Complexity Classes

Complexity Theory

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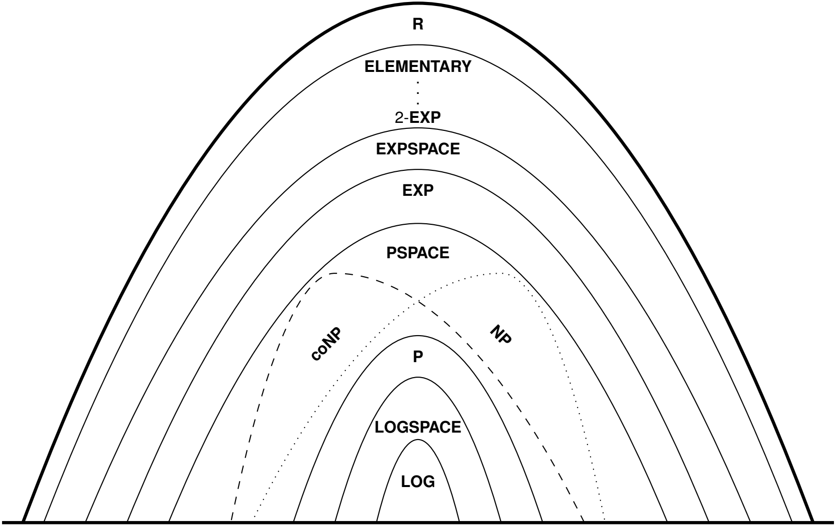
Complexity Classes – Motivation

- We have studied the relation of a complexity class of a certain type on a function which is used for its definition.
- Now, we will focus on investigating the relation between complexity classes of different types.
- Although a lot effort has been invested in this area in recent years, some problems (such as $P \stackrel{?}{=} NP$) remain unsolved.

Complexity Classes – Motivation



Inclusion of Complexity Classes

- We will show that the following chain of inclusions holds:

$$\mathbf{DLOG} \subseteq \mathbf{NLOG} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP} \subseteq \mathbf{NEXP}$$

- To do this, we will prove the statements shown below:
  a) $\mathbf{DSPACE}(f(n)) \subseteq \mathbf{NSPACE}(f(n))$,
  b) $\mathbf{DTIME}(f(n)) \subseteq \mathbf{NTIME}(f(n))$,
  c) $\mathbf{NTIME}(f(n)) \subseteq \bigcup_{c>0} \mathbf{DTIME}(c^{f(n)})$,
  d) $\mathbf{NTIME}(f(n)) \subseteq \mathbf{DSPACE}(f(n))$,
  e) $\mathbf{NSPACE}(f(n)) \subseteq \mathbf{DTIME}(2^{O(f(n))})$ for $f(n) \geq \log n$,
  f) $\mathbf{NSPACE}(f(n)) \subseteq \mathbf{DSPACE}(f(n)^2)$ for $f(n) \geq \log n$ (*Savitch's theorem*).

  where $f(n)$ is a time and space constructible function.

Proof of the Inclusion of Complexity Classes (1/4)

Proof (Parts a, b, c, and d)

- *Parts a)* ($\mathbf{DSPACE}(f(n)) \subseteq \mathbf{NSPACE}(f(n))$) *and b)* ($\mathbf{DTIME}(f(n)) \subseteq \mathbf{NTIME}(f(n))$) *are trivial since each deterministic TM is a special case of a nondeterministic TM.*
- *Part c)* ($\mathbf{NTIME}(f(n)) \subseteq \bigcup_{c>0} \mathbf{DTIME}(c^{f(n)})$) *has been proven in the previous lecture.*
- *The inclusion d)* ($\mathbf{NTIME}(f(n)) \subseteq \mathbf{DSPACE}(f(n))$) *is a consequence of the proof of the proposition* $\mathbf{NTIME}(f(n)) \subseteq \mathbf{DTIME}(2^{f(n)})$. *A nondeterministic TM M_N can make at most $f(n)$ nondeterministic choices. The deterministic TM M_D simulating M_N uses $f(n)$ cells of the tape to represent the choices of M_N and at most $f(n)$ cells for simulating the tape of M_N. M_D simulates the possible runs of M_N one by one.*

Proof of the Inclusion of Complexity Classes (2/4)

Proof (Part e)

- **NSPACE**$(f(n)) \subseteq$ **DTIME**$(2^{O(f(n))})$ *for* $f(n) \geq \log n$
- *We will define the configuration graph* $G(M, w)$ *of an NTM* M *with the input* w*. The vertices of the graph* $G(M, w)$ *are configurations that may occur during a computation of* M *on* w*. There is an edge between the vertices* C_1 *and* C_2 *iff* M *can do a transition from the configuration* C_1 *to* C_2 *in a single step.*
- *We ask whether there is a path from the vertex denoting the initial configuration to the vertex representing the accepting one.*
- *For each input* w*, the number of vertices of* $G(M, w)$ *can be upper-bounded by* $2^{O(\log |w| + f(|w|))}$*.*
- *The already known algorithm REACHABILITY can solve the problem in time* $O(m^2)$ *for graphs with* m *vertices. Therefore, the overall time complexity of a DTM simulating* M *is* $2^{O(\log n + f(n))}$*.*

