NP-completeness

Complexity Theory

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Completeness

The concept of completeness is one of the most important in complexity theory.

Definition (Hardness, Completeness)

Let **C** be a complexity class. We call a language L

C-hard if for all $L' \in \mathbf{C}$, $L' \leq L$,

C-complete if L is **C**-hard and $L \in \mathbf{C}$.

Note: We use $L_1 \leq L_2$ to denote that there exists a polynomial reduction from L_1 to L_2, i.e. that there exists a **PTIME** Turing Machine computing a function $R : \Sigma^* \to \Sigma^*$ s.t. $w \in L_1 \iff R(w) \in L_2$.

This means that **C**-complete problems are the hardest problems of **C**.

Motivation

Proving that a problem A is **NP**-complete means that:

- there is probably no fast algorithm for solving A,
- naïve ways for solving A will probably not work,
- heuristics may be necessary for practical algorithms,
- or we may just try to find an approximate solution,
- Richard M. Karp. *Reducibility Among Combinatorial Problems.*

SAT

SAT: Is a given propositional formula ψ satisfiable?

Theorem (Cook-Levin)

SAT *is* **NP***-complete.*

Proof.

- SAT $\in$ **NP** — by constructing an **NPTIME** TM accepting SAT.
- SAT is **NP**-hard — by showing that for any **NPTIME** TM M and its input w, there is a **PTIME** reduction to a propositional formula ψ s.t. ψ is satisfiable iff $w \in L(M)$. □

CNF

CNF: Is a given propositional formula φ in the conjunctive normal form satisfiable?

Theorem

CNF *is* **NP***-complete.*

Proof.

- CNF is **NP**-hard — from SAT using Tseitin transformation
  - transforms ψ into an equisatisfiable formula φ in CNF,
  - the size of φ grows linearly with the size of ψ,
  - naïve transformation (using De Morgan's laws and distribution) yields exponentially larger formula in the worst case. □

Note: in practice, "SAT" is often used to mean "CNF".

k-CNF (k-SAT)

k-CNF: A restricted version of CNF where each clause has exactly k literals.

Theorem

2CNF $\in$ **P**.

Proof.

Clauses can be rewritten to implications which can be viewed as Horn clauses. There is a **PTIME** algorithm for solving HORNSAT. □

k-CNF (k-SAT)

Theorem

*k-CNF is **NP**-complete for $k \geq 3$.*

Proof.

- 3-CNF is **NP**-hard — by reduction from CNF (similarly for other k). We can transform every clause

$$(a \vee b \vee c \vee \cdots \vee f \vee g)$$

into the conjunction

$$(a \vee b \vee x) \wedge (\neg x \vee c \vee y) \wedge \cdots \wedge (\neg z \vee f \vee g)$$

which is equisatisfiable and only linearly larger. □

3-CNF (3-SAT) is interesting because it is the variant of k-CNF with the lowest k that is **NP**-complete.

CLIQUE

CLIQUE: Given a graph $G = (V, E)$ and $k \in \mathbb{N}$, does G contain a clique (a complete subgraph) of size k?

Theorem

CLIQUE *is* **NP**-*complete.*

Proof.

- CLIQUE is **NP**-hard — by reduction from CNF. For a formula $C_1 \wedge \cdots \wedge C_n$ we set $k = n$ and construct an undirected graph $G = (V, E)$ such that

$$V = \{(\sigma, i) \mid \sigma \text{ is a literal and occurs in } C_i\}$$
$$E = \{\{(\sigma, i), (\delta, j)\} \mid i \neq j \wedge \sigma \neq \neg\delta\}$$

□

INDEPENDENT SET

INDEPENDENT SET: Given a graph $B = (W, J)$ and $m \in \mathbb{N}$, does B contain an independent set of vertices (a set of vertices no two of which are adjacent) of size at least m?

Theorem

INDEPENDENT SET *is* **NP**-*complete.*

Proof.

- INDEPENDENT SET is **NP**-hard — by reduction from CLIQUE. For a graph $G = (V, E)$ and k, we set $m = k$ and construct

$$B = (V, V^2 \setminus E)$$

□

Note that cliques are independent sets in graphs' complements.

