Journal of Computer and System Sciences 95 (2018) 143–150

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

Categorical aspects of inducing closure operators on graphs by sets of walks

Josef Šlapal

IT4Innovations Centre of Excellence, Brno University of Technology, 612 66 Brno, Czechia

ARTICLE INFO

Article history:
Received 30 August 2016
Received in revised form 2 January 2017
Accepted 18 February 2017
Available online 7 March 2017

Keywords:
Simple graph
Walk
Closure operator
Galois correspondence between concrete categories
Diagonal walk set
Digital topology

ABSTRACT

We study closure operators on graphs which are induced by sets of walks of identical lengths in these graphs. It is shown that the induction gives rise to a Galois correspondence between the category of closure spaces and that of graphs with walk sets. We study the two isomorphic subcategories resulting from the correspondence, in particular, the one that is a full subcategory of the category of graphs with walk sets. As examples, we discuss closure operators that are induced by path sets on some natural graphs on the digital plane $\mathbb{Z}^2$. These closure operators are shown to include the well known Marcus–Wyse and Khalimsky topologies, thus indicating the possibility of using them as convenient background structures on the digital plane for the study of geometric and topological properties of digital images.

1. Introduction

As a basic field of discrete mathematics, graph theory finds a wide spectrum of applications. In particular, a special branch of discrete geometry devised for the study of geometric and topological properties of digital images, digital topology, is based on the use of topological aspects of graph theory such as graph connectedness. There are two approaches to digital topology, traditional [12,14] and topological [8,10,19]. While the former employs purely graph-theoretic tools, the latter is based on topological methods. Since both approaches have their specific advantages, it is most desirable to find bridges between them by studying the relationships between graph theory and topology. Such relationships were dealt with by several authors who investigated correspondences between directed graphs (i.e., sets with a binary relation) and topologies or closure operators, see e.g. [2,5,13,16]. The correspondences usually considered associate an Alexandroff topology (or completely additive closure operator) with graphs in a very natural way, thus obtaining the so-called left topology (left closure operator), in which the closure of a set A equals $A \cup \{x;$ there is an edge (a, x) with $a \in A\}$, and, dually, the so-called right topology (right closure operator). However, up to now, only little effort has been exerted to investigate correspondences between simple graphs and spaces more general than the Alexandroff ones. The aim of this note is to proceed with such an investigation. We will focus on studying relationships between (simple) graphs and closure operators that generalize topologies (given by Kuratowski closure operators). It was shown in [17] that such closure operators provide richer scale of instruments for the needs of digital topology than the two topologies usually used, the Khalimsky [8] and Marcus–Wyse [13] ones.

The present paper is a continuation of the author's study of the topic started in [18] and [20]. We will deal with graphs each having a set specified of walks with identical lengths. Such graphs, with special walk sets called path partitions, were

E-mail address: slapal@fme.vutbr.cz.

http://dx.doi.org/10.1016/j.jcss.2017.02.005

introduced and studied in [18] where their geometric properties were discussed based on a special concept of connectedness. The closure operators induced, in a certain way, on graphs by walk sets were discussed in [20] with some interesting relationships between the graphs and the induced closure operators shown. Building on [18], we will discover some more relationships between graphs with walk sets and the induced closure operators and, moreover, such relationships will be regarded in terms of category theory. More precisely, we will show that the induction gives rise to a Galois correspondence between the category of closure spaces and that of graphs with walk sets. The Galois correspondence will be studied and the results achieved will be demonstrated by examples of closure operators induced on (graphs on) the digital plane by certain walk sets.

2. Preliminaries

For the graph-theoretic terminology, we refer to [7]. By a *graph* $G = (V, E)$ we understand an undirected simple graph without loops with $V \neq \emptyset$ the *vertex* set and $E \subseteq \{\{x, y\};\ x, y \in V,\ x \neq y\}$ the set of *edges*. We will say that G is a graph *on* V. As usual, two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$. A key role will be played by the concept of a walk. Unlike the usual walks, in the present paper, the walks are allowed to be transfinite. More precisely, given an ordinal $\alpha > 1$, by an α*-walk* (briefly, a *walk*) in G we understand a sequence (of type α) $(x_i|\ i < \alpha)$ of vertices of V such that x_i is adjacent to x_{i+1} whenever $i + 1 < \alpha$. If $\alpha > 1$ is a finite ordinal, then $\alpha - 1$ is called the *length* of the walk $(x_i|\ i < \alpha)$. An α-walk is called an α*-path* (briefly, a *path*) if its members are pairwise different.

By a *closure operator* u on a set X, we mean a topology in Čech's sense [3], i.e., a map u: $\exp X \to \exp X$ (where $\exp X$ denotes the power set of X) that is

(i) grounded (i.e., $u\emptyset = \emptyset$),
(ii) extensive (i.e., $A \subseteq X \Rightarrow A \subseteq uA$), and
(iii) monotone (i.e., $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$).

The pair (X, u) is then called a *closure space*. Thus, the usual topologies (i.e., Kuratowski closure operators — cf. [6]) are the closure operators u on X that are

(iv) additive (i.e., $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq X$) and
(v) idempotent (i.e., $uuA = uA$ whenever $A \subseteq X$).

Given closure spaces (X, u) and (Y, v), a map $f : X \to Y$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

As usual, we identify cardinals with initial ordinals (accepting so the Axiom of Choice). Given an ordinal α, we denote by $\langle\alpha\rangle$ the least cardinal n with $\alpha \le n$.

Let $m > 1$ be a cardinal. A closure operator u on a set X and the closure space (X, u) are called an S_m*-closure operator* and an S_m*-closure space* (briefly, an S_m*-space*), respectively, if the following condition is satisfied:

$A \subseteq X \Rightarrow uA = \bigcup\{uB;\ B \subseteq A, \operatorname{card} B < m\}$.

In [4], S_2-closure operators and S_2-spaces are called *quasi-discrete*. S_2-topologies (S_2-topological spaces) are called *Alexandroff topologies* (*Alexandroff spaces*) — cf. [8]. Clearly, every S_2-closure operator is additive and, if $m \le \aleph_0$, then every additive S_m-closure operator is an S_2-closure operator. Since every S_m-closure operator is an S_n-closure operator whenever $m \le n$, it is useful to know, for a given closure operator u on X, the minimal cardinal m for which u is an S_m-closure operator. Such a minimal cardinal is an important invariant of the closure space (X, u) as mentioned in [3].

We will use some basic topological concepts (see e.g. [6]) naturally extended from topological to closure spaces. In particular, given a closure space (X, u), a subset $A \subseteq X$ is said to be *closed* if $uA = A$ (and it is said to be *open* if its complement in X is closed). If u, v are closure operators on a set X, then we put $u \le v$ if $uA \subseteq vA$ for every subset $A \subseteq X$. Clearly, $\le$ is a partial order on the set of all closure operators on X. If $u \le v$, then u is said to be *finer* than v and v is said to be *coarser* than u. Note that, for topologies given by open sets, just the converse partial order is usually used.

