Stable Marriage

Stable marriage problem

The setting:

- There are n boys $b_1, b_2, \dots, b_n$ and n girls $g_1, g_2, \dots, g_n$. We assume the number of boys and girls is the same.
- Each boy has his own ranked preference list of girls and each girl has her own ranked preference list of boys.
- The lists are complete and have no ties. Each boy ranks every girl and vice versa.

The goal:

Pair each boy with a unique girl so that there do not exist boys b_i, b_j, and girls g_k, g_l where b_i and g_k are paired up, but boy b_i prefers girl g_l to g_k, and girl g_k prefers boy b_j to b_i.

Stable marriage problem

$$b_1 \rightarrow (g_3, g_2, g_5, g_1, g_4)$$
$$b_2 \rightarrow (g_1, g_2, g_5, g_3, g_4)$$
$$b_3 \rightarrow (g_4, g_3, g_2, g_1, g_5)$$
$$b_4 \rightarrow (g_1, g_3, g_4, g_2, g_5)$$
$$b_5 \rightarrow (g_1, g_2, g_4, g_5, g_3)$$

$$g_1 \rightarrow (b_3, b_5, b_2, b_1, b_4)$$
$$g_2 \rightarrow (b_5, b_2, b_1, b_4, b_3)$$
$$g_3 \rightarrow (b_4, b_3, b_5, b_1, b_2)$$
$$g_4 \rightarrow (b_1, b_2, b_3, b_4, b_5)$$
$$g_5 \rightarrow (b_2, b_3, b_4, b_1, b_5)$$

Gale-Shapley algorithm

Each Day

Morning:

Each girl stands on her balcony. Each boy stands under the balcony of his favorite girl whom he has not yet crossed off his list and serenades. If there are no girls left on his list, he stays home and does graph algorithms homework.

Afternoon:

Girls who have at least one suitor say to their favorite from among the suitors that day: „Maybe, come back tomorrow." To the others, they say „No, I will never marry you!"

Evening:

Any boy who hears „No" crosses that girl off his list.

Gale-Shapley algorithm

Termination Condition:

If there is a day when every girl has at most one suitor, we stop and each girl marries her current suitor (if any).

Gale-Shapley algorithm

$$b_1 \rightarrow (g_3, g_2, g_5, g_1, g_4)$$
$$b_2 \rightarrow (g_1, g_2, g_5, g_3, g_4)$$
$$b_3 \rightarrow (g_4, g_3, g_2, g_1, g_5)$$
$$b_4 \rightarrow (g_1, g_3, g_4, g_2, g_5)$$
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$$g_1 \rightarrow (b_3, b_5, b_2, b_1, b_4)$$
$$g_2 \rightarrow (b_5, b_2, b_1, b_4, b_3)$$
$$g_3 \rightarrow (b_4, b_3, b_5, b_1, b_2)$$
$$g_4 \rightarrow (b_1, b_2, b_3, b_4, b_5)$$
$$g_5 \rightarrow (b_2, b_3, b_4, b_1, b_5)$$

Gale-Shapley algorithm

- the Gale-Shapley algorithm terminates,
- the Gale-Shapley algorithm terminates quickly,
- at termination, there are no rogue couples,
- everyone is married.

Optimality

Let S be the set of all stable matchings. Since the Gale-Shapley algorithm gives a stable matching, we know that $S \neq \emptyset$. For each person p, we define the realm of possibility for p to be

$$\{q | \{p, q\} \in M \text{ for some } M \in S\}.$$

That is, q is within the realm of possibility for p if and only if there is a stable matching where p marries q.

Optimality

- The Gale-Shapley algorithm pairs every boy with his optimal mate, i.e., with his favorite from the realm of possibility!
- The Gale-Shapley algorithm pairs every girl with her pessimal mate, i.e., with her least favorite from the realm of possibility!

Indifference

We have a set of n boys and a set of n girls. Assume each boy and each girl ranks the members of the opposite gender, but now we allow ties in the ranking.

For example (with $n = 4$), a girl could say that b_1 is ranked in first place; second place is a tie between b_2 and b_3 (she has no preference between them); and b_4 is in last place.

We will say that g prefers b to b' if b is ranked higher than b' on her preference list (they are not tied).

With indifferences in the rankings, there could be two natural notions for stability.

Indifference

- A strong instability in a perfect matching M consists of a boy b and a girl g, such that each of b and g prefers the other to their partner in M. Does there always exist a perfect matching with no strong instability?

Indifference

- A weak instability in a perfect matching M consists of a boy b and a girl g, such that their partners in M are g' and b', respectively, and one of the following holds:
  - b prefers g to g', and g either prefers b to b' or is indifferent between these two choices; or
  - g prefers b to b', and b either prefers g to g' or is indifferent between these two choices.

  Does there always exist a perfect matching with no weak instability?

College admission

There are n students $s_1, s_2, \dots, s_n$ and m universities $u_1, u_2, \dots, u_m$.

University u_i has n_i slots for students, and we're guaranteed that

$$n_1 + n_2 + \cdots + n_m = n.$$

Each student ranks all universities (no ties) and each university ranks all students (no ties).

Design an algorithm to assign students to universities with the following properties

College admission

- Every student is assigned to one university.
- University u_i gets assigned n_i students.
- There do not exist students s_i, s_j, and universities u_k, u_l where student s_i is assigned to university u_k, student s_j is assigned to university u_l, student s_j prefers university u_k to university u_l, and university u_k prefers student s_j to student s_i.
- It is student-optimal. This means that of all possible assignments satisfying the first three properties, every student gets his/her top choice of university amongst these assignments.

College admission

Each Day

Morning:

Each university asks which students are interested in applying. Each student applies to his/her favorite university that has not yet rejected him/her. If there are no universities left on the student's list, the student takes some time off to think about life and the future.

College admission

Afternoon:

Each university u_i tells its favorite n_i applicants „Maybe, we are still processing your application." If u_i has less than n_i applicants, it tells all of its applicants this message. If u_i has more than n_i applicants, it tells the remaining ones „Sorry, there were a large number of very qualified students applying this year, yet we can only accept a very limited number. We regret to inform you that you were not accepted. Thank you for applying to our university."

College admission

Evening:

Any student who hears „Sorry, …" from some university, crosses off that university from his/her list.

Termination Condition:

If there is a day when each university u_i has at most n_i applicants, we stop and each university accepts all of its applicants (if any).