Closures

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Reflexive Relation

Definition

Let Q be a set and $R \subseteq Q \times Q$ be a relation over Q. If aRa for every $a \in R$, then R is a *reflexive* relation.

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Example

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Example

As an example of a relation that is not reflexive, consider the relation $>$ on integers. Indeed, $i \not> i$ for every integer i. For example, $6 \not> 6$.

Transitive Relation

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Let Q be a set and $R \subseteq Q \times Q$ be a relation over Q. If every $a, b, c \in Q$ satisfies that if aRb and bRc implies that aRc, then R is a *transitive* relation.

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Example

Once again, consider the standard relation $\geq$ on integers. We have seen that it is a reflexive relation. Is it also transitive? If it is, then for every integers i, j, k, if $i \geq j$ and $j \geq k$, then $i \geq k$. This is true. For example, $6 \geq 4$ and $4 \geq 3$, and $6 \geq 3$.

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Example

Consider the set $P = \{\text{Diana}, \text{Sarah}, \text{Elinor}\}$ and the relation

$$mother = \{(\text{Diana}, \text{Sarah}), (\text{Sarah}, \text{Elinor})\}$$

over P. This relation is not transitive.

Reflexive-Transitive Closure

Definition

Let Q be a set and $R \subseteq Q \times Q$ be a relation over Q. The *reflexive-transitive closure* of R is denoted by R^* and it is the relation with the following three properties:

1. *Reflexivity and transitivity*: R^* is both reflexive and transitive.
2. *Containment*: $R \subseteq R^*$
3. *Minimality*: There is no other reflexive and transitive relation relation R_2 such that $R_2 \subset R^*$ satisfies (1) and (2).

Example

Consider the following set of cities

$$C = \{\text{Prague}, \text{Vienna}, \text{New York}, \text{Ottawa}\}$$

and a relation that says you can get from a city to another one by taking a direct flight:

$$F = \{(\text{Prague}, \text{Vienna}), (\text{Vienna}, \text{New York}), (\text{New York}, \text{Ottawa})\}$$

For simplicity, we assume that there is no way back—that is, even though you can fly from Prague to Vienna, there is no direct flight from Vienna back to Prague.
In what follows, we will now construct a reflexive-transitive closure of F, which will be denoted by F^*. While F means that you can take a *single* flight from a city to another city, F^* means that you can get from a city to another city by taking *as many flights as needed*. This is the general meaning of the reflexive-transitive closure.

Reflexive-Transitive Closure (Example 2/3)

Example

First, to satisfy (ii), we include all elements of F into F:

$$F^* = F$$

Therefore, we have covered the possibilities of getting into a city by taking a single flight. The next step is to make F^* reflexive. To this end, we extend it in the following way:

$$F^* = F^* \cup \{(\text{Prague}, \text{Prague}), (\text{Vienna}, \text{Vienna}), (\text{New York}, \text{New York}), (\text{Ottawa}, \text{Ottawa})\}$$

Even though it may seems strange, this extension of F^* says that when your are in a city X, then you do not have to take *any* flights (or 0 flights) to get into X. This makes sense, right? Now, F^* is reflexive and satisfies (ii) and a half of (i).

Reflexive-Transitive Closure (Example 3/3)

Example

To complete the second half of (i), we have to make F^* transitive:

$$F^* = F^* \cup \{(\text{Prague}, \text{New York}), (\text{Prague}, \text{Ottawa}), (\text{Vienna}, \text{Ottawa})\}$$

After this extension, we see, for example, that we may fly from Prague into Ottawa—that is, Prague F^* Ottawa. Observe that Prague F Ottawa does not hold because there is no direct flight from Prague to Ottawa.

References

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Discussion