For the categorical terminology used see [1]. All categories are considered to be constructs, i.e., concrete categories over *Set* (the category of sets and maps), and all functors are assumed to be concrete, i.e., to preserve the underlying sets and to be identities for morphisms (so that the functors are given by determining them as maps on objects). Recall that, given a pair of objects $A = (X, \rho)$ and $B = (X, \sigma)$ of a category, we write $\rho \le \sigma$ if $\mathrm{id}_X : (X, \rho) \to (X, \sigma)$ is a morphism. We will also write $A \le B$ in this case. Given categories $\mathcal{X}$, $\mathcal{Y}$ and functors $F, G : \mathcal{X} \to \mathcal{Y}$, we write $F \le G$ if $F(A) \le G(A)$ for every object $A \in \mathcal{X}$. A Galois correspondence between categories $\mathcal{X}$ and $\mathcal{Y}$ is a pair of functors (L, R), $L : \mathcal{Y} \to \mathcal{X}$ and $R : \mathcal{X} \to \mathcal{Y}$, such that $L \circ R \le \mathrm{id}_{\mathcal{X}}$ and $R \circ L \ge \mathrm{id}_{\mathcal{Y}}$.

3. A categorical Galois correspondence that arises from inducing closure operators on graphs by walk sets

Given a graph G and an ordinal $\alpha > 1$, we denote by $\mathcal{W}_\alpha(G)$ the set of all α-walks in G. Every subset $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$ will be called an α*-walk set* or, briefly, a *walk set* in G. If every element of $\mathcal{B}$ is even a path, then $\mathcal{B}$ will be called an α*-path set* or, briefly, a *path set* in G.

Let X be a set, $\alpha > 1$ an ordinal, and $\mathcal{B} \subseteq X^\alpha$ (where X^α denotes the set of all sequences of type α with all members from X) a subset such that $(x_i|\ i < \alpha) \in \mathcal{B}$ implies $x_i \neq x_{i+1}$ whenever $0 < i + 1 < \alpha$. Put $E_{\mathcal{B}} = \{\{x_i, x_{i+1}\};\ 0 < i + 1 < \alpha\}$.

Then, the graph $(X, E_{\mathcal{B}})$ is said to be *generated* by $\mathcal{B}$. Clearly, $(X, E_{\mathcal{B}})$ is the graph on X with the minimal (with respect to set inclusion) set of edges such that $\mathcal{B}$ is an α-walk set in $(X, E_{\mathcal{B}})$.

Given a graph $G = (V, E)$ and an α-walk set $\mathcal{B}$ in G, we put

$f_\alpha(\mathcal{B})A = A \cup \{x \in V;$ there exist $(x_i|\ i < \alpha) \in \mathcal{B}$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $\{x_i;\ i < i_0\} \subseteq A$ and $x_{i_0} = x\}$ for every $A \subseteq V$.

It may easily be seen that $f_\alpha(\mathcal{B})$ is an $S_{\langle\alpha\rangle}$-closure operator on G. It will be said to be *induced* by $\mathcal{B}$.

Conversely, given a closure operator u on a set X, we put $g_\alpha(u) = \{(x_i|\ i < \alpha) \in X^\alpha;\ x_i \neq x_{i+1}$ whenever $0 < i + 1 < \alpha$ and $x_j \in u\{x_i;\ i < j\}$ for every j with $0 < j < \alpha\}$.

We denote by *Clo* the category of closure spaces and injective continuous maps and by Gra_α a category whose objects are pairs $(G, \mathcal{B})$ where $G = (V, E)$ is a graph and $\mathcal{B}$ is an α-walk set in G ($\alpha > 1$ an ordinal) and whose morphisms $f : ((V, E), \mathcal{B}) \to ((V', E'), \mathcal{B}')$ are the injective maps $f : V \to V'$ such that $(x_i|\ i < \alpha) \in \mathcal{B}$ implies $(f(x_i)|\ i < \alpha) \in \mathcal{B}'$. Note that Gra_α is a concrete category over *Set* because every object $((V, E), \mathcal{B}) \in Gra_\alpha$ may be regarded as a pair $(V, (E, \mathcal{B}))$.

Let $\alpha > 1$ be an ordinal. For every object $((V, E), \mathcal{B}) \in Gra_\alpha$, we put $F_\alpha((V, E), \mathcal{B}) = (V, f_\alpha(\mathcal{B}))$ and, for every object $(X, u) \in Clo$, we put $G_\alpha(X, u) = ((X, E_{g_\alpha(u)}), g_\alpha(u))$.

Theorem 1. *(F_α, G_α) is a Galois correspondence between Clo and Gra_α for every ordinal $\alpha > 1$.*

Proof. Let $\varphi : ((V, E), \mathcal{B}) \to ((V,' E'), \mathcal{B}')$ be a morphism in Gra_α, let $A \subseteq V$ be a subset and $y \in \varphi(f_\alpha(\mathcal{B})A)$ a point. Then, there is a point $x \in f_\alpha(\mathcal{B})A$ with $y = \varphi(x)$. Consequently, there are a walk $(x_i|\ i < \alpha) \in \mathcal{B}$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $x_{i_0} = x$ and $x_i \in A$ for every $i < i_0$. Thus, we have $(\varphi(x_i)|\ i < \alpha) \in \mathcal{B}'$, $\varphi(x_{i_0}) = \varphi(x) = y$ and $\varphi(x_i) \in \varphi(A)$ for every $i < i_0$. Hence, $y \in f_\alpha(\mathcal{B}')(\varphi(A))$ so that $\varphi(f_\alpha(\mathcal{B})A) \subseteq f_\alpha(\mathcal{B}')(\varphi(A))$. Therefore, $\varphi : (V, f_\alpha(\mathcal{B})) \to (V', f_\alpha(\mathcal{B}'))$ is a continuous map, i.e., a morphism in *Clo*. We have shown that $F_\alpha : Gra_\alpha \to Clo$ is a functor.

Conversely, let $\varphi : (X, u) \to (Y, v)$ be a morphism in *Clo* and let $(x_i|\ i < \alpha) \in g_\alpha(u)$. Then, $x_i \in X$ for every $i < \alpha$, $x_i \neq x_{i+1}$ whenever $0 < i + 1 < \alpha$ and $x_j \in u\{x_i;\ i < j\}$ for every j with $0 < j < \alpha$. Consequently, $\varphi(x_i) \in Y$ for every $i < \alpha$, $\varphi(x_i) \neq \varphi(x_{i+1})$ whenever $0 < i + 1 < \alpha$ and $\varphi(x_j) \in \varphi(u\{x_i;\ i < j\}) \subseteq v\{\varphi(x_i);\ i < j\}$ for every j with $0 < j < \alpha$. This yields $(\varphi(x_i)|\ i < \alpha) \in g_\alpha(v)$ so that $\varphi : ((X, E_{g_\alpha(u)}), g_\alpha(u)) \to ((X, E_{g_\alpha(v)}), g_\alpha(v))$ is a morphism in Gra_α. We have shown that $G_\alpha : Clo \to Gra_\alpha$ is a functor.