VERTEX COVER

VERTEX COVER: Given a graph $H = (U, F)$ and $l \in \mathbb{N}$, does H have a vertex cover of size at most l? I.e., is there a set of vertices $S \subseteq U$ of size $|S| \leq l$ such that all edges of H are incident with at least one vertex from S?

Theorem

VERTEX COVER *is* **NP***-complete.*

Proof.

- VERTEX COVER is **NP**-hard — by reduction from INDEPENDENT SET. For a graph $B = (W, J)$ and m, we set $l = |W| - m$ and $H = B$. □

Note that a set is independent iff its complement is a vertex cover.

GRAPH COLOURING

GRAPH COLOURING: Given a graph $M = (Y, L)$ and $p \in \mathbb{N}$, can the vertices of M be coloured using p colours such that no two adjacent vertices are assigned the same colour?

Theorem

GRAPH COLOURING $\in$ **P** *for* $p = 2$.

Proof.

- A graph is 2-colourable iff it is bipartite, which can be determined using BFS in linear time. □

GRAPH COLOURING

Theorem

GRAPH COLOURING *is* **NP***-complete for* $p \geq 3$*.*

Proof.

- GRAPH COLOURING for $p \geq 3$ is **NP**-hard — by reduction from 3-CNF. For a formula $\varphi_1 \wedge \cdots \wedge \varphi_k$ over variables $x_1, \ldots, x_r$, we set $p = r + 1$ and construct the graph $M = (Y, L)$ in the following way: *Assume the formula*

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2)$$

GRAPH COLOURING

Theorem

GRAPH COLOURING *is* **NP***-complete for* $p \geq 3$*.*

Proof.

- GRAPH COLOURING for $p \geq 3$ is **NP**-hard — by reduction from 3-CNF. For a formula $\varphi_1 \wedge \dots \wedge \varphi_k$ over variables $x_1, \dots, x_r$, we set $p = r + 1$ and construct the graph $M = (Y, L)$ in the following way: *Assume the formula*

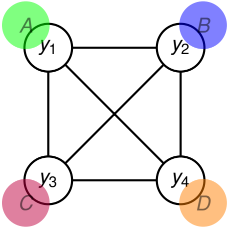
$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2)$$

1. Make sure there are at least 4 variables ($r \geq 4$), otherwise add.
   - *we add* x_4 *to the set of variables* $\rightarrow \{x_1, x_2, x_3, x_4\}$*, and*
   - *set the number of colours* $p = 5$*, call them* $\{A, B, C, D, E\}$*.*

GRAPH COLOURING

Proof (cont).

2. Create a clique with a node y_i for every variable x_i.

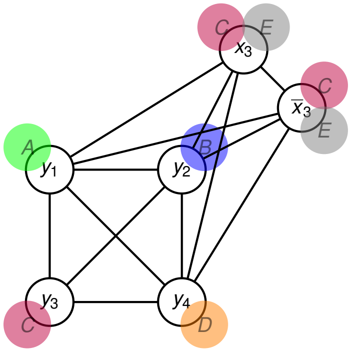


Each node of the clique needs to be coloured with a different colour.

GRAPH COLOURING

Proof (cont).

3. For every variable x_i, add nodes labelled with x_i and $\overline{x_i}$ and connect them with each other and with all y_j, $i \neq j$, from the clique.



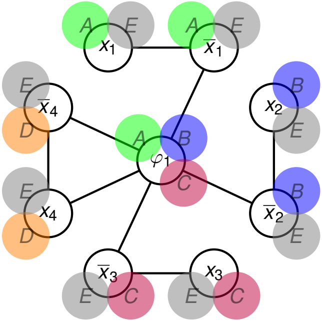
The node x_3 is coloured either by C (which stands for x_3 = *true*) or by E (for x_3 = *false*). The node $\overline{x}_3$ is coloured with the opposite colour.

GRAPH COLOURING

Proof (cont).

4. Add a node for every clause φ_i. For every x_j, connect φ_i with x_j if $x_j \notin \varphi_i$, and with $\overline{x}_j$ if $\neg x_j \notin \varphi_i$.

$$\underbrace{(x_1 \vee x_2 \vee x_3)}_{\varphi_1} \wedge \underbrace{(x_1 \vee \neg x_2 \vee \neg x_3)}_{\varphi_2} \wedge \underbrace{(\neg x_1 \vee x_2)}_{\varphi_3}$$



φ_1 can be coloured only if the colour of at least one of $\overline{x}_1, \overline{x}_2, \overline{x}_3$ is E.

