Absolutely Unlimited Deep Pushdown Automata

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Abstract. This paper introduces an absolutely unlimited deep pushdown automata and studies their computational power. These automata are generalized versions of recently introduced deep pushdown automata in the terms of the depth of expansions. They can expand nonterminal pushdown symbol despite its depth. It is shown that propagating and erasing versions of absolutely unlimited deep pushdown automata characterize type 1 and type 0 languages, respectively.

Keywords: deep pushdown automata, unlimited deep pushdown automata, computational power, absolutely unlimited deep expansions

1 Introduction

Deep pushdown automata (DPDA for short) as a generalization of classical pushdown automata and the automata counterpart to n-limited state grammars were introduced in [3]. State grammars were introduced in [2], however, not only their n-limited versions. The main idea behind DPDA comes from LL syntax analysis, where the topmost symbol on the pushdown can be either popped or expanded. As it was proved in [3], allowing the expansions to be performed deeper in the pushdown will increase the accepting power of classical pushdown automata. With this feature, LL parsers are able to parse languages that are not context-free, but not every context-sensitive language, since the depth of expansions is limited. As a natural extension of DPDA, in this study we consider DPDA with no limit imposed on the depth of expansions, which are also a counterpart to unlimited state grammars.

We introduce unlimited deep pushdown automata (UDPDA for short), which stays as an automata counterpart to unlimited state grammars. An unlimited deep pushdown automaton can expand nonterminal pushdown symbols, regardless how deep in the pushdown stack they occur. We distinguish two types of determination of nonterminal to be expanded based on its position within the pushdown—absolute and relative. However, in this paper, we consider only the absolutely unlimited deep pushdown expansions, the study on the relatively unlimited deep pushdown expansions is currently being prepared.

Absolutely unlimited deep pushdown expansions are very natural generalization of n-limited deep pushdown expansions. The topmost nonterminal specified by applied rule is expanded independently of its depth. It is proved that absolutely unlimited deep pushdown expansions provide the power of linear bounded automata for propagating versions and the power of Turing machines for erasing versions.

2 Preliminaries

We assume that the reader is familiar with formal language theory (see [4, 5]). For a set W, $\mathrm{card}(W)$ denotes its cardinality. Let V be an alphabet (finite nonempty set). V^* is the set of all strings over V. Algebraically, V^* represents the free monoid generated by V under the operation of concatenation. The unit of V^* is denoted by ε. Set $V^+ = V^* - \{\varepsilon\}$. Algebraically, V^+ is thus the free semigroup generated by V under the operation of concatenation. For $w \in V^*$, $|w|$ denotes the length of w. The alphabet of w, denoted by $\mathrm{alph}(w)$, is the set of symbols appearing in w.

Let $\Rightarrow$ be a relation over V^*. The transitive and transitive-reflexive closure of $\Rightarrow$ are denoted $\Rightarrow^+$ and $\Rightarrow^*$, respectively. Unless we explicitly stated otherwise, we write $x \Rightarrow y$ instead of $(x, y) \in \Rightarrow$.

The families of context-free, context-sensitive and recursively enumerable languages are denoted by **CF**, **CS** and **RE**, respectively.

A *state grammar* (see [2]) is a 6-tuple $G = (K, V, T, P, S, s)$, where K is a nonempty finite set of states, V is a total alphabet, $T \subset V$ is a terminal alphabet, $P \subseteq K \times (V - T) \times K \times V^*$ is a finite relation called the set of productions, $S \in V - T$ is the initial symbol, and $s \in K$ is the initial state. Instead of $(p, A, q, x) \in P$, we write $(p, A) \to (q, x) \in P$. Let $\Rightarrow$ be a relation of direct derivation on $K \times V^*$ defined as follows: $(p, uAv) \Rightarrow (q, uxv)$ iff $(p, A) \to (q, x) \in P$ and $(p, A') \to \alpha \notin P$, where $p, q \in K$, $A \in V - T$, $u, v, x \in V^*$, $A' \in \mathrm{alph}(u) - T$, and $\alpha \in K \times V^*$. By $(p, uAv) \Rightarrow (q, uxv)\,[(p, A) \to (q, x)]$, we express that (p, uAv) directly derives (q, uxv) according to $(p, A) \to (q, x)$. In the standard manner, we extend $\Rightarrow$ to $\Rightarrow^m$, where $m \geq 0$; then, based on $\Rightarrow^m$, we define $\Rightarrow^+$ and $\Rightarrow^*$. The language generated by G, denoted as $L(G)$, is defined as $L(G) = \{w \in T^* \mid (s, S) \Rightarrow^* (q, w), q \in K\}$. A state grammar G is called propagating, if for every production $(p, A) \to (q, w) \in P$, $w \neq \varepsilon$. By **ST** and **pST**, we denote the family of languages of all state grammars and the family of languages of all propagating state grammars, respectively.