Let $G = (V, E)$ be a graph, $\mathcal{B}$ an α-walk set in G and $(y_i|\ i < \alpha) \in \mathcal{B}$ a walk. Put $u = f_\alpha(\mathcal{B})$. Then, for every ordinal j with $0 < j < \alpha$, we have $y_j \in u\{y_i;\ i < j\}$. Consequently, $(y_i|\ i < \alpha) \in g_\alpha(u)$. Therefore, $\mathcal{B} \subseteq g_n(u) = g_n(f_n(\mathcal{B}))$. This yields $G_\alpha \circ F_\alpha \geq \mathrm{id}_{Gra_\alpha}$.

Conversely, let (X, u) be a closure space and let $A \subseteq X$ be a subset. Put $\mathcal{B} = g_\alpha(u)$ and let $x \in f_\alpha(g_\alpha(u))A$ be a point. If $x \in A$, then $x \in uA$. Let $x \notin A$. Then, there are $(x_i|\ i < \alpha) \in \mathcal{B} = g_\alpha(u)$ and i_0, $0 < i_0 < \alpha$, such that $x_{i_0} = x_i$ and $x_i \in A$ for every $i < i_0$. Thus, $x_j \in u\{x_i;\ i < j\}$ for every j, $0 < j < \alpha$. In particular, $x = x_{i_0} \in u\{x_i;\ i < i_0\} \subseteq uA$. Therefore, $f_\alpha(g_\alpha(u)) \leq u$. This yields $F_\alpha \circ G_\alpha \leq \mathrm{id}_{Clo}$. □

In consequence of Theorem 1, $F_\alpha(Gra_\alpha)$ is a full coreflective subcategory of *Clo*, $G_\alpha(Clo)$ is a full reflective subcategory of Gra_α, and the restrictions of F_α and G_α to $G_\alpha(Clo)$ and $F_\alpha(Gra_\alpha)$, respectively, are isomorphisms inverse to each other. Thus, for every $(G, \mathcal{B}) \in G_\alpha(Clo)$, we have $g_\alpha(f_\alpha(\mathcal{B})) = \mathcal{B}$ and, for every $(X, u) \in F_\alpha(Gra_\alpha)$, we have $f_\alpha(g_\alpha(u)) = u$. In what follows, we will investigate, for a given ordinal $\alpha > 1$, the isomorphic categories $F_\alpha(Gra_\alpha)$ and $G_\alpha(Clo)$.

Theorem 2. *Let $G = (V, E)$ be a graph and $\mathcal{B}$ an α-walk set in G ($\alpha > 1$ an ordinal). Then, $(G, \mathcal{B}) \in G_\alpha(Clo)$ if and only if the following two conditions are satisfied:*

($*$) *If $(x_i|\ i < \alpha) \in \mathcal{W}_\alpha(G)$ has the property that, for every ordinal i_0 with $0 < i_0 < \alpha$, $x_{i_0} = x_0$ or there exist $(y_j|\ j < \alpha) \in \mathcal{B}$ and j_0, $0 < j_0 < \alpha$, such that $x_{i_0} = y_{j_0}$ and $\{y_j; j < j_0\} \subseteq \{x_i; i < i_0\}$, then $(x_i|\ i < \alpha) \in \mathcal{B}$.*

($**$) *$E = E_{\mathcal{B}}$.*

Proof. Let $(G, \mathcal{B}) \in G_\alpha(Clo)$, let $(x_i|\ i < \alpha) \in \mathcal{W}_\alpha(G)$, and let for any $i_0, 0 < i_0 < \alpha$, $x_{i_0} = x_0$ or there are $(y_j|j < \alpha) \in \mathcal{B}$ and $j_0, 0 < j_0 < \alpha$, such that $x_{i_0} = y_{j_0}$ and $\{y_j; j < j_0\} \subseteq \{x_i; i < i_0\}$. Then, $x_{i_0} \in \{x_i;\ i < i_0\}$ or $x_{i_0} \in f_\alpha(\mathcal{B})\{y_j; j < j_0\}$. Since $\{x_i;\ i < i_0\} \subseteq f_\alpha(\mathcal{B})\{x_i;\ i < i_0\}$ and also $f_\alpha(\mathcal{B})\{y_j;\ j < j_0\} \subseteq f_n(\mathcal{B})\{x_i; i < i_0\}$, we have $x_{i_0} \in f_\alpha(\mathcal{B})\{x_i;\ i < i_0\}$ for every i_0, $0 < i_0 < \alpha$. Therefore, $(x_i|\ i < \alpha) \in g_\alpha(f_\alpha(\mathcal{B})) = \mathcal{B}$. Thus, the condition ($*$) is satisfied. The condition ($**$) follows immediately from the definition of G_α.

Conversely, let the conditions ($*$) and ($**$) be satisfied and let $(x_i|i < \alpha) \in g_\alpha(f_\alpha(\mathcal{B}))$. Then, $x_{i_0} \in f_\alpha(\mathcal{B})\{x_i;\ i < i_0\}$ for each i_0, $0 < i_0 < \alpha$. Hence, for every i_0, $0 < i_0 < \alpha$, there exist $(y_j|\ j < \alpha) \in \mathcal{B}$ and j_0, $0 < j_0 < \alpha$, such that $x_{i_0} = y_{j_0}$ and $\{y_j;\ j < j_0\} \subseteq \{x_i;\ i < i_0\}$. Therefore, $(x_i|\ i < \alpha) \in \mathcal{B}$ and we have shown that $g_\alpha(f_\alpha(\mathcal{B})) \subseteq \mathcal{B}$. But $G_\alpha \circ F_\alpha \geq \mathrm{id}_{Gra_\alpha}$ implies $g_\alpha(f_\alpha(\mathcal{B}) \supseteq \mathcal{B}$ so that $\mathcal{B} = g_\alpha(f_\alpha(\mathcal{B}))$. Consequently, $(G, \mathcal{B}) \in G_\alpha(Clo)$ and the proof is complete. □

Example 1. Note that every subset $\mathcal{B} \subseteq \mathcal{W}_2(G)$ satisfies the condition ($*$) in Theorem 2. It may easily be seen that a subset $\mathcal{B} \subseteq \mathcal{W}_3(G)$ satisfies ($*$) if and only if each of the following four conditions implies $(x, y, z) \in \mathcal{B}$:

(1) $(x, y, t) \in \mathcal{B}$, $(x, z, u) \in \mathcal{B}$,
(2) $(x, y, t) \in \mathcal{B}$, $(y, z, u) \in \mathcal{B}$,
(3) $(x, y, t) \in \mathcal{B}$, $(y, x, z) \in \mathcal{B}$,
(4) $(x, y, t) \in \mathcal{B}$, $x = z$.

Note that $(x, y, z) \in \mathcal{W}_3(G)$ implies $x \neq y \neq z$ (because G has no loops).

The following statement is evident:

Theorem 3. *Let (X, u) be a closure space and $\alpha > 1$ an ordinal. Then, $(X, u) \in F_\alpha(Gra_\alpha)$ if and only if the following condition is satisfied:*

if $A \subseteq X$ and $x \in uA - A$, then there exist a sequence $(x_i|i < \alpha) \in X^\alpha$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $\{x_i;\ i < i_0\} \subseteq A$, $x_j \in u\{x_i;\ i < j\}$ for each j, $0 < j < \alpha$, and $x = x_{i_0}$.

