to (\#_{n+i-1}, \langle t;\kappa;[\![r,i+1]\!]\rangle) \in P$, where $1 \leq n \leq \kappa$, $1 \leq i \leq \kappa - n$, $2 \leq n+i \leq \kappa$, and r is a rule of the form $t'\,{}_n\# \to tx'$ with $t' \in Q$ and $x' \in T^*$. In this case, $\mathscr{X}' = \langle t;\text{occur}(\bar\tau(uA_{\#}v),\#)\rangle \bullet Z'$, $\mathscr{X} = \langle q;\kappa\rangle \bullet Z$, $t = q$, $Z' = [\![r,i]\!]$, $Z = [\![r,i+1]\!]$, $\text{occur}(\bar\tau(u),\#) = (n+i)-1$, $A_{\#} = \#_{n+i}$, $x = \#_{n+i-1}$, $v = v'$, $\text{occur}(\bar\tau(v),\#) = \kappa - (n+i)$, $z = \beta$, and $r \in R$. Clearly, $\bar\tau(\#_{n+i}) = \bar\tau(\#_{n+i-1})$ and then

$$tw'\bar\tau(uA_{\#}v)\$z \Rightarrow^* qw'\bar\tau(uxv)\$z \Rightarrow^* qw\bar\tau(\alpha)\$\beta$$

in M.

Case 4 G performs $(w'uA_{\#}vz, \mathscr{X}')\ {}_k\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using $(\#_{\kappa+1}, \langle t;\kappa;[\![r,\kappa-n+1]\!]\rangle) \to (\#_\kappa, \langle t;\kappa\rangle) \in P$, where $1 \leq n \leq \kappa$ and r is a rule of the form $t'\,{}_n\# \to tx'$ with $t' \in Q$ and $x' \in T^*$. In this case, $\mathscr{X}' = \langle t;\text{occur}(\bar\tau(uA_{\#}v),\#)\rangle \bullet Z'$, $\mathscr{X} = \langle q;\kappa\rangle \bullet Z$, $t = q$, $Z' = [\![r,\kappa-n+1]\!]$, $Z = \lambda$, $\text{occur}(\bar\tau(u),\#) = \kappa - 1$, $A_{\#} = \#_{\kappa+1}$, $x = \#_\kappa$, $v = v'$, $\text{occur}(\bar\tau(v),\#) = 0$, $z = \beta$, and $r \in R$. Clearly, as in previous case,

$$tw'\bar\tau(uA_{\#}v)\$z \Rightarrow^* qw'\bar\tau(uxv)\$z \Rightarrow^* qw\bar\tau(\alpha)\$\beta$$

in M.

Case 5 G performs $(w'uA_{\#}vz, \mathscr{X}')\ {}_k\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using

$$(\#_n, \langle t;\kappa\rangle) \to (x, \langle q;\kappa\rangle) \in P$$

where $1 \leq n \leq \kappa$ and $x = x_1\#_n x_2$ with $x_1, x_2 \in T^*$. In this case, $\mathscr{X}' = \langle t;\text{occur}(\bar\tau(uA_{\#}v),\#)\rangle \bullet Z'$, $\mathscr{X} = \langle q;\kappa\rangle \bullet Z$, $Z' = Z = \lambda$, $\text{occur}(\bar\tau(u),\#) = n-1$, $A_{\#} = \#_n$, $v = v'$, $\text{occur}(\bar\tau(v),\#) = \kappa - n$, and $z = \beta$. Following the construction of P, there is a rule $t\,{}_n\# \to q\bar\tau(x) \in R$ and then

$$tw'\bar\tau(uA_{\#}v)\$z \Rightarrow qw'\bar\tau(uxv)\$z \Rightarrow^* qw\bar\tau(\alpha)\$\beta$$

in M.

Case 6 G performs $(w'uA_{\#}vz, \mathscr{X}')\ {}_k\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using

$$(\#_n, \langle t;\kappa\rangle) \to (\#_n, \langle t;\kappa;[\![r,1]\!]\rangle) \in P$$

where $1 \leq n \leq \kappa$ and r is a rule of the form $t\,{}_n\# \to q'x'$ with $q' \in Q$, $x' \in (\Sigma - \{\$\})^*$, and $\text{occur}(x',\#) \geq 2$. In this case, $\mathscr{X}' = \langle t;\text{occur}(\bar\tau(uA_{\#}v),\#)\rangle \bullet$

Z', $\mathscr{X} = \langle q;\kappa\rangle \bullet Z$, $t = q$, $Z' = \lambda$, $Z = [\![r, 1]\!]$, $\mathrm{occur}(\bar{\tau}(u), \#) = n - 1$, $A_\# = x = \#_n$, $v = v'$, $\mathrm{occur}(\bar{\tau}(v), \#) = \kappa - n$, $z = \beta$, and $r \in R$. Clearly,

$$tw'\bar{\tau}(uA_\#v)\$z \Rightarrow^* qw'\bar{\tau}(uxv)\$z \Rightarrow^* qw\bar{\tau}(\alpha)\$\beta$$

in M.