Proof of the Inclusion of Complexity Classes (3/4)

Proof (Part f)

- $\mathbf{NSPACE}(f(n)) \subseteq \mathbf{DSPACE}(f(n)^2)$ *for* $f(n) \geq \log n$ *(Savitch's th.)*
- *First, we will show that REACHABILITY* $\in \mathbf{DSPACE}(\log^2 n)$.
- *Let G be a graph with n vertices.*
- *Let* x, y *be two distinct vertices from G.*
- *Then, the longest path from* x *to* y *can have at most length n.*
- *We use a DTM to implement the procedure path*(x, y, i) *which returns true iff there exists a path from* x, y *of length at most* 2^i.

```
procedure path(x, y, i)
if (i = 0) then return ((x = y) ∨ ((x,y) ∈ G))
else for all vertices z ∈ G do
    if (path(x, z, i-1) ∧ path(z, y, i-1)) then
        return true
return false
```

Proof of the Inclusion of Complexity Classes (4/4)

Proof (Part f)

- *Recursive calls of path(x, y, i) create a tree of depth i.*
- *For our problem, it suffices to check whether path$(x, y, \lceil \log\ n \rceil)$ holds.*
- *The DTM implementing the path procedure will have to store call stack with at most $\lceil \log\ n \rceil$ triplets of length $3 \cdot \log\ n$.*
- *Therefore, the space complexity of REACHABILITY is $O(\log^2 n)$.*
- *For a NTM M_N we can create a DTM M_D which uses another DTM deciding REACHABILITY for each accepting configuration of M_N.*
- *The number of vertices of $G(M_N, w)$ cannot be bigger than $c^{f(|w|)}$.*
- *Thus, the space complexity of M_D is $O(\log^2 c^{f(n)}) = O(f(n)^2)$.*

Inclusion of Complexity Classes: Conclusion

- We have shown that

$$\mathbf{DLOG} \subseteq \mathbf{NLOG} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP} \subseteq \mathbf{NEXP}$$

- We expect all inclusion to be proper but for none of them has this been either proved or refuted yet.
- However, because of Time/Space Hierarchy Theorems (not covered here), it is known that some inclusions need to be proper. It is known that:

$$\mathbf{DLOG} \subset \mathbf{PSPACE}$$

$$\mathbf{P} \subset \mathbf{EXP}$$

$$\mathbf{NP} \subset \mathbf{NEXP}$$

Definition of Complements of Complexity Classes (1/2)

Definition

Let $L \subseteq \Sigma^*$ be a language. The complement of L denoted co–L is the language

$$co\text{–}L = \Sigma^* \setminus L.$$

- The same approach can be extended for decision problems.

Definition

The complement of a decision problem A, denoted A–COMPL, is a problem for which the solution is:

$$\begin{aligned} \textit{yes} &\iff \textit{the solution for } A \textit{ is no} \\ \textit{no} &\iff \textit{the solution for } A \textit{ is yes} \end{aligned}$$

- Formally, *A–COMPL* is not a complement of a language because $A \cup A\text{–}COMPL \neq \Sigma^*$.

Definition of Complements of Complexity Classes (2/2)

Definition

Let $\mathcal{C}$ be a complexity class. The complement of $\mathcal{C}$, denoted co–$\mathcal{C}$, is the complexity class

$$co\text{–}\mathcal{C} = \{co\text{–}L \mid L \in \mathcal{C}\}.$$

- Clearly, deterministic time and space complexity classes are closed under complement:
- each DTM accepting L can be transformed into a DTM accepting $co\text{–}L$ by swapping accepting and rejecting states.

$$\mathbf{DTIME}(f(n)) = co\text{–}\mathbf{DTIME}(f(n))$$

$$\mathbf{DSPACE}(f(n)) = co\text{–}\mathbf{DSPACE}(f(n))$$

Complements of Nondeterministic Complexity Classes

- Nondeterminism introduces an asymmetry in acceptance of a given input by NTM M:

$$w \in L(M) \iff \text{there is an accepting run of } M \text{ on } w,$$
$$w \notin L(M) \iff \text{there is no accepting run of } M \text{ on } w.$$

- Thus, it suffices that one accepting computation exists for the former case but it is required that all computations are rejecting for the latter one.