Clearly, $u_\mathcal{B}$ is neither additive nor idempotent in general. On the other hand, for every α-walk set in a graph (V, E), the union of a system of closed subsets of $(V, u_\mathcal{B})$ is a closed subset of $(V, u_\mathcal{B})$. For finite ordinal numbers α, this assertion is proved in [20] and, for an arbitrary $\alpha > 1$, the proof is analogous.

We denote by Clo_U the full subcategory of Clo whose objects are the closure spaces (X, u) with u idempotent and, by Top_S, the full subcategory of Clo (and also of Clo_U) whose objects are the Alexandroff topological spaces.

Proposition 1. *For every ordinal $\alpha > 1$, $F_\alpha(Gra_\alpha) \cap Clo_U = F_\alpha(Gra_\alpha) \cap Top_S$.*

Proof. Let $((X, E), \mathcal{B}) \in F_\alpha(Gra_\alpha) \cap Clo_U$ be an object an put $u = f_\alpha(\mathcal{B})$. Let $A \subseteq X$ be a subset and $x \in uA$ a point. If $x \in A$, then $x \in \bigcup_{x\in A} u\{x\}$ because of the extensiveness of u. Let $x \notin A$. Then, by Theorem 3, there exist a sequence $(x_i|i < \alpha) \in X^\alpha$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $\{x_i;\ i < i_0\} \subseteq A$, $x_j \in u\{x_i;\ i < j\}$ for each j, $0 < j < \alpha$, and $x = x_{i_0}$. Thus, we have $x_1 \in u\{x_0\}$ where $x_0 \in A$. If $\alpha = 2$, then $x = x_1$ so that $x \in \bigcup_{y\in A} u\{y\}$. Suppose that $\alpha > 2$. Let i be an ordinal with $1 < i < \alpha$ such that $x_j \in u\{x_0\}$ for every j with $0 < j < i$. Then, $\{x_j;\ j < i\} \subseteq u\{x_0\}$ and $x_i \in u\{x_j;\ j < i\}$. Thus, $x_i \in uu\{x_0\} = u\{x_0\}$. Consequently, $x_i \in u\{x_0\}$ for every $i < \alpha$. In particular, $x = x_{i_0} \in u\{x_0\}$ so that $x \in \bigcup_{y\in A} u\{y\}$. We have shown that $uA \subseteq \bigcup_{y\in A} u\{x\}$. As the converse inclusion follows from the monotonicity of u, the proof is complete. □

The previous statement says that, for every α-walk set $\mathcal{B}$ in a graph (V, E), $f_\alpha(\mathcal{B})$ is idempotent if and only if $(V, f_\alpha(\mathcal{B}))$ is an Alexandroff topological space.

Let α, β, $1 < \alpha \leq \beta$, be ordinals. For every object $((X, E), \mathcal{B}) \in G_\alpha(Clo)$, put $H_{\alpha,\beta}((X, E), \mathcal{B}) = ((X, E), \hat{\mathcal{B}})$ where $\mathcal{B}' \subseteq X^\beta$ is given by $(y_j|j < \beta) \in \hat{\mathcal{B}}$ if and only if $(y_j|j < \beta) \in \mathcal{W}_\beta(X, E)$ and, for each j_0, $0 < j_0 < \beta$, $y_{j_0} = y_0$ or there exist $(x_i|i < \alpha) \in \mathcal{B}$ and i_0, $0 < i_0 < \alpha$, such that $y_{j_0} = x_{i_0}$ and $\{x_i; i < i_0\} \subseteq \{y_j; j < j_0\}$.

Let α, β be ordinals, $\alpha > 1$ finite and $\beta \geq \alpha$. For every sequence $(x_i|\ i < \alpha) \in X^\alpha$ (where X is a set) we put $h_{\alpha,\beta}(x_i|\ i < \alpha) = (y_j|\ j < \beta)$ where $y_j = x_j$ for every j with $j < \alpha$ and

$$y_j = \begin{cases} x_{\alpha-2} & \text{if } j = \delta + n \text{ where } \delta = \alpha \text{ or } \delta \text{ is a limit ordinal and } n \text{ is an even finite ordinal,} \\ x_{\alpha-1} & \text{if } j = \delta + n \text{ where } \delta = \alpha \text{ or } \delta \text{ is a limit ordinal and } n \text{ is an odd finite ordinal} \end{cases}$$

for every j with $\alpha \leq j < \beta$. In the sequel, we will use the obvious fact that, given an α-walk set $\mathcal{B}$ in a graph, $(x_i|\ i < \alpha) \in \mathcal{B}$ implies $h_{\alpha,\beta}(x_i|\ i < \alpha) \in \hat{\mathcal{B}}$. (Indeed, if $(y_j|\ j < \beta) = h_{\alpha,\beta}(x_i|\ i < \alpha)$, putting $i_0 = j_0$ for every j_0 such that $0 < j_0 < \alpha$ and $i_0 = \alpha - 2$ ($i_0 = \alpha - 1$) for every j_0 such that $y_{j_0} \neq y_0$ and $j_0 = \delta + n$ where $\delta = \alpha$ or δ is a limit ordinal and n is an even (odd) finite ordinal, we get $y_{j_0} = x_{i_0}$ and $\{x_i;\ i < i_0\} = \{y_j;\ j < j_0\}$.)

Theorem 4. *Let α, β be ordinals, $\alpha > 1$ finite and $\beta \geq \alpha$. Then, $H_{\alpha,\beta}$ is a full concrete embedding of $G_\alpha(Clo)$ into $G_\beta(Clo)$ having the property $F_\beta \circ H_{\alpha,\beta} = F_\alpha$.*