$\rightarrow$ M is p-colourable iff

$$\varphi_1 \wedge \cdots \wedge \varphi_k$$

is satisfiable. □

SUBSET SUM

SUBSET SUM: Let S be a finite set of elements and w be the weight function $w : S \to \mathbb{Z}$. Is there a subset S' of elements of S, $S' \subseteq S$, s.t. the total weight of elements from S' is W, i.e.

$$\sum_{s \in S'} w(s) = W \; ?$$

Theorem

SUBSET SUM *is* **NP**-*complete.*

Proof.

- SUBSET SUM is **NP**-hard — by reduction from 3-SAT. For a formula $\varphi_1 \wedge \cdots \wedge \varphi_k$ over variables $x_1, \ldots, x_n$, we set $S = \{t_1, \ldots, t_n, f_1, \ldots, f_n, c_1, \ldots, c_k, c'_1, \ldots, c'_k\}$ and assign values to w and W in the following way: *(next slide)*

SUBSET SUM

Proof (cont).

Assume the formula

$$\underbrace{(x_1 \vee x_2 \vee x_3)}_{\varphi_1} \wedge \underbrace{(x_1 \vee \neg x_2 \vee \neg x_3)}_{\varphi_2} \wedge \underbrace{(\neg x_1 \vee x_2)}_{\varphi_3}$$

- We consider decimal encoding of w and W of length $n + k$.
- Each variable x_i is assigned a pair of elements t_i and f_i.
- Each clause φ_j is assigned a pair of elements c_j and c'_j.

SUBSET SUM

Proof (cont).

Assume the formula

$$\underbrace{(x_1 \vee x_2 \vee x_3)}_{\varphi_1} \wedge \underbrace{(x_1 \vee \neg x_2 \vee \neg x_3)}_{\varphi_2} \wedge \underbrace{(\neg x_1 \vee x_2)}_{\varphi_3}$$

- We consider decimal encoding of w and W of length $n + k$.
- Each variable x_i is assigned a pair of elements t_i and f_i.
- Each clause φ_j is assigned a pair of elements c_j and c'_j.

	x_1	x_2	x_3	φ_1	φ_2	φ_3
t_1	1	0	0	1	1	0
f_1	1	0	0	0	0	1
t_2	0	1	0	1	0	1
f_2	0	1	0	0	1	0
t_3	0	0	1	1	0	0
f_3	0	0	1	0	1	0
c_1	0	0	0	1	0	0
c'_1	0	0	0	1	0	0
c_2	0	0	0	0	1	0
c'_2	0	0	0	0	1	0
c_3	0	0	0	0	0	1
c'_3	0	0	0	0	0	1
W	1	1	1	3	3	3

□

PARTITION

PARTITION: Let T be a finite set of elements and v be the weight function $v : T \to \mathbb{Z}$. Can T be partitioned into two sets T' and $T \setminus T'$ of equal total weight, i.e.

$$\sum_{t \in T'} v(t) = \sum_{t \in T \setminus T'} v(t) \ ?$$

Theorem

PARTITION *is* **NP***-complete.*

Proof.

- PARTITION is **NP**-hard — by reduction from SUBSET SUM. For the elements S, weight function w and target weight W, we set $T = S \cup \{z\}$ where $z \notin S$, and $v = w \cup \{z \mapsto (w(S) - 2W)\}$ where $w(S) = \sum_{s \in S} w(s)$. □

KNAPSACK

KNAPSACK: Let R be a finite set of elements, u be the weight function $u : R \to \mathbb{Z}$, and v be the value function $v : R \to \mathbb{Z}$. Is there a subset R' of elements of R, $R' \subseteq R$, s.t. the total weight of elements from R' is at most U and their total value is at least V, i.e.

$$\sum_{r \in R'} u(r) \leq U \;\wedge\; \sum_{r \in R'} v(r) \geq V\ ?$$

Theorem

KNAPSACK *is* **NP***-complete.*

Proof.

- KNAPSACK is **NP**-hard — by reduction from SUBSET SUM. For the elements S, weight function w and target weight W, we set $R = S$, $u = w$, $v = w$, $U = W$, and $V = W$. □