Recall that **ST** = **RE** (see [1]) and **pST** = **CS** (see [2]).

3 Definitions and Examples

In this section, we define an absolutely unlimited deep pushdown automata and demonstrate them by examples.

Informally, during every computational step an absolutely unlimited deep pushdown automaton either pops or expands its pushdown. In the case, the topmost pushdown symbol is a terminal, it is compared with the current input symbol and if they correspond, the pushdown symbol is popped and the input symbol is read. Otherwise, the pushdown may be expanded. With an absolutely unlimited deep pushdown expansion, a nonterminal symbol is chosen and its topmost occurrence is rewritten.

The following definition is based on the original definition of deep pushdown automata, which can be found in [3].

Definition 1. An *absolutely unlimited deep pushdown automaton* (AUDPDA for short) is an 8-tuple $M = (Q, \Sigma, \Gamma, \#, R, s, S, F)$, where Q is a finite *set of states*, Σ is an *input alphabet*, Γ is a *pushdown alphabet*, $\Gamma \cap Q = \emptyset$, $\Sigma \subset \Gamma$, $\# \in \Gamma - \Sigma$ is the special symbol called *bottom marker*, $R \subseteq Q \times (\Gamma - (\Sigma \cup \{\#\})) \times Q \times (\Gamma - \{\#\})^* \cup Q \times \{\#\} \times Q \times (\Gamma - \{\#\})^*\{\#\}$ is a finite relation called the *set of rules*, $s \in Q$ is the *initial state*, $S \in \Gamma$ is the *initial pushdown symbol*, and $F \subseteq Q$ is the set of *final states*. Instead of $(p, A, q, x) \in R$, we write $pA \to qx \in R$. Set $\chi = Q \times \Sigma^* \times (\Gamma - \{\#\})^*\{\#\}$. A *configuration of* M is any element of χ.

Let ${}^{a}_{p}\vdash$ be a relation on χ such that $(q, cw, cz) \, {}^{a}_{p}\vdash (q, w, z)$, where $q \in Q$, $c \in \Sigma$, $w \in \Sigma^*$, and $z \in \Gamma^*$. Then, we say that M *pops* its pushdown from (q, cw, cz) to (q, w, z). Similarly, let ${}^{a}_{e}\vdash$ be a relation on χ such that $(p, w, uAv) \, {}^{a}_{e}\vdash (q, w, uxv)$ iff $pA \to qx \in R$, where $p, q \in Q$, $w \in \Sigma^*$, $u, x, v \in \Gamma^*$, $A \in \Gamma - \Sigma$, $A \notin \mathrm{alph}(u)$ and for every $A' \in \mathrm{alph}(u) - \Sigma$, $pA' \to q'x' \notin R$, where $q' \in Q$ and $x' \in \Gamma^*$. By $(p, w, uAv) \, {}^{a}_{e}\vdash (q, w, uxv) \, [pA \to qx]$, we express that M *expands* its pushdown with *absolutely unlimited deep pushdown expansion* from (p, w, uAv) to (q, w, uxv) according to $pA \to qx$. A *direct move* relation on χ, denoted ${}^{a}\vdash$, is defined as ${}^{a}\vdash \, = {}^{a}_{p}\vdash \cup {}^{a}_{e}\vdash$.

In the standard manner, extend ${}^{a}_{p}\vdash$, ${}^{a}_{e}\vdash$, and ${}^{a}\vdash$ to ${}^{a}_{p}\vdash^m$, ${}^{a}_{e}\vdash^m$, and ${}^{a}\vdash^m$, respectively, for $m \geq 0$; then, based on ${}^{a}_{p}\vdash^m$, ${}^{a}_{e}\vdash^m$, and ${}^{a}\vdash^m$, define ${}^{a}_{p}\vdash^+$, ${}^{a}_{p}\vdash^*$, ${}^{a}_{e}\vdash^+$, ${}^{a}_{e}\vdash^*$, ${}^{a}\vdash^+$, and ${}^{a}\vdash^*$.