Proof. Let $((X, E), \mathcal{B}) \in G_\alpha(Clo)$ be an object and let $(y_j|\ j < \beta) \in \mathcal{W}_\beta(X, E)$ be a walk having the property that, for every j_0, $0 < j_0 < \beta$, we have $y_{j_0} = y_0$ or there exist $(x_i|\ i < \beta) \in \hat{\mathcal{B}}$ and i_0, $0 < i_0 < \beta$, such that $y_{j_0} = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq \{y_j;\ j < j_0\}$. Let j_0, $0 < j_0 < \beta$, be an ordinal with $y_{j_0} \neq y_0$. Then, $(x_i|\ i < \beta) \in \hat{\mathcal{B}}$ implies that $x_{i_0} = x_0$ or there exist $(z_k|\ k < \alpha) \in \mathcal{B}$ and k_0, $0 < k_0 < \alpha$, such that $x_{i_0} = z_{k_0}$ and $\{z_k;\ k < k_0\} \subseteq \{x_i;\ i < i_0\}$. If $x_{i_0} \neq x_0$, then we have $y_{j_0} = z_{k_0}$ and $\{z_k;\ k < k_0\} \subseteq \{y_j;\ j < j_0\}$. Suppose that $x_{i_0} = x_0$. Since $x_0 \in \{y_i;\ i < i_0\}$ and $x_0 = x_{i_0} = y_{j_0} \neq y_0$, there exists the least index (ordinal) j_1, $0 < j_1 < j_0$, such that $x_0 = y_{j_1} \neq y_0$. Therefore, there exist $(t_l|\ l < \alpha) \in \hat{\mathcal{B}}$ and l_0, $0 < l_0 < \alpha$, such that $y_{j_0} = x_{i_0} = x_0 = y_{j_1} = t_{l_0}$ and $\{t_l;\ l < l_0\} \subseteq \{y_j;\ j < j_1\} \subseteq \{y_j;\ j < j_0\}$. Since $t_0 \in \{y_j;\ j < j_1\}$, we have $t \neq t_{l_0}$ (because $t_{l_0} = x_0$ and j_1 is the smallest index with $y_{j_1} = x_0$). Consequently, there exist $(u_m|\ m < \alpha) \in \mathcal{B}$ and m_0, $0 < m_0 < \alpha$, such that $t_{l_0} = u_{m_0}$ and $\{u_m;\ m < m_0\} \subseteq \{t_l;\ l < l_0\}$. We have $y_{j_0} = x_{i_0} = x_0 = t_{l_0} = u_{m_0}$ and

$\{u_m;\ m < m_0\} \subseteq \{y_j;\ j < j_0\}$. Therefore, $(y_j|\ j < \beta) \in \hat{\mathcal{B}}$. We have shown that $\hat{\mathcal{B}}$ satisfies the condition $(*)$ in Theorem 2.

Let $\{x, y\} \in E = E_{\mathcal{B}}$. Then, there exist $(x_i|\ i < \alpha) \in \mathcal{B}$ and i_0, $0 < i_0 + 1 < \alpha$, such that $\{x, y\} = \{x_{i_0}, x_{i_0+1}\}$. Put $(y_j|\ j < \beta) = h_{\alpha,\beta}(x_i|\ i < \alpha)$. Then, $(y_j|\ j < \beta) \in \hat{\mathcal{B}}$ and, since $\{x_{i_0}, x_{i_0+1}\} = \{y_{i_0}, y_{i_0+1}\}$, we have $\{x, y\} = \{y_{i_0}, y_{i_0+1}\}$. Thus, $\{x, y\} \in E_{\hat{\mathcal{B}}}$. Hence, $E_{\mathcal{B}} \subseteq E_{\hat{\mathcal{B}}}$ and, since $\hat{\mathcal{B}} \in \mathcal{W}_\beta(X, E)$, we have $E = E_{\mathcal{B}} = E_{\hat{\mathcal{B}}}$. We have shown that also the condition $(**)$ in Theorem 2 is satisfied. Consequently, $((X, E), \hat{\mathcal{B}}) = H_{\alpha,\beta}((X, E), \mathcal{B}) \in G_\beta(Clo)$.

Let $\varphi : ((X_1, E_1), \mathcal{B}_1) \to ((X_2, E_2), \mathcal{B}_2)$ be a morphism in $G_\alpha(Clo)$ and let $(y_j|\ j < \beta) \in \hat{\mathcal{B}}_1$. Then, for every j_0, $0 < j_0 < \beta$, we have $y_{j_0} = y_0$ or there exist $(x_i|\ i < \alpha) \in \mathcal{B}_1$ and i_0, $0 < i_0 < \alpha$, such that $y_{j_0} = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq \{y_j;\ j < j_0\}$. We then have $(\varphi(x_i)|\ i < \alpha) \in \mathcal{B}_2$, $\varphi(y_{j_0}) = \varphi(x_{i_0})$ and $\{\varphi(x_i);\ i < i_0\} \subseteq \{\varphi(y_j);\ j < j_0\}$. Therefore, $(\varphi(y_j)|\ j < \beta) \in \hat{\mathcal{B}}_2$ so that $\varphi : ((X_1, E_1), \hat{\mathcal{B}}_1) \to ((X_2, E_2), \hat{\mathcal{B}}_2)$ is a morphism in $G_\beta(Clo)$. We have shown that $H_{\alpha,\beta}$ is a functor from $G_\alpha(Clo)$ into $G_\beta(Clo)$.

Conversely, let $((X_1, E_1), \mathcal{B}_1)$, $((X_2, E_2), \mathcal{B}_2) \in G_\alpha(Clo)$ be objects and let $\varphi : ((X_1, E_1), \hat{\mathcal{B}}_1) \to ((X_2, E_2), \hat{\mathcal{B}}_2)$ be a morphism in $G_\beta(Clo)$. Let $(x_i|\ i < \alpha) \in \mathcal{B}_1$ and let $(y_j|\ j < \beta) = h_{\alpha,\beta}(x_i|\ i < \alpha)$. Then, $(y_j|\ j < \beta) \in \hat{\mathcal{B}}_1$ so that $(\varphi(y_j)|\ j < \beta) \in \hat{\mathcal{B}}_2$. Therefore, for every j_0, $0 < j_0 < \alpha$, we have $\varphi(y_{j_0}) = \varphi(y_0)$ or there are $(z_i|\ i < \alpha) \in \mathcal{B}_2$ and i_0, $0 < i_0 < \alpha$, such that $\varphi(y_{j_0}) = z_{i_0}$ and $\{z_i;\ i < i_0\} \subseteq \{\varphi(y_j);\ j < j_0\}$. Since $\mathcal{B}_2$ satisfies the condition $(*)$ in Theorem 2, we have $(\varphi(y_j)|\ j < \alpha) \in \mathcal{B}_2$. But $(\varphi(y_j)|\ j < \alpha) = (\varphi(x_i)|\ i < \alpha)$ so that $\varphi : ((X_1, E_1), \mathcal{B}_1) \to ((X_2, E_2), \mathcal{B}_2)$ is a morphism in $G_\alpha(Clo)$. We have shown that the functor $H_{\alpha,\beta} : G_\alpha(Clo) \to G_\beta(Clo)$ is full.

Let $((X_1, E_1), \mathcal{B}_1)$, $((X_2, E_2), \mathcal{B}_2) \in G_\alpha(Clo)$ be objects and suppose that $\hat{\mathcal{B}}_1 = \hat{\mathcal{B}}_2$. Let $(x_i|\ i < \alpha) \in \mathcal{B}_1$ and let $(y_j|\ j < \beta) = h_{\alpha,\beta}(x_i|\ i < \alpha)$. Then, $(y_j|\ j < \beta) \in \hat{\mathcal{B}}_1$ so that, for every j_0, $0 < j_0 < \alpha$, we have $y_{j_0} = y_0$ or there are $(z_i|\ i < \alpha) \in \mathcal{B}_2$ and i_0, $0 < i_0 < \alpha$, such that $y_{j_0} = z_{i_0}$ and $\{z_i;\ i < i_0\} \subseteq \{y_j;\ j < j_0\}$. Since $\mathcal{B}_2$ satisfies the condition $(*)$ in Theorem 2, we have $(y_j|\ j < \alpha) \in \mathcal{B}_2$. But $(y_j|\ j < \alpha) = (x_i|\ i < \alpha)$ so that $\mathcal{B}_1 \subseteq \mathcal{B}_2$. We may show in an analogous way that $\mathcal{B}_2 \subseteq \mathcal{B}_1$. Therefore, the functor $H_{\alpha,\beta} : G_\alpha(Clo) \to G_\beta(Clo)$ is injective on objects. Consequently, $H_{\alpha,\beta} : G_\alpha(Clo) \to G_\beta(Clo)$ is a full concrete embedding.

