Workspace Theorems for Regular-Controlled Grammars

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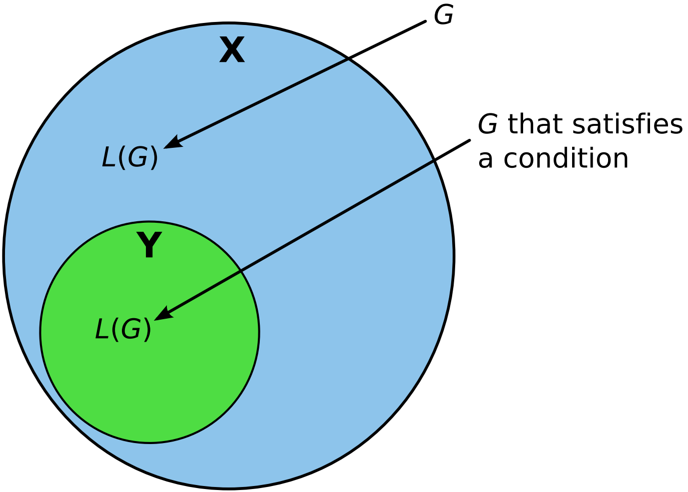
Outline

- **Introduction**
- **Classical Workspace Theorem**
- **New Workspace Theorem**
- **Idea Behind the Proof**
- **Concluding Remarks and Discussion**


G
X
L(G)
G that satisfies
a condition
Y
L(G)

Phrase-Structure Grammars

Definition

A *phrase-structure grammar* is a quadruple

$$G = (N, T, P, S)$$

where

- N is an alphabet of *nonterminals*,
- T is an alphabet of *terminals* ($N \cap T = \emptyset$),
- P is a finite set of *rules* of the form

  $$x \to y$$

  where $x \in (N \cup T)^* N (N \cup T)^*$ and $y \in (N \cup T)^*$,
- $S \in N$ is the *starting nonterminal*.

Phrase-Structure Grammars

Definition

The relation of a *direct derivation*, denoted by $\Rightarrow$, is defined as follows:

$$uxv \Rightarrow uyv \text{ in } G$$

if and only if

- $u, v \in (N \cup T)^*$,
- $x \to y \in P$.

Definition

The *language of G*, denoted by $L(G)$, is defined as

$$L(G) = \{w \in T^* \mid S \Rightarrow^* w\}$$

where $\Rightarrow^*$ is the reflexive-transitive closure of $\Rightarrow$.

RE: the family of recursively enumerable languages

Context-Sensitive Grammars

Definition

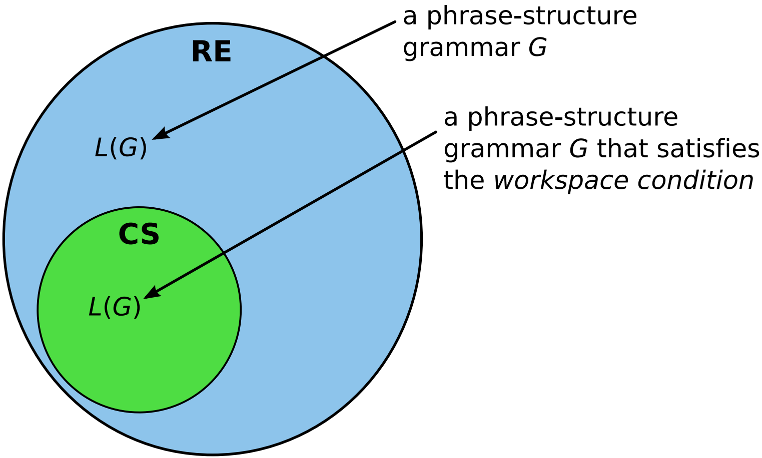
A phrase-structure grammar $G = (N, T, P, S)$ is a *context-sensitive grammar* if every $x \to y \in P$ is of the form

$$uAv \to uzv$$

where $u, v \in (N \cup T)^*$, $A \in N$, and $z \in (N \cup T)^+$.

