# Part VII. Models for Context-Free Languages

# Context-Free Grammar (CFG)

**Gist: A *grammar* is based on a finite set of grammatical rules, by which it generates strings of its language.**

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**Illustration:** Start nonterminal

**Grammar *G*:**

**Nonterminals:** $A$, $B$, $S$

**Terminals:** $a$, $b$, $c$, $d$

**Rules:**
$S \rightarrow AB$,
$A \rightarrow aAb$,
$A \rightarrow ab$,
$B \rightarrow bBa$,
$B \rightarrow ba$

$S$

⇓ Rule: $S \rightarrow AB$

$AB$

⇓ Rule: $A \rightarrow ab$

$abB$

⇓ Rule: $B \rightarrow bBa$

$abbBa$

⇓ Rule: $B \rightarrow ba$

$abbbaa \in L(G)$

# Context-Free Grammar: Definition

**Definition:** *A context-free grammar* (CFG) is a quadruple $G = (N, T, P, S)$, where

- $N$ is an alphabet of *nonterminals*
- $T$ is an alphabet of *terminals*, $N \cap T = \emptyset$
- $P$ is a finite set of *rules* of the form $A \rightarrow x$, where $A \in N$, $x \in (N \cup T)^*$
- $S \in N$ is the *start nonterminal*

**Mathematical Note on Rules:**

- Strictly mathematically, $P$ is a relation from $N$ to $(N \cup T)^*$
- Instead of $(A, x) \in P$, we write $A \rightarrow x \in P$

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- $A \rightarrow x$ means that $A$ can be replaced with $x$
- $A \rightarrow \varepsilon$ is called ***ε-rule***

# Convention

- $A, \ldots, F, S$ : nonterminals
- $S$ : the start nonterminal
- $a, \ldots, d$ : terminals
- $U, \ldots, Z$ : members of $(N \cup T)$
- $u, \ldots, z$ : members of $(N \cup T)^*$
- $\pi$ : sequence of productions

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A subset of rules of the form:

$$A \to x_1, A \to x_2, \ldots, A \to x_n$$

can be simply written as:

$$A \to x_1 \mid x_2 \mid \ldots \mid x_n$$

# Derivation Step

**Gist: A change of a string by a rule.**

**Definition:** Let $G = (N, T, P, S)$ be a CFG. Let $u, v \in (N \cup T)^*$ and $p = A \to x \in P$. Then, $uAv$ *directly derives* $uxv$ *according to* $p$ in $G$, written as $uAv \Rightarrow uxv$ [$p$] or, simply, $uAv \Rightarrow uxv$.

**Note:** If $uAv \Rightarrow uxv$ in $G$, we also say that $G$ makes a *derivation step* from $uAv$ to $uxv$.

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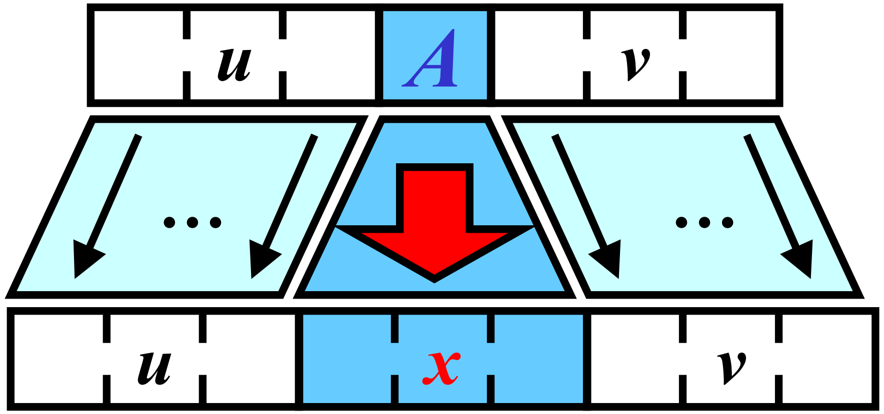
**Rule:** $A \to x$



# Sequence of Derivation Steps 1/2

**Gist: Several consecutive derivation steps.**

**Definition:** Let $u \in (N \cup T)^*$. $G$ makes a *zero-step derivation* from $u$ to $u$; in symbols,

$$u \Rightarrow^0 u\ [\varepsilon] \text{ or, simply, } u \Rightarrow^0 u$$

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**Definition:** Let $u_0,\ldots,u_n \in (N \cup T)^*$, $n \geq 1$, and $u_{i-1} \Rightarrow u_i\ [p_i]$, $p_i \in P$, for all $i = 1,\ldots, n$; that is

$$u_0 \Rightarrow u_1\ [p_1] \Rightarrow u_2\ [p_2] \ldots \Rightarrow u_n\ [p_n]$$

Then, $G$ makes *$n$ derivation steps* from $u_0$ to $u_n$,

$$u_0 \Rightarrow^n u_n\ [p_1 \ldots p_n] \text{ or, simply, } u_0 \Rightarrow^n u_n$$

# Sequence of Derivation Steps 2/2

If $u_0 \Rightarrow^n u_n$ [π] for some $n \geq 1$, then $u_0$ *properly derives* $u_n$ in $G$, written as $u_0 \Rightarrow^+ u_n$ [π].

If $u_0 \Rightarrow^n u_n$ [π] for some $n \geq 0$, then $u_0$ *derives* $u_n$ in $G$, written as $u_0 \Rightarrow^* u_n$ [π].

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**Example:** Consider

$aAb \Rightarrow aaBbb$ [**1**: $A \rightarrow aBb$], and

$aaBbb \Rightarrow aacbb$ [**2**: $B \rightarrow c$].