Let $((X, E), \mathcal{B}) \in G_\alpha(Clo)$ and put $(X, u) = F_\beta(H_{\alpha,\beta}((X, E), \mathcal{B})) = (X, f_\beta(\hat{\mathcal{B}}))$, $(X, v) = F_\alpha((X, E), \mathcal{B}) = (X, f_\alpha(\mathcal{B}))$. Let $A \subseteq X$ and $y \in uA$. Then, $y \in A$ or there exist $(y_j|j < \beta) \in \hat{\mathcal{B}}$ and $j_0, 0 < j_0 < \beta$, such that $y = y_{j_0}$ and $y_j \in A$ for all $j < j_0$. If $y \in A$, then $y \in vA$. Suppose that $y \notin A$. Then, for each $j_0, 0 < j_0 < \beta$, there exist $(x_i|i < \alpha) \in \mathcal{B}$ and $i_0, 0 < i_0 < \alpha$, such that $y_{j_0} = x_{i_0}$ and $\{x_i; i < i_0\} \subseteq \{y_j; j < j_0\}$. Thus, $y = x_{i_0}$ and $x_i \in A$ for all $i < i_0$, which yields $y \in f_\alpha(\mathcal{B})A = vA$. We have shown that $u \le v$.

On the contrary, suppose that $y \in vA$. If $y \in A$, then $y \in uA$. Suppose that $y \notin A$. Then, there exist $(x_i|i < \alpha) \in \mathcal{B}$ and $i_0, 0 < i_0 < \alpha$, such that $y = x_{i_0}$ and $x_i \in A$ for all $i < i_0$. Let $(y_j|\ j < \beta) = h_{\alpha,\beta}(x_i|\ i < \alpha)$. Then, $(y_j|\ j < \beta) \in \hat{\mathcal{B}}$, $y = y_0$ and $y_j \in A$ for all $j < i_0$. Therefore, $y \in f_\beta(\hat{\mathcal{B}})A = uA$ and we have shown that $v \le u$. Thus, $u = v$, which yields $F_\beta \circ H_{\alpha,\beta} = F_\alpha$. The proof is complete. $\square$

Theorem 4 immediately results in

Corollary 1. *$F_\alpha(Gra_\alpha)$ is a full subcategory of $F_\beta(Gra_\beta)$ whenever $\alpha > 1$ is finite and $\beta \ge \alpha$.*

Remark 1. In digital image processing, it is important to have the digital plane equipped with a structure providing a convenient concept of connectedness. The connectedness in the closure spaces $(V, f_n(\mathcal{B}))$, where (V, E) is a graph, $n > 1$ a finite ordinal and $\mathcal{B}$ an n-walk set in (V, E), was studied in [20] where it was shown that a space $(V, f_n(\mathcal{B}))$ is connected (i.e., for every pair A, B of disjoint closed subsets of V, $V = A \cup B$ implies that $A = \emptyset$ or $B = \emptyset$) if and only if the following condition is satisfied:

For every pair of two different vertices $x, y \in V$, there exist a sequence $(x_i|\ i < m) \in V^m$, $m > 1$ a finite ordinal, with $x_0 = x$ and $x_{m-1} = y$, and an increasing sequence $(j_k|\ k < p)$ of natural numbers with $j_0 = 0$ and $j_{p-1} = m - 1$ such that $j_{k+1} - j_k < n$ for all $k < p - 1$ and there is an n-walk $(y_i|\ i < n) \in \mathcal{B}_n$ with $x_i = y_{i-j_k}$ for every $i = j_k, j_k + 1, ..., j_{k+1}$ or $x_i = y_{j_{k+1}} - i$ for every $i = j_k, j_k + 1, ..., j_{k+1}$.

This result provides a useful tool for investigating connectedness in $(V, f_n(\mathcal{B}))$.

4. Graphs with diagonal walk sets

Definition 1. Let $\alpha > 1$ be an ordinal. An α-walk set $\mathcal{B}$ in a graph (X, E) is said to be *diagonal* if, whenever $x_{ij} \in X$ for all ordinals $i, j < \alpha$, from $(x_{ij}|j < \alpha) \in \mathcal{B}$ for each $i < \alpha$ and $(x_{ij}|i < \alpha) \in \mathcal{B}$ for each $j < \alpha$, it follows that $(x_{ii}|i < \alpha) \in \mathcal{B}$.

Thus, an α-walk set $\mathcal{B}$ in a graph (X, E) is diagonal if, whenever a matrix of type $\alpha \times \alpha$ over X has the property that all its rows and columns belong to $\mathcal{B}$, its diagonal belongs to $\mathcal{B}$, too.

Let $G_\alpha(Clo)^*$ ($\alpha > 1$ an ordinal) be the full subcategory of $G_\alpha(Clo)$ given by the objects $(G, \mathcal{B}) \in G_\alpha(Clo)$ with $\mathcal{B}$ diagonal. Obviously, a 2-walk in a graph (X, E) is diagonal if and only if it is a transitive binary relation on X. Therefore, $G_2(Clo)^*$ is the category of graphs (X, E) with 2-walk sets that are irreflexive and transitive binary relations on X.

For every ordinal $\alpha > 1$, we denote by F^*_α the restriction of F_α onto $G_\alpha(Clo)^*$. Lemma 1 below immediately follows from the well-known fact that the subcategory of the category of topological spaces and continuous maps whose objects are the Alexandroff spaces is isomorphic to the category of preordered sets and preorder preserving maps where the isomorphism is given by the specialization preorder — cf. [2] (recall that the specialization preorder of a topology u on a set X is the binary relation ρ on X defined by $x\rho y \Leftrightarrow x \in u\{y\}$ whenever $x, y \in X$).