The *language accepted by* M, denoted as $L(M)$, is defined as

$$L(M) = \{w \in \Sigma^* \mid (s, w, S\#) \, {}^{a}\vdash^* (f, \varepsilon, \#), f \in F\}.$$

In addition, we define the *language that* M *acceps by empty pushdown*, $L(M)_\varepsilon$, as

$$L(M)_\varepsilon = \{w \in \Sigma^* \mid (s, w, S\#) \, {}^{a}\vdash^* (q, \varepsilon, \#), q \in Q\}.$$

An AUDPDA M is called *propagating* (pAUDPDA for short), if for every rule $pA \to qx \in R$, $x \neq \varepsilon$. ■

Let **AUDPD** and **pAUDPD** denote the *family of all absolutely unlimited deep pushdown automata languages* and the *family of all propagating absolutely unlimited deep pushdown automata languages*, respectively.

Example 1. Consider the absolutely unlimited deep pushdown automaton

$$M = (Q, \{a\}, \{S, A, X, A', X', \#, a\}, \#, R, \langle s\rangle, S, \{\langle f\rangle\})$$

with $Q = \{\langle s\rangle, \langle c\rangle, \langle 1\rangle, \langle 2\rangle, \langle 1'\rangle, \langle 2'\rangle, \langle f\rangle\}$ and R containing rules

$$\begin{array}{rlrl}
\langle s\rangle S \to \langle c\rangle aSAX, & & \langle c\rangle S \to \langle c\rangle aSA, \\
\langle c\rangle S \to \langle 1\rangle, & & \langle 1\rangle A \to \langle 2\rangle, \\
\langle 1\rangle X \to \langle 1'\rangle X', & & \langle 2\rangle A \to \langle 1\rangle A', \\
\langle 2\rangle X \to \langle f\rangle, & & \langle 1'\rangle A' \to \langle 2'\rangle, \\
\langle 1'\rangle X' \to \langle 1\rangle X, & & \langle 2'\rangle A' \to \langle 1'\rangle A, \\
\langle 2'\rangle X' \to \langle f\rangle. & &
\end{array}$$

On $aaaa$, M makes

$$\begin{array}{rll}
(\langle s\rangle, aaaa, S\#) & {}_e^a\vdash (\langle c\rangle, aaaa, aSAX\#) & [\langle s\rangle S \to \langle c\rangle aSAX] \\
& {}_p^a\vdash (\langle c\rangle, aaa, SAX\#) & \\
& {}_e^a\vdash (\langle c\rangle, aaa, aSAAX\#) & [\langle c\rangle S \to \langle c\rangle aSA] \\
& {}_p^a\vdash (\langle c\rangle, aa, SAAX\#) & \\
& {}_e^a\vdash (\langle c\rangle, aa, aSAAAX\#) & [\langle c\rangle S \to \langle c\rangle aSA] \\
& {}_p^a\vdash (\langle c\rangle, a, SAAAX\#) & \\
& {}_e^a\vdash (\langle c\rangle, a, aSAAAAX\#) & [\langle c\rangle S \to \langle c\rangle aSA] \\
& {}_p^a\vdash (\langle c\rangle, \varepsilon, SAAAAX\#) & \\
& {}_e^a\vdash (\langle 1\rangle, \varepsilon, AAAAX\#) & [\langle c\rangle S \to \langle 1\rangle] \\
& {}_e^a\vdash (\langle 2\rangle, \varepsilon, AAAX\#) & [\langle 1\rangle A \to \langle 2\rangle] \\
& {}_e^a\vdash (\langle 1\rangle, \varepsilon, A'AAX\#) & [\langle 2\rangle A \to \langle 1\rangle A'] \\
& {}_e^a\vdash (\langle 2\rangle, \varepsilon, A'AX\#) & [\langle 1\rangle A \to \langle 2\rangle] \\
& {}_e^a\vdash (\langle 1\rangle, \varepsilon, A'A'X\#) & [\langle 2\rangle A \to \langle 1\rangle A'] \\
& {}_e^a\vdash (\langle 1'\rangle, \varepsilon, A'A'X'\#) & [\langle 1\rangle X \to \langle 1'\rangle X'] \\
& {}_e^a\vdash (\langle 2'\rangle, \varepsilon, A'X'\#) & [\langle 1'\rangle A' \to \langle 2'\rangle] \\
& {}_e^a\vdash (\langle 1'\rangle, \varepsilon, AX'\#) & [\langle 2'\rangle A' \to \langle 1'\rangle A] \\
& {}_e^a\vdash (\langle 1\rangle, \varepsilon, AX\#) & [\langle 1'\rangle X' \to \langle 1\rangle X] \\
& {}_e^a\vdash (\langle 2\rangle, \varepsilon, X\#) & [\langle 1\rangle A \to \langle 2\rangle] \\
& {}_e^a\vdash (\langle f\rangle, \varepsilon, \#) & [\langle 2\rangle X \to \langle f\rangle]
\end{array}$$