Then, $aAb \Rightarrow^2 aacbb$ [**1 2**],

$aAb \Rightarrow^+ aacbb$ [**1 2**],

$aAb \Rightarrow^* aacbb$ [**1 2**]

# Generated Language

**Gist:** ***G generates*** **a terminal string** ***w*** **by a sequence of derivation steps from** ***S*** **to** ***w***

**Definition:** Let $G = (N, T, P, S)$ be a CFG. The *language generated by G*, $L(G)$, is defined as

$$L(G) = \{w: w \in T^*, S \Rightarrow^* w\}$$

**Illustration:**

$G = (N, T, P, S)$, let $w = a_1a_2\ldots a_n$; $a_i \in T$ for $i = 1..n$

**if** $S \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \underbrace{a_1a_2\ldots a_n}_{w}$ **then** $w \in L(G)$;

**otherwise,** $w \notin L(G)$

# Context-Free Language (CFL)

**Gist: A language generated by a CFG.**

**Definition:** Let $L$ be a language. $L$ is a *context-free language* (CFL) if there exists a context-free grammar that generates $L$.

**Example:**

$G = (N, T, P, S)$, where $N = \{S\}$, $T = \{a, b\}$,

$P = \{1: S \to aSb, 2: S \to \varepsilon\}$

$S \Rightarrow \varepsilon$ [2]

$S \Rightarrow aSb$ [1] $\Rightarrow ab$ [2]

$S \Rightarrow aSb$ [1] $\Rightarrow aaSbb$ [1] $\Rightarrow aabb$ [2]

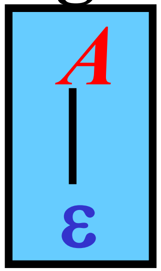
⋮

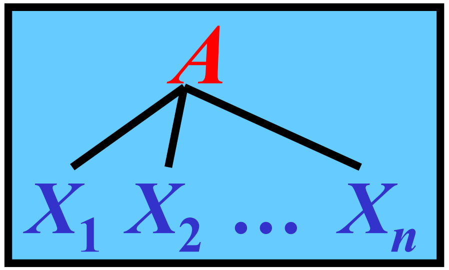
$L(G) = \{a^n b^n : n \geq 0\}$

**$L = \{a^n b^n : n \geq 0\}$ is a CFL.**

# Rule Tree

- **Rule tree graphically represents a rule**

**1)** $A \to \varepsilon$**:** 


**2)** $A \to X_1X_2\ldots X_n$**:** 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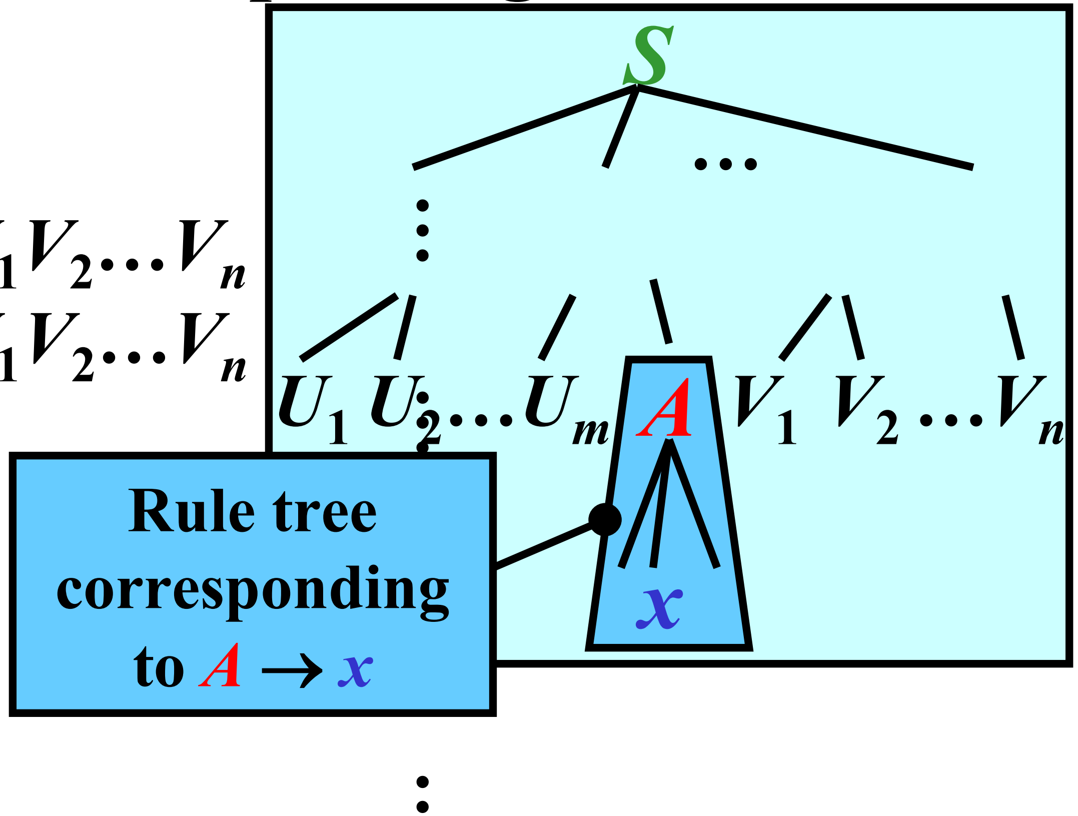
- **Derivation tree corresponding to a derivation**

$S \Rightarrow \ldots$

$\vdots$

$\Rightarrow U_1U_2\ldots U_m A V_1V_2\ldots V_n$

$\Rightarrow U_1U_2\ldots U_m x V_1V_2\ldots V_n$



# Derivation Tree: Example

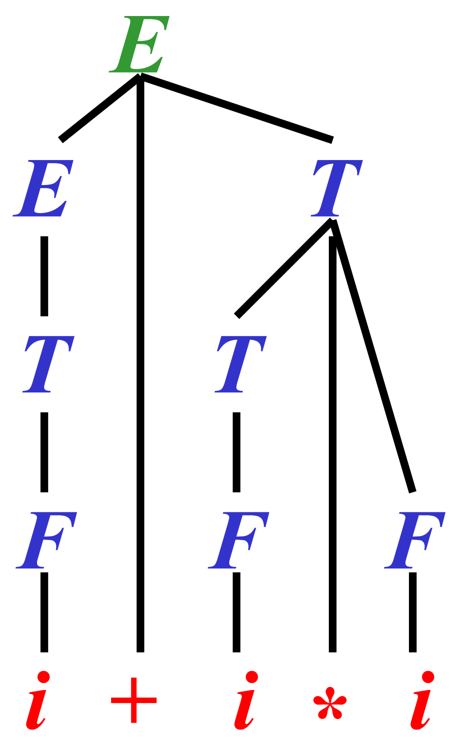
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (, )\}$,