Lemma 1. *F^*_2 is an isomorphism of $G_2(Clo)^*$ onto Clo_S.*

Lemma 2. *Let $\alpha > 1$ be an ordinal and let $\mathcal{B}$ be a diagonal 2-walk set in a graph (X, E). Then, for every $(x_i|\ i < \alpha) \in X^\alpha$, we have $(x_i|\ i < \alpha) \in \hat{\mathcal{B}} \subseteq X^\alpha$ if and only if, for every i_0 with $0 < i_0 < \alpha$, $x_{i_0} = x_0$ or $(x_0, x_{i_0}) \in \mathcal{B}$.*

Proof. Let $(x_i|\ i < \alpha) \in \hat{\mathcal{B}} \subseteq X^\alpha$. Then, for every i_0, $0 < i_0 < \alpha$, $x_{i_0} = x_0$ or there exists $j_0 < i_0$ such that $(x_{j_0}, x_{i_0}) \in \mathcal{B}$. Thus, we have $(x_0 x_1) \in \mathcal{B}$. Let i_0, $1 < i_0 < \alpha$, be an arbitrary ordinal with $x_{i_0} \neq x_0$ and suppose that $x_j = x_0$ or $(x_0, x_j) \in \mathcal{B}$ for each $j < i_0$. Then, $(x_0, x_{i_0}) \in \mathcal{B}$ where $x_{i_0} = x_0$ or $(x_0, x_{i_0}) \in \mathcal{B}$. Hence, we have always have $(x_0, x_{i_0}) \in \mathcal{B}$ by the diagonality (transitivity) of $\mathcal{B}$. Thus, according to the principle of transfinite induction, $x_{i_0} = x_0$ or $(x_0, x_{i_0}) \in \mathcal{B}$. Therefore, $(x_i|\ i < \alpha) \in \hat{\mathcal{B}}$ implies $x_{i_0} = x_0$ or $(x_0, x_{i_0}) \in \mathcal{B}$ for every i_0 with $0 < i_0 < \alpha$. Since the converse implication is evident, the proof is complete. □

Given an ordinal $\alpha > 1$, we denote by $H^*_{2,\alpha}$ the restriction of $H_{2,\alpha}$ onto $G_2(Clo)^*$.

Proposition 2. *For every ordinal $\alpha > 1$, $H^*_{2,\alpha} : G_2(Clo)^* \to G_\alpha(Clo)^*$ is a full concrete embedding.*

Proof. Let $\alpha > 1$ be an ordinal, let $((X, E), \mathcal{B}) \in G_2(Clo)^*$ be an object and let $(x_{ij}|\ i, j < \alpha)$ be a matrix of type $\alpha \times \alpha$ over X such that $(x_{ij}|\ i < \alpha), (x_{ij}|\ j < \alpha) \in \hat{\mathcal{B}} \subseteq X^\alpha$. Let i_0, $0 < i_0 < \alpha$, be an ordinal and suppose that $x_{i_0 i_0} \neq x_{00}$. If $x_{0i_0} = x_{00}$, then we have $(x_{00}, x_{i_0 i_0}) \in \mathcal{B}$. Next, if $x_{0i_0} \neq x_{00}$, then $(x_{00}, x_{0i_0}) \in \mathcal{B}$ and we have either $x_{i_0 i_0} = x_{0i_0}$, in which case $(x_{00}, x_{i_0 i_0}) \in \mathcal{B}$, or $x_{i_0 i_0} \neq x_{0i_0}$, in which case $(x_{0i_0}, x_{i_0 i_0}) \in \mathcal{B}$ and, by the diagonality (transitivity) of $\mathcal{B}$, $(x_{00}, x_{i_0 i_0}) \in \mathcal{B}$. Thus, we have $x_{i_0 i_0} = x_{00}$ or $(x_{00}, x_{i_0 i_0}) \in \mathcal{B}$ whenever $0 < i_0 < \alpha$. Therefore, by Lemma 2, $(x_{ii}|\ i < \alpha) \in \hat{\mathcal{B}}$ so that $((X, E), \hat{\mathcal{B}}) = H_{2,\alpha}((X, E), \mathcal{B}) \in G_\alpha(Clo)^*$. Hence, the statement follows from Theorem 4. □

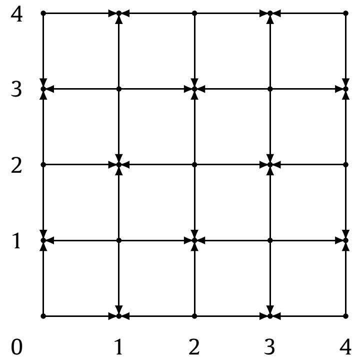
Note that, in consequence of Theorem 4 and Proposition 2, we have $F^*_2 = F_\alpha \circ H^*_{2,\alpha}$. Thus, by Proposition 1 and Lemma 1, we get:

Theorem 5. *Let $\alpha > 1$ be an ordinal and $(G, \mathcal{B}) \in G_\alpha(Clo)$ an object. If $f_\alpha(\mathcal{B})$ is idempotent, then $\mathcal{B}$ is diagonal.*

5. Walk-set induced closure operators on the digital plane

In digital topology, one of the basic problems is to find a structure on the digital plane $\mathbb{Z}^2$ convenient for processing digital images. The classical approach to digital topology is based on using the 4-adjacency and 8-adjacency graphs — cf. [11]. A disadvantage of this approach is that neither of these two graphs allows for a digital analogue of the Jordan curve theorem so that a combination of them must be used — cf. [15]. In [8], the authors proposed a new, purely topological approach to the problem that uses a topology, called the *Khalimsky* topology, for structuring the digital plane for the needs of digital image processing. Another topology that may be used for structuring the digital plane is the *Marcus–Wyse* one [14], which provides connectedness identical to the graph connectedness in the 4-adjacency graph (it is well known that no topology on the digital plane provides connectedness identical to the graph connectedness in the 8-adjacency graph). It was shown in [17] that closure operators, which are more general than topologies, may be used as convenient structures on the digital plane. In the following examples, we will also propose using closure operators for structuring the digital plane, namely closure operators induced by path sets in certain graphs on $\mathbb{Z}^2$. Employing such closure operators is a combination of the classical approach to digital topology and the topological one, and has the advantages of both of them. We will show that both the Marcus–Wyse and Khalimsky topologies on $\mathbb{Z}^2$ may be obtained as closure operators on graphs with the vertex set $\mathbb{Z}^2$ induced by certain walk sets belonging to $G_\alpha(Clo)^*$, $\alpha > 1$ an ordinal.

Example 2. Let $G_4 = (\mathbb{Z}^2, E)$ where $E = \{\{(x, y), (z, t)\};\ \ (x, y), (z, t) \in \mathbb{Z}^2, |x - z| + |y - t| = 1\}$. Then, G_4 is called the *4-adjacency graph* on $\mathbb{Z}^2$. Put $\mathcal{B} = \{((x_i, y_i)|\ i < 2);\ \ (x_i, y_i) \in \mathbb{Z}^2$ for every $i < 2$, $|x_0 - x_1| + |y_0 - y_1| = 1$ and $x_0 + y_0$ is even$\}$. Then, $\mathcal{B}$ satisfies the conditions $(*)$ and $(**)$ in Theorem 2 so that $(G_4, \mathcal{B}) \in G_2(Clo)$ and it is evident that we even have $(G_4, \mathcal{B}) \in G_2(Clo)^*$. A portion of $\mathcal{B}$ is shown in the following figure where the 2-paths from $\mathcal{B}$ are represented by arrows directed from first to last vertices.