In brief, $(\langle s\rangle, aaaa, S\#) \,{}^a\vdash^* (\langle f\rangle, \varepsilon, \#)$. Observe that $L(M) = L(M)_\varepsilon = \{a^{2^n} \mid n \geq 0\}$, which belongs to $\mathbf{CS} - \mathbf{CF}$. ■

4 Results

In this section, we prove that $\mathbf{AUDPD} = \mathbf{ST} = \mathbf{RE}$ and $\mathbf{pAUDPD} = \mathbf{pST} = \mathbf{CS}$.

Lemma 1. *For every state grammar, G, there exists an absolutely unlimited deep pushdown automaton, M, such that $L(G) = L(M)$.*

Proof. Let $G = (K, V, T, P, S, s)$ be a state grammar. Set $N = V - T$. Introduce the AUDPDA, $M = (K \cup \{\bar{f}\}, T, V \cup \{\#\}, \#, R, s, S, \{\bar{f}\})$, where R is constructed by performing the following steps:

(i) for every $(p, A) \to (q, x) \in P$, where $p, q \in K$, $A \in N$, and $x \in V^*$, add $pA \to qx$ to R;

(ii) for every $p \in K$, add $p\# \to \bar{f}\#$ to R.

Claim 2. *Let $(p, S) \Rightarrow^m (q, wy)$ in G, where $p, q \in K$, $w \in T^*$, $y \in (NT^*)^*$, and $m \geq 0$. Then, $(p, w, S\#)\, {}^a\!\vdash^*(q, \varepsilon, y\#)$ in M.*

Proof. This claim is proved by induction on $m \geq 0$.

Basis. Let $m = 0$, so $(p, S) \Rightarrow^0 (p, S)$ in G, $w = \varepsilon$ and $y = S$. Then,

$$(p, \varepsilon, S\#)\, {}^a\!\vdash^0(p, \varepsilon, S\#)$$

in M, so the basis holds.

Induction Hypothesis. Assume that the claim holds for all $0 \leq m \leq k$, where k is a non-negative integer.

Induction Step. Let $(p, S) \Rightarrow^{k+1} (q, wy)$ in G, where $p, q \in K$, $w \in T^*$, and $y \in (NT^*)^*$. Since $k+1 \geq 1$, express $(p, S) \Rightarrow^{k+1} (q, wy)$ as $(p, S) \Rightarrow^k (t, w'uAv) \Rightarrow (q, w'uxv)\, [(t, A) \to (q, x)]$, where $t \in K$, $w' \in T^*$, $u \in (NT^*)^*$, $A \in N$, $x, v \in V^*$, $(t, A) \to (q, x) \in P$, $w = w'\hat{w}$, and $\hat{w}y = uxv$ with $\hat{w} \in T^*$. By the induction hypothesis, there exists a move $(p, w', S\#)\, {}^a\!\vdash^*(t, \varepsilon, uAv\#)$ in M which implies that there also exists a move $(p, w'\hat{w}, S\#)\, {}^a\!\vdash^*(t, \hat{w}, uAv\#)$ in M. By the definition of a derivation step in G, there are no other rules with the (t, A') left-hand side in P, for all $A' \in \mathrm{alph}(u) - T$. From the first step of the construction of R follows that there must be a rule $tA \to qx$ in R. Thus, $(t, \hat{w}, uAv\#)\, {}^a_e\!\vdash(q, \hat{w}, uxv\#)$ in M according to $tA \to qx$ and there are no other rules with the tA' left-hand side in R, for all $A' \in \mathrm{alph}(u) - T$. Since $\hat{w}y = uxv$, we have $(q, \hat{w}, \hat{w}y\#)\, {}^a_p\!\vdash^{|\hat{w}|}(q, \varepsilon, y\#)$ in M, which completes the induction step.