Case 7 G performs $(w'uA_\#vz, \mathscr{X}')\ {}_k\!\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using $(\#_{\kappa-i}, \langle t;\kappa;[\![r, i+1]\!]\rangle) \to (\#_{\kappa+\eta-i}, \langle t;\kappa;[\![r, i+2]\!]\rangle) \in P$, where $0 \leq i \leq \kappa - n - 1$, $1 \leq n \leq \kappa - 1$, r is a rule of the form $t\,{}_n\# \to q'x'$ with $q' \in Q$, $x' \in (\Sigma - \{\$\})^*$, and $\mathrm{occur}(x', \#) \geq 2$, and $\eta = \mathrm{occur}(x', \#) - 1$. In this case, $\mathscr{X}' = \langle t; \mathrm{occur}(\bar{\tau}(uA_\#v), \#)\rangle \bullet Z'$, $\mathscr{X} = \langle q;\kappa\rangle \bullet Z$, $t = q$, $Z' = [\![r, i+1]\!]$, $Z = [\![r, i+2]\!]$, $\mathrm{occur}(\bar{\tau}(u), \#) = (\kappa - i) - 1$, $A_\# = \#_{\kappa-i}$, $x = \#_{\kappa+\eta-i}$, $v = v'$, $\mathrm{occur}(\bar{\tau}(v), \#) = i$, $z = \beta$, and $r \in R$. As $\bar{\tau}(A_\#) = \bar{\tau}(x)$, it holds

$$tw'\bar{\tau}(uA_\#v)\$z \Rightarrow^* qw'\bar{\tau}(uxv)\$z \Rightarrow^* qw\bar{\tau}(\alpha)\$\beta$$

in M.

Case 8 G performs $(w'uA_\#vz, \mathscr{X}')\ {}_k\!\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using

$$(\#_n, \langle t;\kappa;[\![r, \kappa - n + 1]\!]\rangle) \to (x, \langle q;\kappa + \eta\rangle) \in P$$

where $1 \leq n \leq \kappa$, $x \in (N_\# \cup T)^*$, $\eta = \mathrm{occur}(\bar{\tau}(x), \#) - 1$, $\eta \geq 1$, and $r = t\,{}_n\# \to q\bar{\tau}(x)$. In this case, $\mathscr{X}' = \langle t; \mathrm{occur}(\bar{\tau}(uA_\#v), \#)\rangle \bullet Z'$, $\mathscr{X} = \langle q;\kappa + \eta\rangle \bullet Z$, $Z' = [\![r, \kappa - n + 1]\!]$, $Z = \lambda$, $\mathrm{occur}(\bar{\tau}(u), \#) = n - 1$, $A_\# = \#_n$, $v = v'$, $\mathrm{occur}(\bar{\tau}(v), \#) = \kappa - n$, and $z = \beta$. As $r \in R$, it is clear that

$$tw'\bar{\tau}(uA_\#v)\$z \Rightarrow qw'\bar{\tau}(uxv)\$z \Rightarrow^* qw\bar{\tau}(\alpha)\$\beta$$

in M.

Case 9 G performs $(w'uA_\#vz, \mathscr{X}')\ {}_k\!\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using $(\#_\kappa, \langle t;\kappa\rangle) \to (y, \langle q;\kappa - 1\rangle) \in P$, where $x \in (V' - N_\#)^*$. In this case, $\mathscr{X}' = \langle t; \mathrm{occur}(\bar{\tau}(uA_\#v), \#)\rangle \bullet Z'$, $\mathscr{X} = \langle q;\kappa - 1\rangle \bullet Z$, $Z' = Z = \lambda$, $\mathrm{occur}(\bar{\tau}(u), \#) = \kappa - 1$, $A_\# = \#_\kappa$, $x = v = v' = \varepsilon$, and $\beta = yz$. Following the construction of P, there is a rule $t\#\$ \to q\$y \in R$ and then

$$tw'\bar{\tau}(uA_\#v)\$z \Rightarrow qw'\bar{\tau}(uxv')\$yz \Rightarrow^* qw\bar{\tau}(\alpha)\$\beta$$

in M.