$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

**Derivation:**

$$\underline{E} \Rightarrow E + \underline{T} \quad [1]$$
$$\Rightarrow E + \underline{T} * F \quad [3]$$
$$\Rightarrow E + \underline{F} * F \quad [4]$$
$$\Rightarrow \underline{E} + i * F \quad [6]$$
$$\Rightarrow T + i * \underline{F} \quad [2]$$
$$\Rightarrow \underline{T} + i * i \quad [6]$$
$$\Rightarrow \underline{F} + i * i \quad [4]$$
$$\Rightarrow i + i * i \quad [6]$$

**Derivation tree:**



# Leftmost Derivation

**Gist: During a *leftmost derivation step*, the leftmost nonterminal is rewritten.**

**Definition:** Let $G = (N, T, P, S)$ be a CFG, let $\boldsymbol{u} \in T^*$, $\boldsymbol{v} \in (N \cup T)^*$. Let $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$ be a rule. Then, $\boldsymbol{uAv}$ directly derives $\boldsymbol{uxv}$ in the *leftmost way* according to $\boldsymbol{p}$ in $G$, written as

$$\boldsymbol{uAv} \Rightarrow_{lm} \boldsymbol{uxv}\ [\boldsymbol{p}]$$

**Note:** We define $\Rightarrow_{lm}^{+}$ and $\Rightarrow_{lm}^{*}$ by analogy with $\Rightarrow^{+}$ and $\Rightarrow^{*}$, respectively.

# Leftmost Derivation: Example

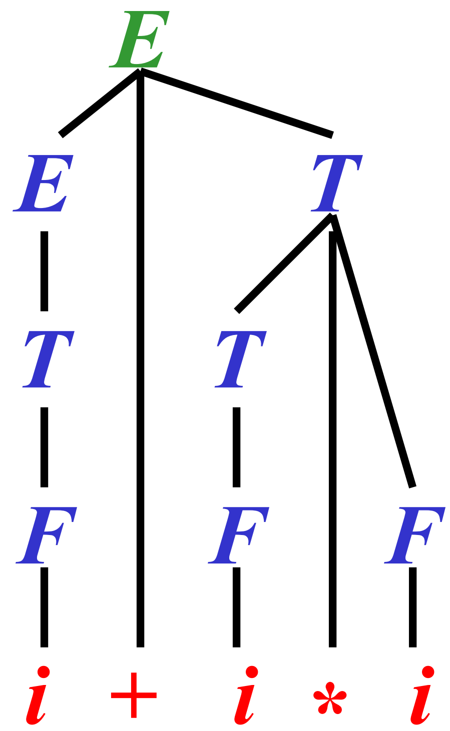
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (, )\}$,

$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

**Leftmost derivation:**

$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]
$\Rightarrow_{lm} \underline{T} + T$ [2]
$\Rightarrow_{lm} \underline{F} + T$ [4]
$\Rightarrow_{lm} i + \underline{T}$ [6]
$\Rightarrow_{lm} i + \underline{T} * F$ [3]
$\Rightarrow_{lm} i + \underline{F} * F$ [4]
$\Rightarrow_{lm} i + i * \underline{F}$ [6]
$\Rightarrow_{lm} i + i * i$ [6]

**Derivation tree:**



# Rightmost Derivation

**Gist: During a *rightmost derivation step*, the rightmost nonterminal is rewritten.**

**Definition:** Let $G = (N, T, P, S)$ be a CFG, let $\boldsymbol{u} \in (N \cup T)^*$, $\boldsymbol{v} \in T^*$. Let $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$ be a rule. Then, $\boldsymbol{uAv}$ directly derives $\boldsymbol{uxv}$ in the *rightmost way* according to $\boldsymbol{p}$ in $G$, written as

$$\boldsymbol{uAv} \Rightarrow_{rm} \boldsymbol{uxv}\ [\boldsymbol{p}]$$

**Note:** We define $\Rightarrow_{rm}^+$ and $\Rightarrow_{rm}^*$ by analogy with $\Rightarrow^+$ and $\Rightarrow^*$, respectively.

# Rightmost Derivation: Example

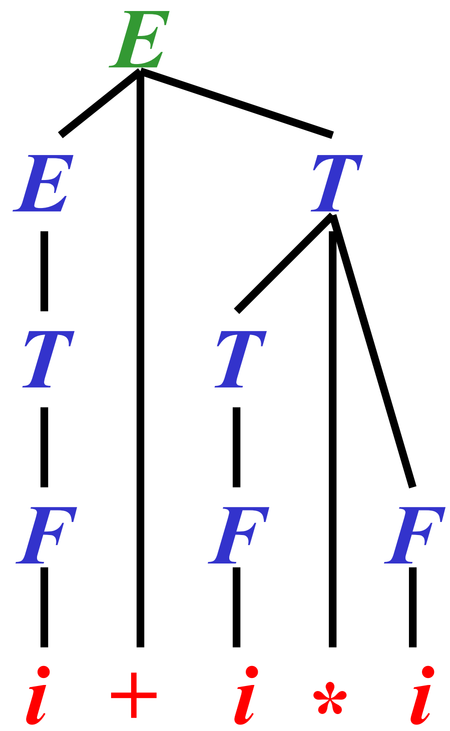
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (, )\}$,

$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

**Rightmost derivation:**

$\underline{E} \Rightarrow_{rm} E + \underline{T}$ [1]
$\Rightarrow_{rm} E + T * \underline{F}$ [3]
$\Rightarrow_{rm} E + \underline{T} * i$ [6]
$\Rightarrow_{rm} E + \underline{F} * i$ [4]
$\Rightarrow_{rm} \underline{E} + i * i$ [6]
$\Rightarrow_{rm} \underline{T} + i * i$ [2]
$\Rightarrow_{rm} \underline{F} + i * i$ [4]
$\Rightarrow_{rm} i + i * i$ [6]

**Derivation tree:**



# Derivations: Summary

- Let $A \rightarrow x \in P$ be a rule.