CS: the family of context-sensitive languages

Basic Idea


RE
CS
$L(G)$
$L(G)$
a phrase-structure grammar G
a phrase-structure grammar G that satisfies the *workspace condition*

Theorem

Let $G = (N, T, P, S)$ be a phrase-structure grammar. If there exists some $k \geq 1$ such that for every (nonempty) $w \in L(G)$, there exists a derivation

$$S \Rightarrow x_1 \Rightarrow x_2 \Rightarrow \cdots \Rightarrow x_n = w$$

where

$$|x_i| \leq k|w|$$

for all $1 \leq i \leq n$, then

$$L(G) \in \mathbf{CS}$$

$$|x_i| \;\leq\; |\overbrace{w\ w\ \cdots\ w}^{k \text{ times}}|$$

Context-Free Grammars

Definition

A phrase-structure grammar $G = (N, T, P, S)$ is a *context-free grammar* if every $x \rightarrow y \in P$ satisfies

$$x \in N$$

If $uAv \Rightarrow uyv$ by $r = (A \rightarrow y) \in P$, then we write

$$uAv \Rightarrow uyv\ [r]$$

If $S \Rightarrow x_1\ [r_1] \Rightarrow \cdots \Rightarrow x_n\ [r_n]$, then we write

$$S \Rightarrow^* x_n\ [r_1 \cdots r_n]$$

CF: the family of context-free languages

Regular-Controlled Grammars

Definition

A *regular-controlled (context-free) grammar* is a pair

$$H = (G, R)$$

where

- $G = (N, T, P, S)$ is a context-free grammar,
- $R \subseteq P^*$ is a regular *control language*.

Definition

The *language generated by H*, denoted by $L(H)$, is defined as

$$L(H) = \{w \in T^* \mid S \Rightarrow^* w\ [\varrho] \text{ with } \varrho \in R\}$$

Example of a Regular-Controlled Grammar

Example

Consider the following regular-controlled grammar H:

$1: S \rightarrow ABC$

$2: A \rightarrow aA$

$3: B \rightarrow bB$

$4: C \rightarrow cC$

$5: A \rightarrow \varepsilon$

$6: B \rightarrow \varepsilon$

$7: C \rightarrow \varepsilon$

$R = \{1\}\{234\}^*\{567\}$

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S

Example of a Regular-Controlled Grammar

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$S \Rightarrow ABC$

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$S \Rightarrow ABC \Rightarrow aABC$

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$S \Rightarrow ABC \Rightarrow aABC \Rightarrow aAbBC$

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$R = \{1\}\{234\}^*\{567\}$

$S \Rightarrow ABC \Rightarrow aABC \Rightarrow aAbBC \Rightarrow aAbBcC \Rightarrow abBcC \Rightarrow abcC \Rightarrow abc$

$$L(H) = \{a^n b^n c^n \mid n \geq 0\}$$

Denotation of Language Families

RC: the family of languages generated by regular-controlled grammars

RC$^{-\varepsilon}$: the family of languages generated by regular-controlled grammars without *erasing rules*

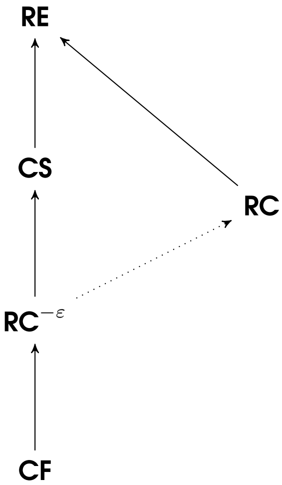
$$A \to \varepsilon$$

RE: the family of recursively enumerable languages

CS: the family of context-sensitive languages

CF: the family of context-free languages

Relationships Between Language Families



Basic Idea

Definition

A regular-controlled grammar H *works within a k-limited workspace* provided that for all (nonempty) $w \in L(H)$, there exists a derivation $S \Rightarrow^* w$ in which every sentential form x satisfies the following condition:

for every occurrence of a symbol in x which is not erased, x contains at most k occurrences of symbols which are erased.