$\square$

By the previous claim for $y = \varepsilon$, if $(s, S) \Rightarrow^* (q, w)$, where $q \in K$ and $w \in T^*$, then $(s, w, S\#)\, {}^a\!\vdash^*(q, \varepsilon, \#)$ in M. Since $q\# \to \bar{f}\# \in R$, we also have $(s, w, S\#)\, {}^a\!\vdash^*(\bar{f}, \varepsilon, \#)$ in M. Thus, $w \in L(G)$ implies $w \in L(M)$, so $L(G) \subseteq L(M)$.

Claim 3. *Let $(p, w, S\#)\, {}^a\!\vdash^m(q, \varepsilon, \hat{w}y\#)$ in M, where $p, q \in K$, $w, \hat{w} \in T^*$, $y \in (NT^*)^*$, and $m \geq 0$. Then, $(p, S) \Rightarrow^* (q, w\hat{w}y)$ in G.*

Proof. This claim is proved by induction on $m \geq 0$.

Basis. Let $m = 0$. Then, $w = \hat{w} = \varepsilon$, $y = S$, and

$$(p, \varepsilon, S\#)\, {}^a\!\vdash^0(p, \varepsilon, S\#)$$

in M. As $(p, S) \Rightarrow^0 (p, S)$ in G, the basis holds.

Induction Hypothesis. Assume that the claim holds for all $0 \leq m \leq k$, where k is a non-negative integer.

Induction Step. Let $(p, w, S\#)\, {}^a\!\vdash^{k+1}(q, \varepsilon, \hat{w}y\#)$ in M, where $p, q \in K$, $w, \hat{w} \in T^*$, and $y \in (NT^*)^*$. Since $k+1 \geq 1$, we can express $(p, w, S\#)\, {}^a\!\vdash^{k+1}(q, \varepsilon, \hat{w}y\#)$ as

$$(p, w, S\#)\, {}^a\!\vdash^k \chi\, {}^a\!\vdash(q, \varepsilon, \hat{w}y\#),$$

where χ is a configuration of M whose form depend on whether the last move is a popping move or an expansion.

(I) Assume that $\chi\ {}_p^a\vdash (q, \varepsilon, \hat{w}y\#)$ in M. In the greater detail, let

$$\chi = (q, a, a\hat{w}y\#)$$

with $a \in T$ such that $w = w'a$, where $w' \in T^*$. Thus,

$$(p, w, S\#)\ {}^a\vdash^k (q, a, a\hat{w}y\#)\ {}_p^a\vdash (q, \varepsilon, \hat{w}y\#).$$

Since $(p, w, S\#)\ {}^a\vdash^k (q, a, a\hat{w}y\#)$, we have

$$(p, w', S\#)\ {}^a\vdash^k (q, \varepsilon, a\hat{w}y\#).$$

By the induction hypothesis, $(p, S) \Rightarrow^* (q, w'a\hat{w}y)$ in G. As $w = w'a$, $(p, S) \Rightarrow^* (q, w\hat{w}y)$ in G.

(II) Assume that $\chi\ {}_e^a\vdash (q, \varepsilon, \hat{w}y\#)$ in M. If this expansion is made by the rule introduced in step (ii), then $q = \bar{f}$, $\hat{w} = \varepsilon$, $y = \varepsilon$, and the induction step follows from the induction hypothesis. Therefore, suppose that this expansion is made by a rule introduced in step (i). In greater detail, suppose that $\chi = (t, \varepsilon, uAv\#)$ and

$$(t, \varepsilon, uAv\#)\ {}_e^a\vdash (q, \varepsilon, uxv\#)$$

by using $tA \to qx \in R$, where $t \in K$, $A \in N$, $u \in (NT^*)^*$, $v, x \in V^*$, and $\hat{w}y = uxv$. By the induction hypothesis, $(p, w, S\#)\ {}^a\vdash^k (t, \varepsilon, uAv\#)$ implies $(p, S) \Rightarrow^* (t, wuAv)$ in G. As $tA \to qx \in R$, $(t, A) \to (q, x) \in P$ and for every $A' \in \mathrm{alph}(u) - T$, there is no other rule in P with its left-hand side in the form of (t, A'). Thus, $(p, S) \Rightarrow^* (t, wuAv) \Rightarrow (q, wuxv)$ in G. Thus, $(p, S) \Rightarrow^* (q, w\hat{w}y)$ in G since $\hat{w}y = uxv$.