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**1) Derivation:**

Let $u, v \in (N \cup T)^*$ : $uAv \Rightarrow uxv$

**Note: Any nonterminal is rewritten**

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**2) Leftmost derivation:**

Let $u \in T^*, v \in (N \cup T)^*$ : $uAv \Rightarrow_{lm} uxv$

**Note: Leftmost nonterminal is rewritten**

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**3) Rightmost derivation:**

Let $u \in (N \cup T)^*, v \in T^*$ : $uAv \Rightarrow_{rm} uxv$

**Note: Rightmost nonterminal is rewritten**

# Reduction of the Number of Derivations

**Gist: Without any loss of generality, we can consider only leftmost or rightmost derivations.**

**Theorem:** Let $G = (N, T, P, S)$ be a CFG.
The next three languages coincide

(1) $\{w: w \in T^*, S \Rightarrow_{lm}^* w\}$

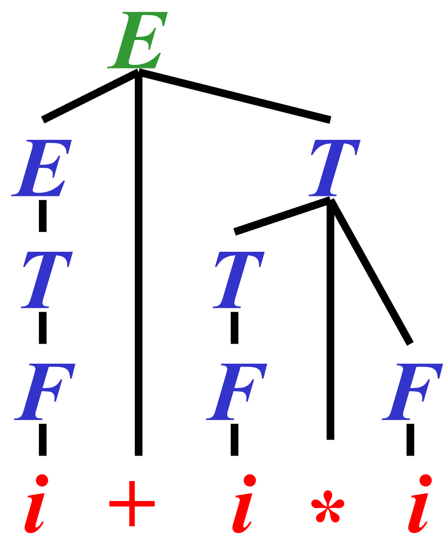
(2) $\{w: w \in T^*, S \Rightarrow_{rm}^* w\}$

(3) $\{w: w \in T^*, S \Rightarrow^* w\} = L(G)$

# Introduction to Ambiguity

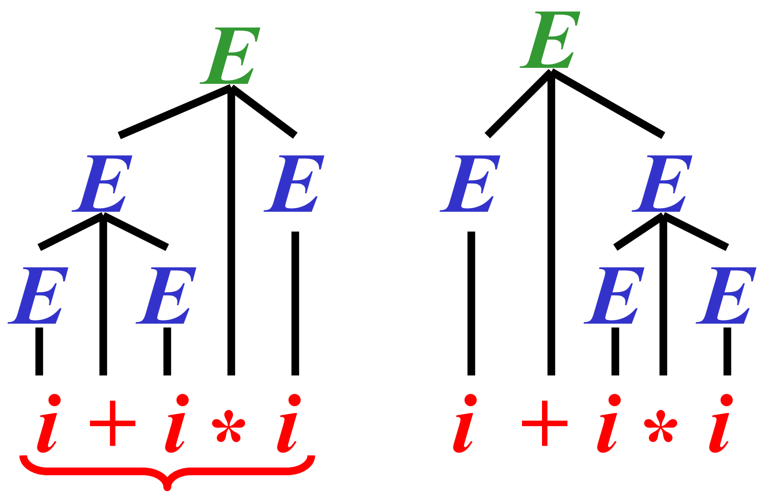
$G_{expr1} = (N, T, P, E)$, where
$N = \{E, F, T\}$, $T = \{i, +, *, (, )\}$,
$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$,
3: $T \rightarrow T*F$, 4: $T \rightarrow F$,
5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

**Theory: ☹ × Practice: ☺**




$G_{expr2} = (N, T, P, E)$, where
$N = \{E\}$, $T = \{i, +, *, (, )\}$,
$P = \{$ 1: $E \rightarrow E+E$, 2: $E \rightarrow E*E$,
3: $E \rightarrow (E)$, 4: $E \rightarrow i$ $\}$

**Theory: ☺ × Practice: ☹**




**Improper during compilation**

**Note:** $L(G_{expr1}) = L(G_{expr2})$

# Grammatical Ambiguity

**Definition:** Let $G = (N, T, P, S)$ be a CFG. If there exists $x \in L(G)$ with more than one derivation tree, then $G$ is *ambiguous*; otherwise, $G$ is *unambiguous*.