are erased

$$x = u_0 a_1 u_1 a_2 u_2 \dots a_n u_n$$

are not erased

$$k|a_1 a_2 \dots a_n| \geq |u_0 u_1 \dots u_n|$$

Example

Example

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For every $a^n b^n c^n \in L(H)$, where $n \geq 1$, there exists

$$S \Rightarrow^* a^n A b^n B c^n C \Rightarrow^* a^n b^n c^n \; [\varrho] \text{ with } \varrho \in R$$

Example

Example

Consider the regular-controlled grammar H:

$1: S \rightarrow ABC$

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For every $a^n b^n c^n \in L(H)$, where $n \geq 1$, there exists

$$S \Rightarrow^* a^n A b^n B c^n C \Rightarrow^* a^n b^n c^n \ [\varrho] \text{ with } \varrho \in R$$

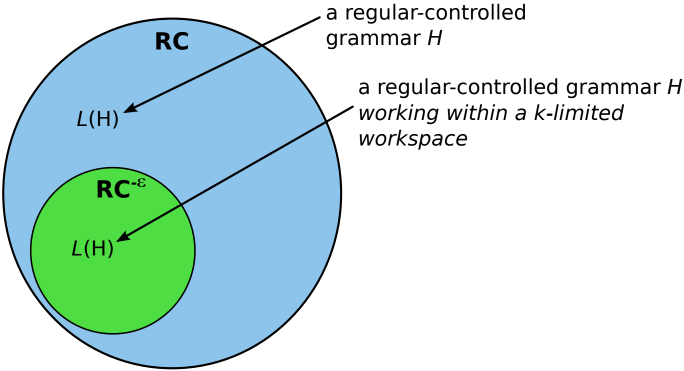
Hence, H works within a 1-limited workspace.

New Workspace Theorem

Theorem

Let H be a regular-controlled grammar working within a k-limited workspace, for some $k \geq 0$*. Then,*

$$L(H) \in \mathbf{RC}^{-\varepsilon}$$



Idea Behind the Proof

We use compound nonterminals of the form

$$\langle X, y \rangle$$

where

- X is a symbol that is not erased throughout the rest of the derivation,
- y is a k-limited string of symbols that are erased throughout the rest of the derivation.

Idea Behind the Proof

We use compound nonterminals of the form

$$\langle X, y \rangle$$

where

- X is a symbol that is not erased throughout the rest of the derivation,
- y is a k-limited string of symbols that are erased throughout the rest of the derivation.

Illustration

$$\langle A, DE \rangle \langle b, \varepsilon \rangle \langle b, A \rangle$$

Idea Behind the Proof

1. For each $A \to x_0 X_1 x_1 X_2 x_2 \cdots X_m x_m$ and $\langle A, y \rangle$, we introduce

$$\langle A, y \rangle \to \langle X_1, y x_0 x_1 \cdots x_m \rangle \langle X_2, \varepsilon \rangle \cdots \langle X_m, \varepsilon \rangle$$

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$$\langle A, y \rangle \rightarrow \langle X_1, y x_0 x_1 \cdots x_m \rangle \langle X_2, \varepsilon \rangle \cdots \langle X_m, \varepsilon \rangle$$

Illustration

For $A \rightarrow aA$ and $\langle A, \varepsilon \rangle$, we introduce

$$\langle A, \varepsilon \rangle \rightarrow \langle a, \varepsilon \rangle \langle A, \varepsilon \rangle$$
$$\langle A, \varepsilon \rangle \rightarrow \langle a, A \rangle$$