□

Consider the previous claim for $\hat{w} = y = \varepsilon$ to see that

$$(s, w, S\#)\ {}^a\vdash^* (q, \varepsilon, \#)$$

in M implies $(s, S) \Rightarrow^* (q, w)$ in G. Let $w \in L(M)$. Then,

$$(S, w, S\#)\ {}^a\vdash^* (\bar{f}, \varepsilon, \#),$$

which can be expressed as $(s, w, S\#)\ {}^a\vdash^* (q, \varepsilon, \#)\ {}_e^a\vdash (\bar{f}, \varepsilon, \#)$. Observe that the last move is made by a rule introduced in step (ii). By the previous claim, $(s, S) \Rightarrow^* (q, w)$, so $w \in L(G)$. Thus, $w \in L(M)$ implies $w \in L(G)$, so $L(M) \subseteq L(G)$.

As $L(M) \subseteq L(G)$ and $L(G) \subseteq L(M)$, $L(G) = L(M)$. Thus, Lemma 1 holds.

□

Lemma 4. *For every absolutely unlimited deep pushdown automaton, M, there exists a state grammar, G, such that $L(M)\{\flat\} = L(G)$, where $\flat$ is a new symbol such that $\flat \notin \bigcup_{x \in L(M)} \mathrm{alph}(x)$.*

Proof. Let $M = (Q, \Sigma, \Gamma, \#, R, s, S, F)$ be an absolutely unlimited deep pushdown automaton. Set $N = \Gamma - \Sigma$. Introduce the state grammar, $G = (K, \Gamma \cup \{Z, \flat\}, \Sigma \cup \{\flat\}, P, \bar{s}, Z)$, where

$$K = Q \cup \{\bar{s}, \bar{f}\}$$

and P is constructed by performing the following steps:

(i) add $(\bar{s}, Z) \rightarrow (s, S\#)$ to P;
(ii) for every $pA \rightarrow qx \in R$, where $p, q \in Q$, $A \in N$, and $x \in \Gamma^*$, add $(p, A) \rightarrow (q, x)$ to P;
(iii) for every $p \in Q$, add $(p, \#) \rightarrow (\bar{f}, \flat)$ to P.

Claim 5. *Let $(p, S\#) \Rightarrow^m (q, wy\#)$ in G, where $p, q \in Q$, $w \in \Sigma^*$, $y \in ((N - \{\#\})\Sigma^*)^*$, and $m \geq 0$. Then, $(p, w, S\#) \, {}^a\!\vdash^* (q, \varepsilon, y\#)$ in M.*

Proof. This claim is proved by induction on $m \geq 0$.

Basis. Let $m = 0$, so $(p, S\#) \Rightarrow^0 (p, S\#)$ in G, $w = \varepsilon$ and $y = S$. Then, $(p, \varepsilon, S\#) \, {}^a\!\vdash^0 (p, \varepsilon, S\#)$ in M, so the basis holds.

Induction Hypothesis. Assume that the claim holds for all $0 \leq m \leq k$, where k is a non-negative integer.

Induction Step. Let $(p, S\#) \Rightarrow^{k+1} (q, wy\#)$ in G, where $p, q \in Q$, $w \in \Sigma^*$, and $y \in ((N - \{\#\})\Sigma^*)^*$. Observe that rules introduced in steps (i) and (iii) are not used.

Since $k + 1 \geq 1$, express $(p, S\#) \Rightarrow^{k+1} (q, wy\#)$ as

$$(p, S\#) \Rightarrow^k (t, w'uAv) \Rightarrow (q, w'uxv) \, [(t, A) \rightarrow (q, x)]$$

where $t \in Q$, $w' \in \Sigma^*$, $u \in ((N - \{\#\})\Sigma^*)^*$, $A \in N$, $x, v \in \Gamma^*$, $(t, A) \rightarrow (q, x) \in P$, and $wy\# = w'uxv$. Express w as $w = w'\hat{w}$ with $\hat{w} \in \Sigma^*$, so $\hat{w}y\# = uxv$. By the induction hypothesis, $(p, S\#) \Rightarrow^k (t, w'uAv)$ implies