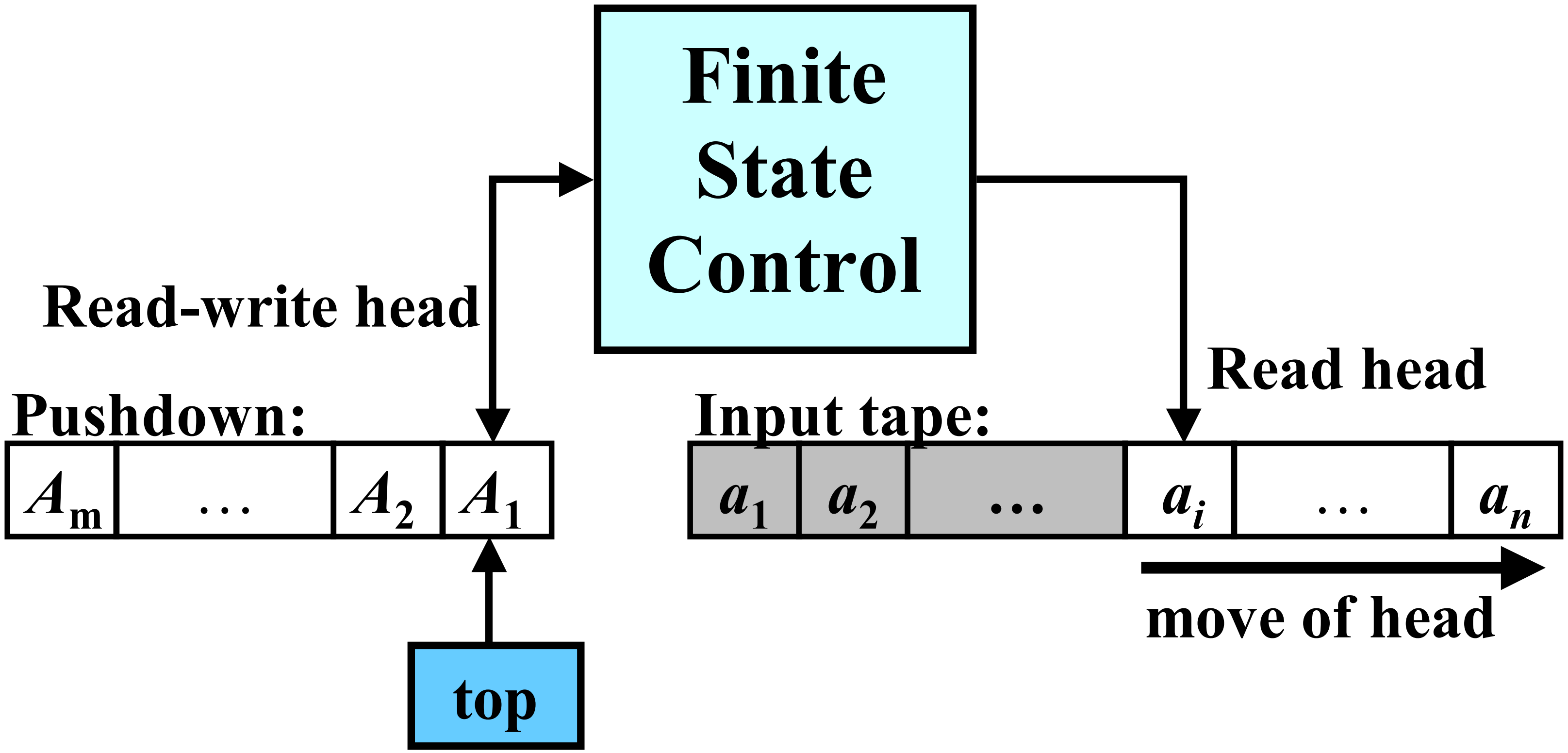
**Definition:** A CFL, $L$, is *inherently ambiguous* if $L$ is generated by no unambiguous grammar.

## Example:

- $G_{expr1}$ is **unambiguous**, because for every $x \in L(G_{expr1})$ there exists **only one derivation tree**
- $G_{expr2}$ is **ambiguous**, because for $i+i*i \in L(G_{expr2})$ there exist **two derivation trees**
- $L_{expr} = L(G_{expr1}) = L(G_{expr2})$ is **not inherently ambiguous** because $G_{expr1}$ **is unambiguous**

# Pushdown Automata (PDA)

**Gist: An FA extended by a pushdown store.**



# Pushdown Automata: Definition

**Definition:** *A pushdown automaton* (PDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where

- $Q$ is a *finite set of states*
- $\Sigma$ is an *input alphabet*
- $\Gamma$ is a *pushdown alphabet*
- $R$ is a *finite set of rules* of the form: $Apa \rightarrow wq$
  where $A \in \Gamma$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $w \in \Gamma^*$
- $s \in Q$ is the *start state*
- $S \in \Gamma$ is the *start pushdown symbol*
- $F \subseteq Q$ is a set of *final states*

# Notes on PDA Rules

**Mathematical note on rules:**

- Strictly mathematically, $R$ is a relation from $\Gamma \times Q \times (\Sigma \cup \{\varepsilon\})$ to $\Gamma^* \times Q$
- Instead of $(\boldsymbol{Apa}, \boldsymbol{wq}) \in R$, however, we write $\boldsymbol{Apa} \rightarrow \boldsymbol{wq} \in R$

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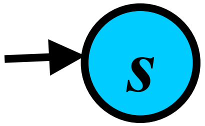
- **Interpretation of $\boldsymbol{Apa} \rightarrow \boldsymbol{wq}$**: if the current state is $\boldsymbol{p}$, current input symbol is $\boldsymbol{a}$, and the topmost symbol on the pushdown is $\boldsymbol{A}$, then $M$ can read $\boldsymbol{a}$, replace $\boldsymbol{A}$ with $\boldsymbol{w}$ and change state $\boldsymbol{p}$ to $\boldsymbol{q}$.
- **Note**: if $\boldsymbol{a} = \varepsilon$, no symbol is read