Idea Behind the Proof

2. For each $A \rightarrow y$ and $\langle X, uAv \rangle$, we introduce

$$\langle X, uAv \rangle \rightarrow \langle X, uyv \rangle$$

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Illustration

For $A \rightarrow BC$ and $\langle a, AD \rangle$, we introduce

$$\langle a, AD \rangle \rightarrow \langle a, BCD \rangle$$

2. For each $A \rightarrow y$ and $\langle X, uAv \rangle$, we introduce

$$\langle X, uAv \rangle \rightarrow \langle X, uyv \rangle$$

Illustration

For $A \rightarrow BC$ and $\langle a, AD \rangle$, we introduce

$$\langle a, AD \rangle \rightarrow \langle a, BCD \rangle$$

Illustration

For $A \rightarrow \varepsilon$ and $\langle a, AD \rangle$, we introduce

$$\langle a, AD \rangle \rightarrow \langle a, D \rangle$$

Idea Behind the Proof

3. For each $\langle X, uAv \rangle$ and $\langle Y, y \rangle$, we introduce

$$\begin{aligned} \langle X, uAv \rangle &\rightarrow \langle X, uv \rangle \\ \langle Y, y \rangle &\rightarrow \langle Y, yA \rangle \end{aligned}$$

+ regulation!

3. For each $\langle X, uAv\rangle$ and $\langle Y, y\rangle$, we introduce

$$\begin{aligned}\langle X, uAv\rangle &\rightarrow \langle X, uv\rangle \\ \langle Y, y\rangle &\rightarrow \langle Y, yA\rangle\end{aligned}$$

+ regulation!

Illustration

For $\langle C, AD\rangle$ and $\langle a, B\rangle$, we introduce

$$\begin{aligned}\langle C, AD\rangle &\rightarrow \langle C, D\rangle \\ \langle a, B\rangle &\rightarrow \langle a, BA\rangle\end{aligned}$$

Idea Behind the Proof

4. For each terminal a, we introduce

$$\langle a, \varepsilon \rangle \rightarrow a$$

4. For each terminal a, we introduce

$$\langle a, \varepsilon \rangle \to a$$

Illustration

For a, b, c, we introduce

$$\begin{aligned} \langle a, \varepsilon \rangle &\to a \\ \langle b, \varepsilon \rangle &\to b \\ \langle c, \varepsilon \rangle &\to c \end{aligned}$$

Example of the Construction

Example

Consider the regular-controlled grammar H:

$1\colon S \rightarrow ABC$

$2\colon A \rightarrow aA$

$3\colon B \rightarrow bB$

$4\colon C \rightarrow cC$

$5\colon A \rightarrow \varepsilon$

$6\colon B \rightarrow \varepsilon$

$7\colon C \rightarrow \varepsilon$

$\langle S, \varepsilon \rangle$

Example

Consider the regular-controlled grammar H:

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$5\colon A \rightarrow \varepsilon$

$6\colon B \rightarrow \varepsilon$

$7\colon C \rightarrow \varepsilon$

$$\langle S, \varepsilon\rangle \quad \Rightarrow \quad \langle A, \varepsilon\rangle\langle B, \varepsilon\rangle\langle C, \varepsilon\rangle \qquad [\langle S, \varepsilon\rangle \rightarrow \langle A, \varepsilon\rangle\langle B, \varepsilon\rangle\langle C, \varepsilon\rangle]$$

Example

Consider the regular-controlled grammar H:

$1: S \rightarrow ABC$

$2: A \rightarrow aA$

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$5: A \rightarrow \varepsilon$

$6: B \rightarrow \varepsilon$

$7: C \rightarrow \varepsilon$

$$\begin{aligned} \langle S, \varepsilon \rangle &\Rightarrow \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle S, \varepsilon \rangle \rightarrow \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle] \\ &\Rightarrow \langle a, A \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle A, \varepsilon \rangle \rightarrow \langle a, A \rangle] \end{aligned}$$