$$(p, w', S\#) \, {}^a\!\vdash^* (t, \varepsilon, uAv)$$

in M, which implies $(p, w'\hat{w}, S\#) \, {}^a\!\vdash^* (t, \hat{w}, uAv)$ in M. As $(t, A) \rightarrow (q, x) \in P$ and there are no other rules with the (t, A') left-hand side in P, for all $A' \in \mathrm{alph}(u) - \Sigma$, $tA \rightarrow qx \in R$ must be the only applicable rule on $(t, \hat{w}, uAv)$. Thus, $(t, \hat{w}, uAv) \, {}^a_e\!\vdash (q, \hat{w}, uxv)$ in M. Since $\hat{w}y\# = uxv$, we have

$$(q, \hat{w}, \hat{w}y\#) \, {}^a_p\!\vdash^{|\hat{w}|} (q, \varepsilon, y\#)$$

in M, which completes the induction step. $\square$

Consider any $w \in L(G)$. Observe that $w = w'\flat$, where $w' \in \Sigma^*$. Next, observe that G generates w as

$$\begin{array}{rll} (\bar{s}, Z) \Rightarrow & (s, S\#) & [(\bar{s}, Z) \rightarrow (s, S\#)] \\ \Rightarrow^* & (q, w'\#) & (\text{Claim 5 with } y = \varepsilon) \\ \Rightarrow & (\bar{f}, w'\flat) & [(q, \#) \rightarrow (\bar{f}, \flat)] \end{array}$$

where $q \in Q$, and $(\bar{s}, Z) \rightarrow (s, S\#) \in P$ and $(q, \#) \rightarrow (\bar{f}, \flat) \in P$ are rules introduced in steps (i) and (iii) of the construction of P, respectively. Thus, $(s, w', S\#) \,{}^{a}{\vdash}^{*} (q, \varepsilon, \#)$ in M. Thus, $w'\flat \in L(G)$ implies $w' \in L(M)$, so $L(G) \subseteq L(M)\{\flat\}$.

Claim 6. *Let $(p, w, S\#) \,{}^{a}{\vdash}^{m} (q, \varepsilon, \hat{w}y\#)$, where $p, q \in Q$, $w, \hat{w} \in \Sigma^*$, $y \in ((N - \{\#\})\Sigma^*)^*$, and $m \geq 0$. Then, $(p, S\#) \Rightarrow^* (q, w\hat{w}y\#)$ in G.*

Proof. This claim is proved by induction on $m \geq 0$.

Basis. Let $m = 0$. Then, $w = \hat{w} = \varepsilon$, $y = S$, and $(p, \varepsilon, S\#) \,{}^{a}{\vdash}^{0} (p, \varepsilon, S\#)$ in M. As $(p, S\#) \Rightarrow^0 (p, S\#)$ in G, the basis holds.

Induction Hypothesis. Assume that the claim holds for all $0 \leq m \leq k$, where k is a non-negative integer.

Induction Step. Let $(p, w, S\#) \,{}^{a}{\vdash}^{k+1} (q, \varepsilon, \hat{w}y\#)$, where $p, q \in Q$, $w, \hat{w} \in \Sigma^*$, and $y \in ((N-\{\#\})\Sigma^*)^*$. Since $k+1 \geq 1$, we can express $(p, w, S\#) \,{}^{a}{\vdash}^{k+1} (q, \varepsilon, \hat{w}y\#)$ as $(p, w, S\#) \,{}^{a}{\vdash}^{k} \chi \,{}^{a}{\vdash} (q, \varepsilon, \hat{w}y\#)$, where χ is a configuration of M whose form depend on whether the last move is a popping move or an expansion.

(I) Assume that $\chi \,{}^{a}_{p}{\vdash} (q, \varepsilon, \hat{w}y\#)$ in M. In a greater detail, let

$$\chi = (q, a, a\hat{w}y\#)$$

with $a \in \Sigma$ such that $w = w'a$, where $w' \in \Sigma^*$. Thus,

$$(p, w, S\#) \,{}^{a}{\vdash}^{k} (q, a, a\hat{w}y\#) \,{}^{a}_{p}{\vdash} (q, \varepsilon, \hat{w}y\#).$$

Since $(p, w, S\#) \,{}^{a}{\vdash}^{k} (q, a, a\hat{w}y\#)$, we have $(p, w', S\#) \,{}^{a}{\vdash}^{k} (q, \varepsilon, a\hat{w}y\#)$. By the induction hypothesis, $(p, S\#) \Rightarrow^* (q, w'a\hat{w}y\#)$ in G. As $w = w'a$, $(p, S\#) \Rightarrow^* (q, w\hat{w}y\#)$ in G.