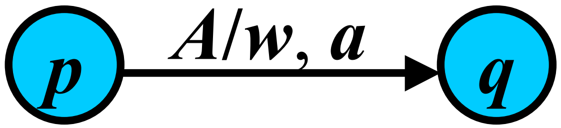
# Graphical Representation

represents $q \in Q$



represents the initial state $s \in Q$

represents a final state $f \in F$



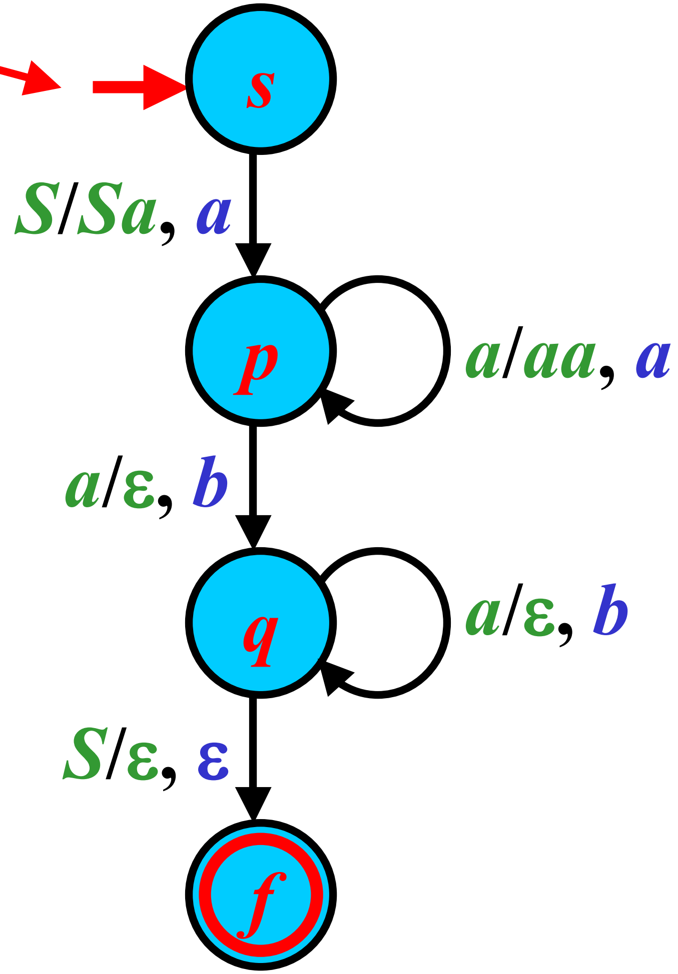
denotes $Apa \to wq \in R$

# Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

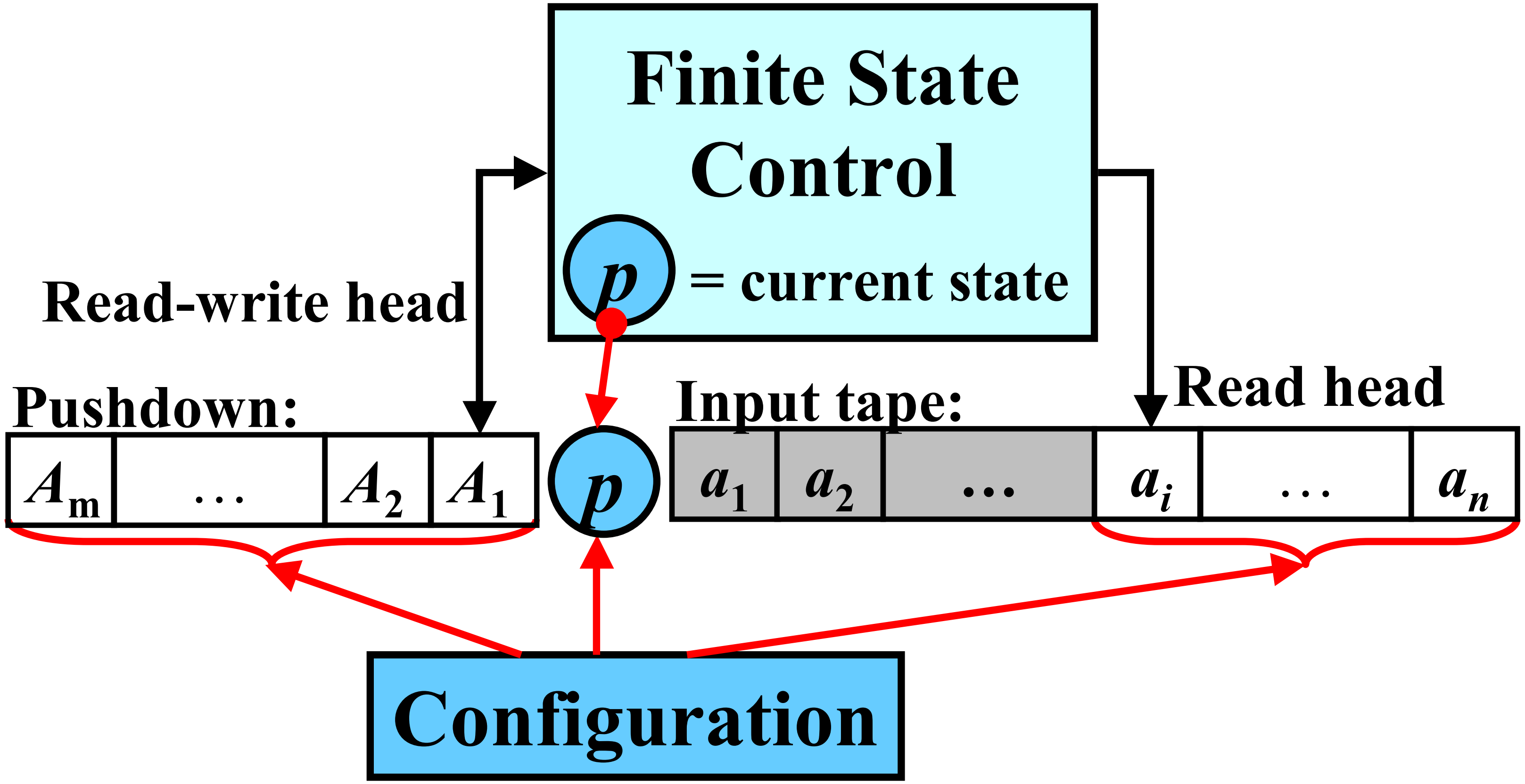
- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$
  $apa \rightarrow aap,$
  $apb \rightarrow q,$
  $aqb \rightarrow q,$
  $Sq \rightarrow f\}$
- $F = \{f\}$



# PDA Configuration

**Gist: Instantaneous description of PDA**

**Definition:** Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA.
*A configuration* of $M$ is a string $\chi \in \Gamma^*Q\Sigma^*$

