Sequences, Relations, and Functions

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Sequence

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Example

A sequence of the numbers for five consecutive dice rolls:

$$R = (1, 3, 2, 6, 6)$$

A sequence of all the letters in the word *sequence*:

$$L = (s, e, q, u, e, n, c, e)$$

A sequence containing the letters in L in an alphabetic order:

$$A = (c, e, e, e, n, q, s, u)$$

Finite and Infinite Sequences

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Example

The Fibonacci sequence, where each member equals to the sum of two previous members, is one of the most known infinite number sequence:

$$F = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$$

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Example

Sequences can be infinite in both directions. For example, the sequence of even numbers ordered by value can be written as

$$E = (\ldots, -4, -2, 0, 2, 4, 6, 8, 10, \ldots)$$

Finite Sequences

Finite sequences are also called *tuples*.

two elements	*pair*
three elements	*triplet*
four elements	*quadruple*
five elements	*quintuple*
six elements	*sextuple*
seven elements	*septuple*

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Example

The coordinates of a point in an n-dimensional space can be represented as n-tuples. We can use pairs such as $(3, 5)$, $(-10, 12)$, or $(1.23, -4.8)$ for two dimensional points, or triplets as $(1, -2, 3)$ or $(99.76, 0.593, 2.67)$ for points in a three dimensional space.

Cartesian Product

Definition

The *Cartesian product* of two sets, P and Q, denoted by $P \times Q$, is defined as

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Example

Consider the two sets $A = \{1, 2\}$ and $B = \{\clubsuit, \diamondsuit, \spadesuit, \heartsuit\}$. Then, their Cartesian product is

$$\begin{aligned} A \times B \quad = \quad & \{(1, \clubsuit), (1, \diamondsuit), (1, \spadesuit), (1, \heartsuit), \\ & (2, \clubsuit), (2, \diamondsuit), (2, \spadesuit), (2, \heartsuit)\} \end{aligned}$$

As you can see, we have included all possible combinations of 1 and 2 with $\clubsuit$, $\diamondsuit$, $\spadesuit$, and $\heartsuit$.

Relation

Definition

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Example

Let us consider four persons: Jane, Paul, Don, and Elizabeth. Jane is 16 years old, Paul is 32 years old, and both Don and Elizabeth are 22 years old. We want to specify this age relation mathematically. To do this, we create a set

$$P = \{\text{Jane}, \text{Paul}, \text{Don}, \text{Elizabeth}\}$$

and define the relation $age \subseteq P \times \mathbb{N}$ as follows:

$$age = \{(\text{Jane}, 16), (\text{Paul}, 32), (\text{Don}, 22), (\text{Elizabeth}, 22)\}$$

In this way, we have formally specified the relation between the four persons and their age.

If a relation has a finite number of elements, it is a *finite relation*; otherwise, it is an *infinite relation*.

Finite and Infinite Relations

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Example

The relation *age* from the previous example is finite while the mathematical relation $>$ over integers is infinite.

Definition

The *Cartesian product* of n sets, P_1 through P_n, where $n \geq 1$, is denoted by

$$P_1 \times P_2 \times \cdots \times P_n$$

and defined as

$$P_1 \times P_2 \times \cdots \times P_n = \{(x_1, x_2, \ldots, x_n) : x_1 \in P_1, x_2 \in P_2, \ldots, x_n \in P_n\}$$

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Definition

For $n \geq 1$, an n-ary *relation* R over sets $P_1, P_2, \ldots, P_n$ is defined as

$$R \subseteq P_1 \times P_2 \times \cdots \times P_n$$

Function

Definition

Let P and Q be two sets. A *function* f from P to Q, is a relation from P to Q such that for every $x \in P$, there is precisely one $y \in Q$ such that $(x, y) \in f$.

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Example

Consider the relation *age* over the sets

$$P = \{\text{Jane}, \text{Paul}, \text{Don}, \text{Elizabeth}\}$$

and the set of all natural numbers $\mathbb{N}$:

$$age = \{(\text{Jane}, 16), (\text{Paul}, 32), (\text{Don}, 22), (\text{Elizabeth}, 22)\}$$

Clearly, as each name have *exactly* one corresponding number, it is a function.

Example

On the other hand, if you consider the relation

$$\begin{aligned} age_2 = \; & \{(\text{Jane}, 16), (\text{Jane}, 20), (\text{Paul}, 32), \\ & (\text{Don}, 22), (\text{Elizabeth}, 22)\} \end{aligned}$$

you can see that as Jane has two corresponding values, 16 and 20 attached to it, it is not a function. Likewise, the relation

$$age_3 = \{(\text{Jane}, 16), (\text{Paul}, 32), (\text{Elizabeth}, 22)\}$$

is not a function over $P \times \mathbb{N}$ because even though it does not have multiple values attached to some name, it is not total. Indeed, Don has no corresponding value attached to itself.

References

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Discussion