(II) Assume that $\chi \,{}^{a}_{e}{\vdash} (q, \varepsilon, \hat{w}y\#)$ in M. In a greater detail, let $\chi = (t, \varepsilon, uAv)$, where $t \in Q$, $A \in N$, $u \in ((N - \{\#\})\Sigma^*)^*$, and $v \in \Gamma^*$. By the induction hypothesis, $(p, w, S\#) \,{}^{a}{\vdash}^{k} (t, \varepsilon, uAv)$ implies $(p, S\#) \Rightarrow^* (t, wuAv)$ in G. Consider that $(t, \varepsilon, uAv) \,{}^{a}_{e}{\vdash} (q, \varepsilon, uxv)$ according to $tA \rightarrow qx \in R$ in M with $\hat{w}y\# = uxv$. Following the construction of P, there must be a rule $(t, A) \rightarrow (q, x)$ in P introduced in step (ii). As there are no other rules with the tA' left-hand side in P, there are also no other rules with the (t, A') left-hand side in R, for all $A' \in \mathrm{alph}(u) - \Sigma$. Thus, $(t, wuAv) \Rightarrow (q, wuxv)$ in G. Thus, putting the previous sequences of derivations together, we obtain $(p, S\#) \Rightarrow^* (q, w\hat{w}y\#)$ in G since $\hat{w}y\# = uxv$.

□

By the previous claim for $y = \hat{w} = \varepsilon$, if $(s, w, S\#) \,{}^{a}{\vdash}^{*} (q, \varepsilon, \#)$ in M, where $q \in Q$ and $w \in \Sigma^*$, then $(s, S\#) \Rightarrow^* (q, w\#)$ in G. As P contains rules introduced in steps (i) and (iii), we also have $(\bar{s}, Z) \Rightarrow (s, S\#) \Rightarrow^* (q, w\#) \Rightarrow (\bar{f}, w\flat)$ in G. Thus, $w \in L(M)$ implies $w\flat \in L(G)$, so $L(M)\{\flat\} \subseteq L(G)$.

As $L(M)\{\flat\} \subseteq L(G)$ and $L(G) \subseteq L(M)\{\flat\}$, $L(M)\{\flat\} = L(G)$. Thus, Lemma 4 holds. □

Theorem 7. $\mathbf{AUDPD} = \mathbf{RE}$.

Proof. In [1] $\mathbf{ST} = \mathbf{RE}$ was proved. From Lemma 1 and 4, $\mathbf{AUDPD} = \mathbf{ST}$, which completes the proof. $\square$

Theorem 8. $\mathbf{pAUDPD} = \mathbf{CS}$.

Proof. Following the construction of R in Lemma 1 and a proof of that lemma, it is clear that every context-sensitive language can be accepted by some propagating AUDPDA. Thus, $\mathbf{CS} \subseteq \mathbf{pAUDPD}$.

Conversely, following the construction of P in Lemma 4 and its proof, we have that for every propagating AUDPDA, M, there exists a propagating state grammar, G, such that $L(M)\{\flat\} = L(G)$, where $\flat$ is a symbol defined in Lemma 4. Let Σ be an alphabet such that $\flat \notin \Sigma$. Since the family of context-sensitive languages is closed under linear erasing (see [4]), for every $L \in 2^{\Sigma^*}$, $L\{\flat\} \in \mathbf{CS}$ implies $L \in \mathbf{CS}$. Thus, for every propagating AUDPDA M, $L(M)\{\flat\} \in \mathbf{CS}$ implies $L(M) \in \mathbf{CS}$, and then $\mathbf{pAUDPD} \subseteq \mathbf{CS}$. $\square$

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References

1. Horváth, G., Meduna, A.: On state grammars. Acta Cybernetica 8, 237–245 (1988)
2. Kasai, T.: An hierarchy between context-free and context-sensitive languages. Journal of Computer and System Science 4, 492–508 (1970)
3. Meduna, A.: Deep pushdown automata. Acta Informatica (98), 114–124 (2006), ISSN: 0001-5903
4. Rozenberg, G., Salomaa, A.: Handbook of Formal Languages, Vol. 1: Word, Language, Grammar. Springer, New York (1997)
5. Salomaa, A.: Formal Languages. Academic Press, London (1973)