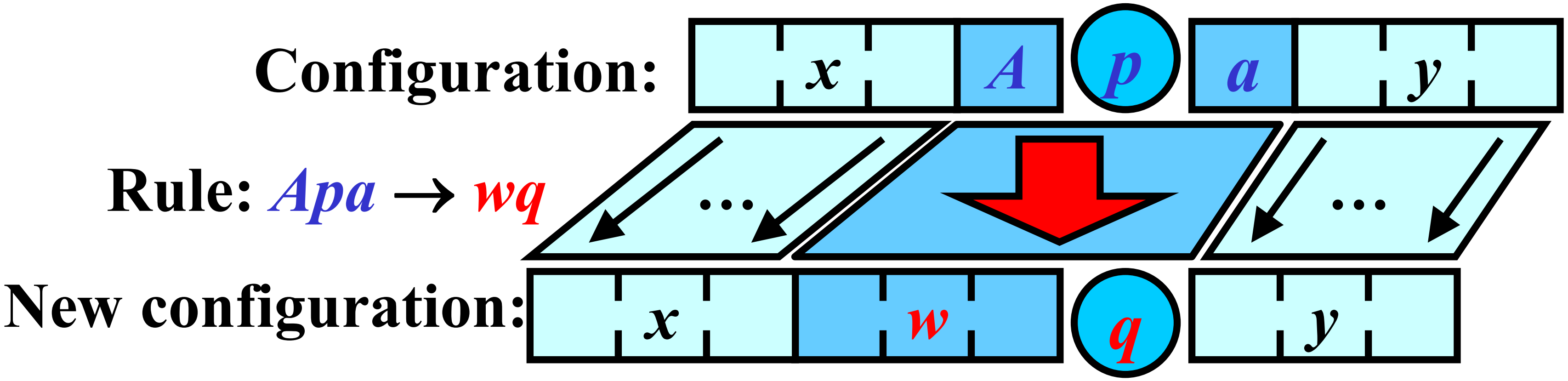


# Move

**Gist: A computational step made by a PDA**

**Definition:** Let $xApay$ and $xwqy$ be two configurations of a PDA, $M$, where
$x, w \in \Gamma^*$, $A \in \Gamma$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $y \in \Sigma^*$.
Let $r = Apa \rightarrow wq \in R$ be a rule. Then, $M$ makes a *move* from $xApay$ to $xwqy$ according to $r$, written as $xApay \vdash xwqy\ [r]$ or, simply, $xApay \vdash xwqy$.

**Note:** if $a = \varepsilon$, no input symbol is read



# Sequence of Moves 1/2

**Gist: Several consecutive computational steps**

**Definition:** Let $\chi$ be a configuration. *M* makes *zero moves* from $\chi$ to $\chi$; in symbols,

$$\chi \vdash^0 \chi \; [\varepsilon] \text{ or, simply, } \chi \vdash^0 \chi$$

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**Definition:** Let $\chi_0, \chi_1, \ldots, \chi_n$ be a sequence of configurations, $n \geq 1$, and $\chi_{i-1} \vdash \chi_i \; [r_i]$, $r_i \in R$, for all $i = 1, \ldots, n$; that is,

$$\chi_0 \vdash \chi_1 \; [r_1] \vdash \chi_2 \; [r_2] \ldots \vdash \chi_n \; [r_n]$$

Then *M* makes *n moves* from $\chi_0$ to $\chi_n$,

$$\chi_0 \vdash^n \chi_n \; [r_1 \ldots r_n] \text{ or, simply, } \chi_0 \vdash^n \chi_n$$

# Sequence of Moves 2/2

If $\chi_0 \;\vdash^n \chi_n\, [\rho]$ for some $n \geq 1$, then

$$\chi_0 \;\vdash^+ \chi_n\, [\rho] \text{ or, simply, } \chi_0 \;\vdash^+ \chi_n$$

If $\chi_0 \;\vdash^n \chi_n\, [\rho]$ for some $n \geq 0$, then

$$\chi_0 \;\vdash^* \chi_n\, [\rho] \text{ or, simply, } \chi_0 \;\vdash^* \chi_n$$

**Example:** Consider

***AApabc*** $\vdash$ ***ABqbc*** [**1**: ***Apa*** $\rightarrow$ ***Bq***], and

***ABqbc*** $\vdash$ ***ABCrc*** [**2**: ***Bqb*** $\rightarrow$ ***BCr***].

Then,

***AApabc*** $\vdash^2$ ***ABCrc*** [**1 2**],

***AApabc*** $\vdash^+$ ***ABCrc*** [**1 2**],

***AApabc*** $\vdash^*$ ***ABCrc*** [**1 2**]

# Accepted Language: Three Types

**Definition:** Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA.

**1)** The *language that M accepts* ***by final state***, denoted by $\boldsymbol{L(M)_f}$, is defined as
$L(M)_f = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z \in \Gamma^*, f \in F\}$

**2)** The *language that M accepts* ***by empty pushdown***, denoted by $\boldsymbol{L(M)_\varepsilon}$, is defined as
$L(M)_\varepsilon = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z = \varepsilon, f \in Q\}$