Nondeterministic Space Complexity Classes (1/8)

Proposition (Immerman-Szelepcsényi Theorem)

$$\mathbf{NSPACE}(f(n)) = co\text{–}\mathbf{NSPACE}(f(n)) \qquad \textit{for } f \geq \log n$$

- We have seen that *REACHABILITY* is in **NLOGSPACE**.
- First, we will demonstrate that the converse of *REACHABILITY*, called *UNREACHABILITY*, is in **NLOGSPACE** too.

Definition

UNREACHABILITY

- *Input: Graph $G = (V, E)$, a pair of vertices $x, y \in V$.*
- *Output: YES if there is no path from x to y, NO otherwise.*

Proposition

$$\textit{UNREACHABILITY} \in \mathbf{NLOGSPACE}$$

Nondeterministic Space Complexity Classes (2/8)

Proof (*UNREACHABILITY* $\in$ **NLOGSPACE**)

- *We will first devise an algorithm that computes the number of reachable nodes in a graph in $O(\log n)$ space,*
  - *and later modify it to solve UNREACHABILITY.*
- *Let $G = (V, E)$ be a graph and $x \in V$.*
- *$S(k)$ is the set of nodes reachable from x in k or less steps,*
  - *$|S(k)|$ is the number of nodes reachable from x in k or less steps.*
- *adjac(v, u) is true iff $v = u \vee (v, u) \in E$, otherwise false.*
- ***procedure*** `numReachable(x, (V, E))`
  - *nondeterministically either **FAIL**s or returns the number of nodes reachable from x,*
  - *we will start with a procedure working in $O(n)$ space and modify it in several steps to use $O(\log n)$ space.*

Nondeterministic Space Complexity Classes (3/8)

Proof (*UNREACHABILITY* $\in$ NLOGSPACE)

```
procedure numReachable(x, (V, E))
|S(0)| := 1
for k := 1 to |V|-1 do
   compute |S(k)| from |S(k-1)|
return |S(|V|-1)|
```

Nondeterministic Space Complexity Classes (4/8)

Proof (*UNREACHABILITY* ∈ NLOGSPACE)

```
procedure numReachable(x, (V, E))
|S(0)| := 1
for k := 1 to |V|-1 do
//compute |S(k)| from |S(k-1)|
  ℓ := 0
  for each node u := 1 to |V| do
    if u ∈ S(k) then ℓ++




  |S(k)| := ℓ
return |S(|V|-1)|
```

Nondeterministic Space Complexity Classes (4/8)

Proof (*UNREACHABILITY* $\in$ NLOGSPACE)

```
procedure numReachable(x, (V, E))
|S(0)| := 1
for k := 1 to |V|-1 do
//compute |S(k)| from |S(k-1)|
  ℓ := 0
  for each node u := 1 to |V| do
    if u ∈ S(k) then ℓ++




  |S(k)| := ℓ
return |S(|V|-1)|
```

Nondeterministic Space Complexity Classes (5/8)

Proof (*UNREACHABILITY* $\in$ NLOGSPACE)

```
procedure numReachable(x, (V, E))
|S(0)| := 1
for k := 1 to |V|-1 do
//compute |S(k)| from |S(k-1)|
  ℓ := 0
  for each node u := 1 to |V| do
//  if u ∈ S(k) then ℓ++
    reply := false
    for each node v := 1 to |V| do
      if v ∈ S(k-1) then
        if adjac(v, u) then reply := true

    if reply then ℓ++
  |S(k)| := ℓ
return |S(|V|-1)|
```

Nondeterministic Space Complexity Classes (5/8)

Proof (*UNREACHABILITY* ∈ **NLOGSPACE**)

```
procedure numReachable(x, (V, E))
|S(0)| := 1
for k := 1 to |V|-1 do
//compute |S(k)| from |S(k-1)|
  ℓ := 0
  for each node u := 1 to |V| do
//   if u ∈ S(k) then ℓ++
    reply := false
    for each node v := 1 to |V| do
      if v ∈ S(k-1) then
        if adjac(v, u) then reply := true

    if reply then ℓ++
  |S(k)| := ℓ
return |S(|V|-1)|
```

Nondeterministic Space Complexity Classes (6/8)

Proof (*UNREACHABILITY* $\in$ **NLOGSPACE**)

```
procedure numReachable(x, (V, E))
|S(0)| := 1
for k := 1 to |V|-1 do
//compute |S(k)| from |S(k-1)|
  ℓ := 0
  for each node u := 1 to |V| do
//  if u ∈ S(k) then ℓ++
    reply := false, m := 0
    for each node v := 1 to |V| do
      if reachNondet(x, v, k-1) then m++    // v ∈ S(k-1)
        if adjac(v, u) then reply := true
    if m < |S(k-1)| then FAIL
    if reply then ℓ++
  |S(k)| := ℓ
return |S(|V|-1)|
```

Nondeterministic Space Complexity Classes (6/8)