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Consider the regular-controlled grammar H:

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\begin{array}{lll}
\langle S, \varepsilon \rangle & \Rightarrow \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle S, \varepsilon \rangle \rightarrow \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle] \\
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& \Rightarrow \langle a, A \rangle \langle b, B \rangle \langle C, \varepsilon \rangle & [\langle B, \varepsilon \rangle \rightarrow \langle b, B \rangle]
\end{array}
$$

Example

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\begin{aligned}
\langle S, \varepsilon \rangle &\Rightarrow \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle S, \varepsilon \rangle \rightarrow \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle] \\
&\Rightarrow \langle a, A \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle A, \varepsilon \rangle \rightarrow \langle a, A \rangle] \\
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&\Rightarrow \langle a, \varepsilon \rangle \langle b, B \rangle \langle c, C \rangle && [\langle a, A \rangle \rightarrow \langle a, \varepsilon \rangle]
\end{aligned}
$$

Example

Consider the regular-controlled grammar H:

$$1: S \rightarrow ABC \qquad 5: A \rightarrow \varepsilon$$
$$2: A \rightarrow aA \qquad 6: B \rightarrow \varepsilon$$
$$3: B \rightarrow bB \qquad 7: C \rightarrow \varepsilon$$
$$4: C \rightarrow cC$$

$$\begin{array}{llll}
\langle S, \varepsilon \rangle & \Rightarrow & \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle S, \varepsilon \rangle \rightarrow \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle] \\
 & \Rightarrow & \langle a, A \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle A, \varepsilon \rangle \rightarrow \langle a, A \rangle] \\
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\end{aligned}
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$2\colon A \to aA$ $\quad 6\colon B \to \varepsilon$

$3\colon B \to bB$ $\quad 7\colon C \to \varepsilon$

$4\colon C \to cC$

$$
\begin{array}{lll}
\langle S, \varepsilon \rangle & \Rightarrow \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle S, \varepsilon \rangle \to \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle] \\
 & \Rightarrow \langle a, A \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle A, \varepsilon \rangle \to \langle a, A \rangle] \\
 & \Rightarrow \langle a, A \rangle \langle b, B \rangle \langle C, \varepsilon \rangle & [\langle B, \varepsilon \rangle \to \langle b, B \rangle] \\
 & \Rightarrow \langle a, A \rangle \langle b, B \rangle \langle c, C \rangle & [\langle C, \varepsilon \rangle \to \langle c, C \rangle] \\
 & \Rightarrow \langle a, \varepsilon \rangle \langle b, B \rangle \langle c, C \rangle & [\langle a, A \rangle \to \langle a, \varepsilon \rangle] \\
 & \Rightarrow \langle a, \varepsilon \rangle \langle b, \varepsilon \rangle \langle c, C \rangle & [\langle b, B \rangle \to \langle b, \varepsilon \rangle] \\
 & \Rightarrow \langle a, \varepsilon \rangle \langle b, \varepsilon \rangle \langle c, \varepsilon \rangle & [\langle c, C \rangle \to \langle c, \varepsilon \rangle] \\
 & \Rightarrow a \langle b, \varepsilon \rangle \langle c, \varepsilon \rangle & [\langle a, \varepsilon \rangle \to a]
\end{array}
$$

Example

Consider the regular-controlled grammar H:

