Sets

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Supported by the FRVŠ MŠMT FR271/2012/G1 grant, 2012.

Contents

What Is a Set?

A *set* is a collection of distinct objects.

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Example

Examples of sets:

- Set V of three basic colors: $V = \{\text{red}, \text{green}, \text{blue}\}$.
- Set A containing four arrows: $A = \{\uparrow, \leftarrow, \downarrow, \rightarrow\}$.

The Basic Property of Sets

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2. Objects in a set have no implicit ordering.

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Two severe implications:

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2. Objects in a set have no implicit ordering.

Example

The sets $\{1, 2, 3\}$, $\{1, 1, 2, 3\}$, and $\{2, 3, 1\}$ are all equal. In fact, these are just three different ways to describe a single set!

Membership

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Denotation:

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Example

Consider the set of all integers, which is usually denoted by $\mathbb{Z}$. The objects $-5, 0, 12456$ are all elements of $\mathbb{Z}$ while the objects 3.25, `car`, and $\heartsuit$ are not elements of this set. Using the notation utilizing $\in$ and $\notin$, we may write, for example, $-5 \in \mathbb{Z}$ and $\heartsuit \notin \mathbb{Z}$.

Describing Sets: Using Natural Languages

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Example

To define that $\mathbb{Z}$ is the set of all integers, we may write: "Let $\mathbb{Z}$ be the set of all integers." Or, as another example, we may say: "Let N be the set of states in Europe whose names do not start with F." The latter example, of course, assumes that we agree what states are part of Europe and what are not.

The second way of describing a set consists of enumerating all the elements of the set as a comma-separated list enclosed in curly brackets.

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Example

To define a set Q consisting of numbers 1 through 5, we may write

$$Q = \{1, 2, 3, 4, 5\}$$

As another example, the set P of all playing card suits may be defined as

$$P = \{\clubsuit, \diamondsuit, \spadesuit, \heartsuit\}$$

Using Ellipsis To Ease the Enumeration

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Example

The set R consisting of numbers between 0 and 100 may be defined as

$$R = \{1, 2, \ldots, 100\}$$

As another example, $\{a, b, \ldots, z\}$ stands for all the lower-case letters of the English alphabet.

The last way of specifying a set is by giving a property that all its elements satisfy:

$$Q = \{x : \text{some property that } x \text{ has to satisfy}\}$$

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Example

The set of all *natural numbers* $\mathbb{N}$ may be written as

$$\mathbb{N} = \{x : x \in \mathbb{Z}, x \geq 0\}$$

Equality of Sets

Two sets are *equal* if they contain precisely the same elements.

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Example

Let M and N be two sets, defined as $M = \{1, 2, 3\}$ and

$$N = \{x : 1 \leq x \leq 3\}$$

Then, $M = N$. Furthermore, let P be a set defined as $P = \{1, 2, 3, 4, 5\}$. Then, $N \neq P$, which also implies that $M \neq P$.

Containment of Sets

Let A and B be two sets. If every element of a set A is also an element of a set B, then we say that A is a *subset* of B, written as $A \subseteq B$.

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Example

Consider the set $P = \{1, 2, 3, 4, 5\}$. We have argued that $P \subseteq \mathbb{Z}$. However, as $P \neq \mathbb{Z}$, we may also write $P \subset \mathbb{Z}$.

The Empty Set

The *empty set* is the set containing no elements. It is denoted by $\emptyset$.

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Example

Define the set E by the following notation:

$$E = \{x : x \geq 0, x + 1 = x\}$$

E is, in fact, empty, which can be written as $E = \emptyset$.

Let Q be a set. The *power set* of Q, denoted by 2^Q, is the set of all subsets of Q, defined as

$$2^Q = \{P : P \subseteq Q\}$$

Example

Consider the set $A = \{1, 2, 3\}$. Then, its power set is

$$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$

Finite and Infinite Sets

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Example

$\mathbb{Z}$ is an infinite set while $P = \{1, 2, 3, 4, 5\}$ is a finite set.

Let P and Q be two sets. The *union* of P and Q, denoted by $P \cup Q$, is defined as

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Set Union

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Example

Consider the sets $A = \{\clubsuit, \spadesuit\}$ and $C = \{\clubsuit, \diamondsuit\}$. Then,

- $A \cup B = \{\clubsuit, \spadesuit, \diamondsuit\}$;
- $A \cup A = A \cup \emptyset = \{\clubsuit, \spadesuit\}$.

Set Intersection

Let P and Q be two sets. The *intersection* of P and Q, denoted by $P \cap Q$, is defined as

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Example

Set $A = \{\clubsuit, \spadesuit\}$, $B = \{\clubsuit, \diamondsuit\}$ and $B = \{\heartsuit, \diamondsuit\}$. Then

- $A \cap B = \{\clubsuit\}$;
- $B \cap C = \{\diamondsuit\}$;
- $A \cap A = \{\clubsuit, \spadesuit\}$;
- $A \cap C = A \cap \emptyset = \emptyset$.

Set Complement

Let P a set, and $\mathbb{U}$ be the universe of all objects. Then, the *complement* of P, denoted by $\overline{P}$, is defined as

$$\overline{P} = \{x : x \in \mathbb{U}, x \notin P\}$$

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Example

Set the universe to $\mathbb{U} = \{\clubsuit, \spadesuit, \heartsuit, \diamondsuit\}$ and consider the set $A = \{\clubsuit, \diamondsuit\}$. Then

- $\overline{A} = \{\heartsuit, \spadesuit\}$;
- $\overline{\emptyset} = \mathbb{U}$;
- $\overline{\mathbb{U}} = \emptyset$.

References

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Discussion