Proof (*UNREACHABILITY* ∈ **NLOGSPACE**)

```
procedure numReachable(x, (V, E))
|S(0)| := 1
for k := 1 to |V|-1 do
//compute |S(k)| from |S(k-1)|
  ℓ := 0
  for each node u := 1 to |V| do
//   if u ∈ S(k) then ℓ++
    reply := false, m := 0
    for each node v := 1 to |V| do
      if reachNondet(x, v, k-1) then m++    // v ∈ S(k-1)
        if adjac(v, u) then reply := true
    if m < |S(k-1)| then FAIL
    if reply then ℓ++
  |S(k)| := ℓ
return |S(|V|-1)|
```

guesses `path` from `x` to `v` of length at most `k-1`: returns ***true*** iff `path` is OK, ***false*** if `path` is wrong

if calls to `reachNondet()` did not guess all the paths correctly, terminate the computation

Nondeterministic Space Complexity Classes (6/8)

Proof (*UNREACHABILITY* ∈ NLOGSPACE)

```
procedure numReachable(x, (V, E))
|S(0)| := 1
for k := 1 to |V|-1 do
//compute |S(k)| from |S(k-1)|
  ℓ := 0
  for each node u := 1 to |V| do
//   if u ∈ S(k) then ℓ++
    reply := false, m := 0
    for each node v := 1 to |V| do
      if reachNondet(x, v, k-1) then m++    // v ∈ S(k-1)
        if adjac(v, u) then reply := true
    if m < |S(k-1)| then FAIL
    if reply then ℓ++
  |S(k)| := ℓ
return |S(|V|-1)|
```

Nondeterministic Space Complexity Classes (7/8)

Proof (*UNREACHABILITY* $\in$ NLOGSPACE)

```
procedure reachNondet(x, v, d)
w_0 := x
for p := 1 to d do
  guess a node w_p
  if ¬adjac(w_{p-1}, w_p) then return false
return (w_d = v)
```

- *For UNREACHABILITY*(G, x, y)*, modify* `numReachable()`*:*

```
  ...
    if m < |S(k-1)| then return false
    if reply then ℓ++
    if k = |V|-1 and u = y then return ¬reply
  ...
```

Nondeterministic Space Complexity Classes (8/8)

Proof (*UNREACHABILITY* $\in$ NLOGSPACE)

- *UNREACHABILITY can be implemented using a NTM M that works in space $O(\log n)$.*
- *M keeps binary values of $\texttt{|S(k-1)|}$, ℓ, $\texttt{k}$, $\texttt{u}$, $\texttt{v}$, $\texttt{m}$, $\texttt{reply}$, $\texttt{p}$, $\texttt{w}_p$, $\texttt{w}_{p-1}$ on separate tapes.*
- *Variables are only incremented or compared with each other.*
- *Values of all variables are at most $|V|$, their length at most $\log |V|$.*

Nondeterministic Space Complexity Classes (7/7)

Proposition (Immerman-Szelepcsényi Theorem)

$$\mathbf{NSPACE}(f(n)) = co\text{–}\mathbf{NSPACE}(f(n)) \qquad \textit{for } f \geq \log n$$

Proof

- *Let* $L \in \mathbf{NSPACE}(f(n))$ *be a language decided by a NTM* M_N *and let* w *be its input.*
- *We can construct a DTM* M_D *working in* $\mathbf{NSPACE}(f(n))$ *deciding co–L.*
- M_D *simulates the UNREACHABILITY algorithm for the configuration graph* $G(M, w)$*.*

Corrolary

$$\mathbf{NPSPACE}(f(n)) = co\text{–}\mathbf{NPSPACE}(f(n)), \ldots$$

Nondeterministic Time Complexity Classes (1/2)

- We have seen that

$$\mathbf{NTIME}(f(n)) \subseteq \bigcup_{c>0} \mathbf{DTIME}(c^{f(n)})$$

- Each problem from class **NTIME**$(f(n))$ can be deterministically decided in time $\bigcup_{c>0} \mathbf{DTIME}(c^{f(n)})$.

Corrolary

$$co\text{–}\mathbf{NTIME}(f(n)) \subseteq \bigcup_{c>0} \mathbf{DTIME}(c^{f(n)})$$

Nondeterministic Time Complexity Classes (2/2)

- Relationship between complements of nondeterministic time complexity classes remains unsolved:

$$\mathbf{NTIME}(f(n)) \stackrel{?}{=} co - \mathbf{NTIME}(f(n))$$

Corrolary

$$\mathbf{NP}(f(n)) \stackrel{?}{=} co\text{–}\mathbf{NP}(f(n))$$

Corrolary

$$\mathbf{NEXP}(f(n)) \stackrel{?}{=} co\text{–}\mathbf{EXP}(f(n))$$