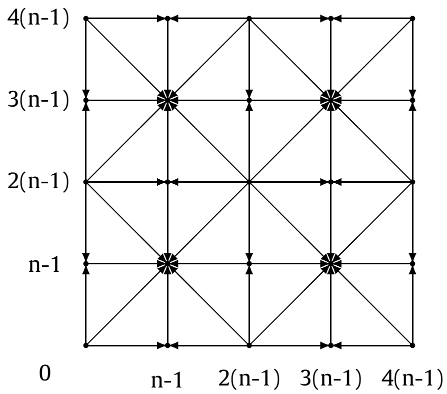


Clearly, $F_2(G_4, \mathcal{B}) = (\mathbb{Z}^2, f_2(\mathcal{B}))$ is a connected Alexandroff topological space in which the points $(x, y) \in \mathbb{Z}^2$ with $x + y$ even are open while those with $x + y$ odd are closed. The closure operator $f_2(\mathcal{B})$ coincides with the Marcus–Wyse topology.

Example 3. Let $n > 1$ be a finite ordinal and let $G_8 = (\mathbb{Z}^2, E)$ be a graph with $E = \{\{(x, y), (z, t)\};\ (x, y), (z, t) \in \mathbb{Z}^2, |x - z| + |y - t| > 0,\ |x - z| \leq 1,\ |y - t| \leq 1\}$. The graph G_8 is called the *8-adjacency graph* on $\mathbb{Z}^2$. Let $\mathcal{D}$ be the set of all sequences $((x_i, y_i)|\ i < n) \in (\mathbb{Z}^2)^n$ such that one of the following eight conditions is satisfied:

(1) $x_0 = x_1 = \ldots = x_{n-1}$ and there is $k \in \mathbb{Z}$ such that $y_i = 2k(n-1) + i$ for all $i < n$,
(2) $x_0 = x_1 = \ldots = x_{n-1}$ and there is $k \in \mathbb{Z}$ such that $y_i = 2k(n-1) - i$ for all $i < n$,
(3) $y_0 = y_1 = \ldots = y_{n-1}$ and there is $k \in \mathbb{Z}$ such that $x_i = 2k(n-1) + i$ for all $i < n$,
(4) $y_0 = y_1 = \ldots = y_{n-1}$ and there is $k \in \mathbb{Z}$ such that $x_i = 2k(n-1) - i$ for all $i < n$,
(5) there is $k \in \mathbb{Z}$ such that $x_i = 2k(n-1) + i$ for all $i < n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2l(n-1) + i$ for all $i < n$,
(6) there is $k \in \mathbb{Z}$ such that $x_i = 2k(n-1) + i$ for all $i < n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2l(n-1) - i$ for all $i < n$,
(7) there is $k \in \mathbb{Z}$ such that $x_i = 2k(n-1) - i$ for all $i < n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2l(n-1) + i$ for all $i < n$,
(8) there is $k \in \mathbb{Z}$ such that $x_i = 2k(n-1) - i$ for all $i < n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2l(n-1) - i$ for all $i < n$.

It may easily be seen that $(G_8, \mathcal{D}) \in G_n(Clo)^*$. A portion of $\mathcal{D}$ can be seen in the following figure. The paths belonging to $\mathcal{D}$ are represented by arrows directed from first to last terms. Between any pair of neighboring parallel horizontal or vertical arrows (having the same direction), there are $n - 2$ more parallel arrows with the same direction that are not displayed.



Clearly, $F_n(G_8, \mathcal{D}) = (\mathbb{Z}^2, f_n(\mathcal{D}))$ is a connected S_n-space. In particular, the closure operator $f_2(\mathcal{D})$ is an Alexandroff topology that coincides with the Khalimsky topology on $\mathbb{Z}^2$.

By the previous examples, the closure operators $f_n(\mathcal{B})$ with $n > 1$ a finite ordinal and $\mathcal{B}$ an n-walk set in a graph on the digital plane, may be regarded as generalizations of both the Marcus–Wyse topology and the Khalimsky topology on $\mathbb{Z}^2$. Both the topologies are known to provide convenient background structures on the digital plane for the study of digital images because they allow for digital analogues of the Jordan curve theorem (for a digital Jordan curve theorem in the Marcus–Wyse topology see [9] and for one in the Khalimsky topology see [8]). To show that the closure operators $f_n(\mathcal{B})$ may also be used in digital topology for structuring the digital plane, it would be beneficial to prove a digital Jordan curve theorem for them starting with proving such a theorem for $f_n(\mathcal{D})$, $n > 1$ a finite ordinal (see Example 3). This will be the aim of our forthcoming research.

Acknowledgment

This work was supported by the Ministry of Education, Youth and Sports of the Czech Republic from the National Programme of Sustainability (NPU II) project "IT4Innovations Excellence in Science — LQ1602".

References

[1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, Wiley & Sons, 1990.
[2] T.J. Ahlborn, T.N. Bhargava, On topological spaces associated with digraphs, Acta Math. Acad. Sci. Hung. 19 (1968) 47–52.
[3] E. Čech, Topological spaces, in: Topological Papers of Eduard Čech, Academia, 1968, pp. 436–472, ch. 28.
[4] E. Čech, Topological Spaces (revised by Z. Frolík and M. Katětov), Academia, 1966.
[5] J. Chvalina, Separation properties of topologies associated with digraphs, Scr. Fac. Sci. Nat. Univ. Purkyn. Brun., Math. 10 (8) (1980) 399–410.
[6] R. Engelking, General Topology, Państwowe Wydawnictwo Naukowe, 1977.
[7] F. Harrary, Graph Theory, Addison-Wesley Publ. Comp, 1969.
[8] E.D. Khalimsky, R. Kopperman, P.R. Meyer, Computer graphics and connected topologies on finite ordered sets, Topol. Appl. 36 (1990) 1–17.
[9] C.O. Kiselman, Digital Jordan curve theorems, Lect. Notes Comput. Sci. 1953 (2000) 46–56.
[10] T.Y. Kong, R. Kopperman, P.R. Meyer, A topological approach to digital topology, Am. Math. Mon. 98 (1991) 902–917.
[11] T.Y. Kong, W. Roscoe, A theory of binary digital pictures, Comput. Vis. Graph. Image Process. 32 (1985) 221–243.
[12] T.Y. Kong, A. Rosenfeld, Digital topology: introduction and survey, Comput. Vis. Graph. Image Process. 48 (1989) 357–393.
[13] M. Malitza, Topology, binary relations and internal operations, Rev. Roum. Math. Pures Appl. 33 (1988) 623–630.
[14] D. Marcus, et al., A special topology for the integers (Problem 5712), Am. Math. Mon. 77 (1970) 1119.
[15] A. Rosenfeld, Digital topology, Am. Math. Mon. 86 (1979) 621–630.
[16] J. Šlapal, On closure operations induced by binary relations, Rev. Roum. Math. Pures Appl. 33 (1988) 623–630.
[17] J. Šlapal, Convenient closure operators on $\mathbb{Z}^2$, Lect. Notes Comput. Sci. 5852 (2009) 425–436.
[18] J. Šlapal, Graphs with a path partition for structuring digital spaces, Inf. Sci. 233 (2013) 305–312.
[19] J. Šlapal, Topological structuring of the digital plane, Discret. Math. Theor. Comput. Sci. 15 (2013) 165–176.
[20] J. Šlapal, Path-set induced closure operators on graphs, Filomat 30 (2016) 863–871.