$1: S \rightarrow ABC$

$2: A \rightarrow aA$

$3: B \rightarrow bB$

$4: C \rightarrow cC$

$5: A \rightarrow \varepsilon$

$6: B \rightarrow \varepsilon$

$7: C \rightarrow \varepsilon$

$$
\begin{array}{llll}
\langle S, \varepsilon\rangle & \Rightarrow & \langle A, \varepsilon\rangle\langle B, \varepsilon\rangle\langle C, \varepsilon\rangle & [\langle S, \varepsilon\rangle \rightarrow \langle A, \varepsilon\rangle\langle B, \varepsilon\rangle\langle C, \varepsilon\rangle] \\
 & \Rightarrow & \langle a, A\rangle\langle B, \varepsilon\rangle\langle C, \varepsilon\rangle & [\langle A, \varepsilon\rangle \rightarrow \langle a, A\rangle] \\
 & \Rightarrow & \langle a, A\rangle\langle b, B\rangle\langle C, \varepsilon\rangle & [\langle B, \varepsilon\rangle \rightarrow \langle b, B\rangle] \\
 & \Rightarrow & \langle a, A\rangle\langle b, B\rangle\langle c, C\rangle & [\langle C, \varepsilon\rangle \rightarrow \langle c, C\rangle] \\
 & \Rightarrow & \langle a, \varepsilon\rangle\langle b, B\rangle\langle c, C\rangle & [\langle a, A\rangle \rightarrow \langle a, \varepsilon\rangle] \\
 & \Rightarrow & \langle a, \varepsilon\rangle\langle b, \varepsilon\rangle\langle c, C\rangle & [\langle b, B\rangle \rightarrow \langle b, \varepsilon\rangle] \\
 & \Rightarrow & \langle a, \varepsilon\rangle\langle b, \varepsilon\rangle\langle c, \varepsilon\rangle & [\langle c, C\rangle \rightarrow \langle c, \varepsilon\rangle] \\
 & \Rightarrow & a\langle b, \varepsilon\rangle\langle c, \varepsilon\rangle & [\langle a, \varepsilon\rangle \rightarrow a] \\
 & \Rightarrow & ab\langle c, \varepsilon\rangle & [\langle b, \varepsilon\rangle \rightarrow b]
\end{array}
$$

Example

Consider the regular-controlled grammar H:

$1: S \rightarrow ABC$
$2: A \rightarrow aA$
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$$
\begin{array}{llll}
\langle S, \varepsilon \rangle & \Rightarrow & \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle S, \varepsilon \rangle \rightarrow \langle A, \varepsilon \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle] \\
 & \Rightarrow & \langle a, A \rangle \langle B, \varepsilon \rangle \langle C, \varepsilon \rangle & [\langle A, \varepsilon \rangle \rightarrow \langle a, A \rangle] \\
 & \Rightarrow & \langle a, A \rangle \langle b, B \rangle \langle C, \varepsilon \rangle & [\langle B, \varepsilon \rangle \rightarrow \langle b, B \rangle] \\
 & \Rightarrow & \langle a, A \rangle \langle b, B \rangle \langle c, C \rangle & [\langle C, \varepsilon \rangle \rightarrow \langle c, C \rangle] \\
 & \Rightarrow & \langle a, \varepsilon \rangle \langle b, B \rangle \langle c, C \rangle & [\langle a, A \rangle \rightarrow \langle a, \varepsilon \rangle] \\
 & \Rightarrow & \langle a, \varepsilon \rangle \langle b, \varepsilon \rangle \langle c, C \rangle & [\langle b, B \rangle \rightarrow \langle b, \varepsilon \rangle] \\
 & \Rightarrow & \langle a, \varepsilon \rangle \langle b, \varepsilon \rangle \langle c, \varepsilon \rangle & [\langle c, C \rangle \rightarrow \langle c, \varepsilon \rangle] \\
 & \Rightarrow & a \langle b, \varepsilon \rangle \langle c, \varepsilon \rangle & [\langle a, \varepsilon \rangle \rightarrow a] \\
 & \Rightarrow & ab \langle c, \varepsilon \rangle & [\langle b, \varepsilon \rangle \rightarrow b] \\
 & \Rightarrow & abc & [\langle c, \varepsilon \rangle \rightarrow c]
\end{array}
$$

Concluding Remarks

- Regular-controlled grammars with *appearance checking*.
- Workspace theorems for other regulated grammars.
- Does the new workspace theorem hold for the classical condition as well?

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Discussion