Case 10 G performs $(w'uA_\#vz, \mathscr{X}')\ {}_k\!\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using

$$(A, \langle t;\kappa\rangle) \to (A, \langle t;\kappa;[\![r, ?]\!]\rangle) \in P$$

where $A \in (V' - N_\# - T)$ and r is a rule of the form $t\$A \to q'\#\$$ with $q' \in Q$. In this case, $\mathscr{X}' = \langle t; \mathrm{occur}(\bar{\tau}(uA_\#v), \#)\rangle \bullet Z'$, $\mathscr{X} = \langle q;\kappa\rangle \bullet Z$, $t = q$, $Z' = \lambda$, $Z = [\![r, ?]\!]$, $A_\# = x = \varepsilon$, $v = v'$, $z = \beta = z_1Az_2$, $z_1, z_2 \in (V' - N_\#)^*$, $\kappa + \mathrm{occur}(z_1, V' - N_\# - T) \leq k - 1$, and $r \in R$. Hence,

$$tw'\bar{\tau}(uA_\#v)\$z \Rightarrow^* qw'\bar{\tau}(uxv')\$\beta \Rightarrow^* qw\bar{\tau}(\alpha)\$\beta$$

in M.

Case 11 G performs $(w'uA_\# vz, \mathscr{X}')\ {}_k\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using

$$(X, \langle t; \kappa; [\![r, ?]\!]\rangle) \to (X, \langle t; \kappa; [\![r, X]\!]\rangle) \in P$$

where $X \in (V' - N_\# - T)$ and r is a rule of the form $t\$A' \to q'\#\$$ with $A' \in (V' - N_\# - T)$ and $q' \in Q$. In this case, $\mathscr{X}' = \langle t; \mathrm{occur}(\bar{\tau}(uA_\# v), \#)\rangle \bullet Z'$, $\mathscr{X} = \langle q; \kappa\rangle \bullet Z$, $t = q$, $Z' = [\![r, ?]\!]$, $Z = [\![r, X]\!]$, $A_\# = x = \varepsilon$, $v = v'$, $z = \beta = z_1 X z_2$, $z_1 \in T^*$, $z_2 \in (V' - N_\#)^*$, $\kappa \le k - 1$, and $r \in R$. Hence,

$$tw'\bar{\tau}(uA_\# v)\$z \Rightarrow^* qw'\bar{\tau}(uxv')\$\beta \Rightarrow^* qw\bar{\tau}(\alpha)\$\beta$$

in M.

Case 12 G performs $(w'uA_\# vz, \mathscr{X}')\ {}_k\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using

$$(Y, \langle t; \kappa; [\![r, Y]\!]\rangle) \to (Y, q_{\mathrm{false}}) \in P$$

where $Y \in (V' - N_\# - T)$ and r is a rule of the form $t\$A' \to q'\#\$$ with $A' \in (V' - N_\# - T)$, $A' \ne Y$, and $q' \in Q$. In this case, $\mathscr{X}' = \langle t; \mathrm{occur}(\bar{\tau}(uA_\# v), \#)\rangle \bullet Z'$, $\mathscr{X} = q_{\mathrm{false}}$, $Z' = [\![r, Y]\!]$, $A_\# = x = \varepsilon$, $v = v'$, $z = \beta = z_1 Y z_2 A' z_3$, $z_1 \in T^*$, $z_2 \in (V' - N_\# - \{A'\})^*$, $z_3 \in (V' - N_\#)^*$, $\kappa + \mathrm{occur}(z_1 Y z_2, V' - N_\# - T) \le k - 1$, and $r \in R$. With $\bar{q} = t$, it follows that

$$tw'\bar{\tau}(uA_\# v)\$z \Rightarrow^* tw'\bar{\tau}(uxv')\$\beta \Rightarrow^* \bar{q}w\bar{\tau}(\alpha)\$\beta$$

in M and there is a rule $r\colon \bar{q}\$A' \to q'\#\$ \in R$ such that r is not applicable on $\bar{q}w\bar{\tau}(\alpha)\$\beta$.

Case 13 G performs $(w'uA_\# vz, \mathscr{X}')\ {}_k\!\Rightarrow (w'uxv'\beta, \mathscr{X})$ using

$$(A, \langle t; \kappa; [\![r, A]\!]\rangle) \to (\#_{\kappa+1}, \langle q; \kappa + 1\rangle) \in P$$

where $A \in (V' - N_\# - T)$ and $r = t\$A \to q\#\$$. In this case, $\mathscr{X}' = \langle t; \mathrm{occur}(\bar{\tau}(uA_\# v), \#)\rangle \bullet Z'$, $\mathscr{X} = \langle q; \kappa + 1\rangle \bullet Z$, $Z' = [\![r, A]\!]$, $Z = \lambda$, $A_\# = x = \varepsilon$, $v' = v\#_{\kappa+1}$, and $z = A\beta$. Following the construction of P, $r \in R$ and then

$$tw'\bar{\tau}(uA_\# v)\$z \Rightarrow qw'\bar{\tau}(uxv')\$\beta \Rightarrow^* qw\bar{\tau}(\alpha)\$\beta$$

in M.

□