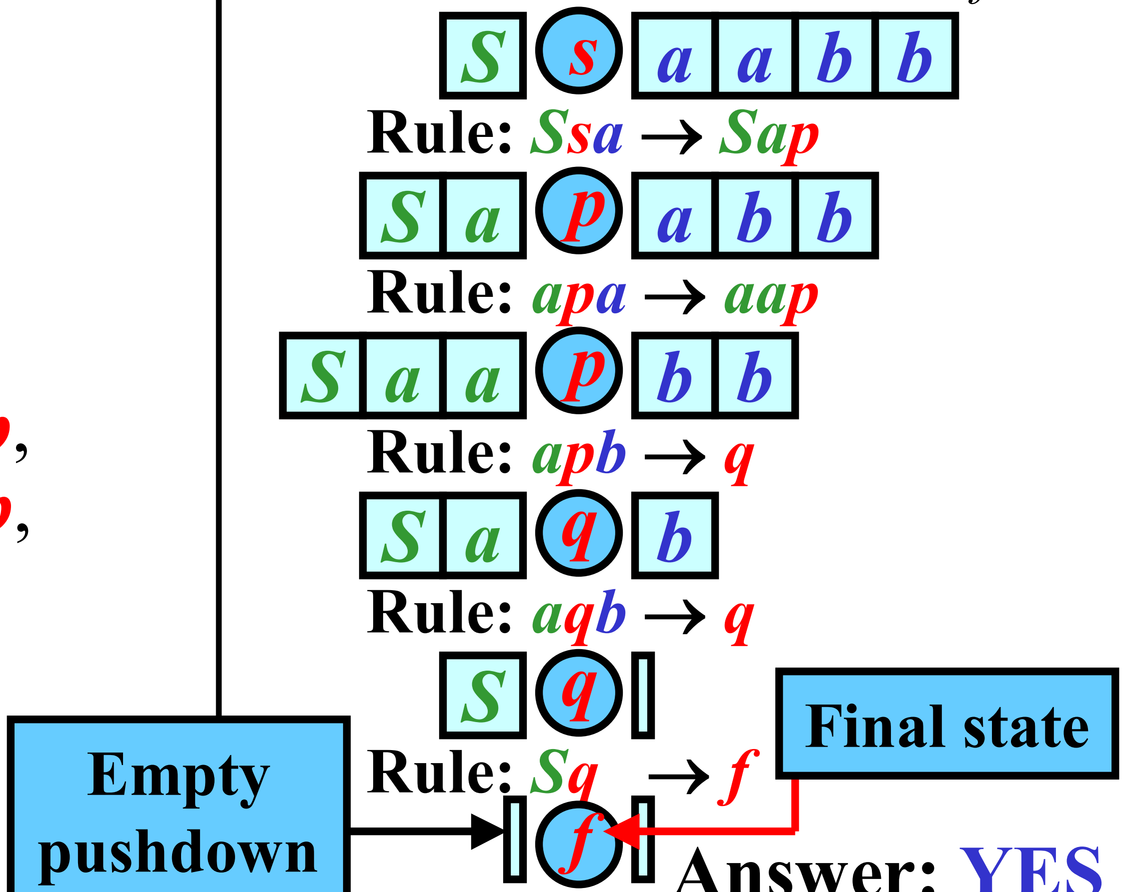
**3)** The *language that M accepts* ***by final state and empty pushdown***, denoted by $\boldsymbol{L(M)_{f\varepsilon}}$, is defined as
$L(M)_{f\varepsilon} = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z = \varepsilon, f \in F\}$

# PDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$
where:

- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$
  $apa \rightarrow aap,$
  $apb \rightarrow q,$
  $aqb \rightarrow q,$
  $Sq \rightarrow f\}$
- $F = \{f\}$

**Question:** $aabb \in L(M)_{f\varepsilon}$?




$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq \vdash f$

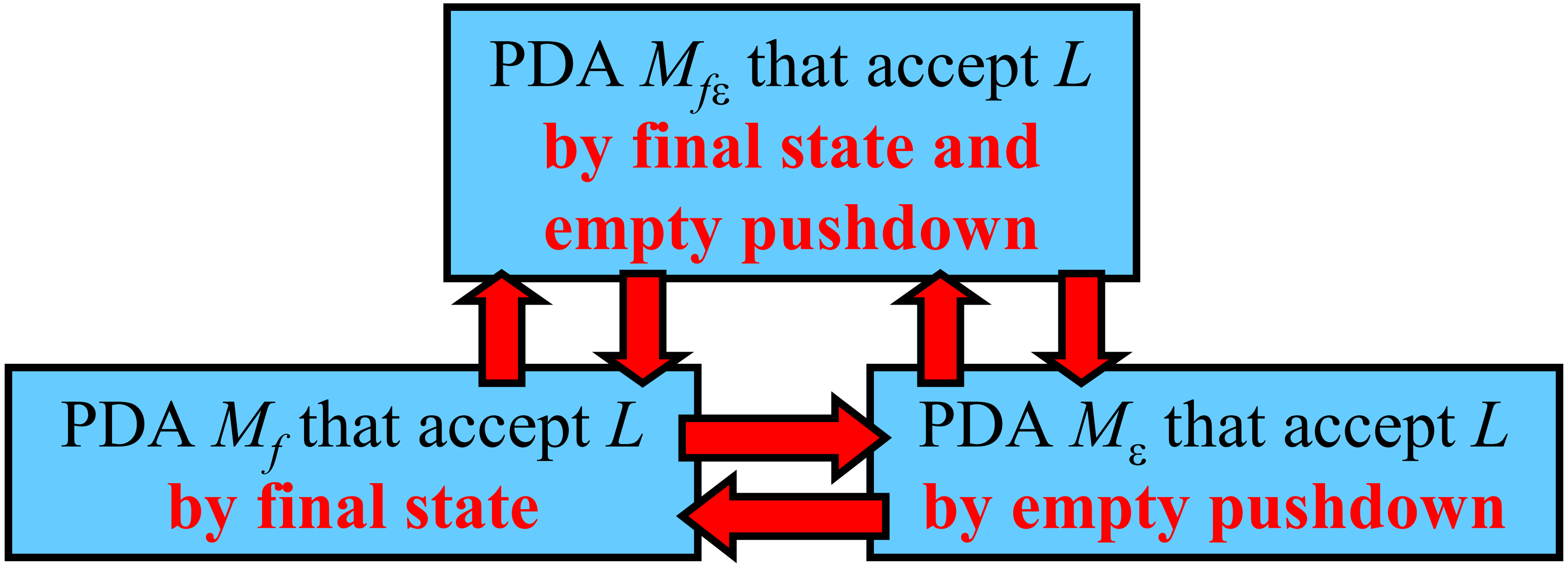
**Note:** $L(M)_f = L(M)_\varepsilon = L(M)_{f\varepsilon} = \{a^n b^n : n \geq 1\}$

# Three Types of Acceptance: Equivalence

**Theorem:**

- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for a PDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_\varepsilon)_\varepsilon$ for a PDA $M_\varepsilon$

**Note:** There exist these conversions:



# Deterministic PDA (DPDA)

**Gist: Deterministic PDA makes no more than one move from any configuration.**

**Definition:** Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA. $M$ is a *deterministic PDA* if for each rule $Apa \rightarrow wq \in R$, it holds that $R - \{Apa \rightarrow wq\}$ contains no rule with the left-hand side equal to $Apa$ or $Ap$.

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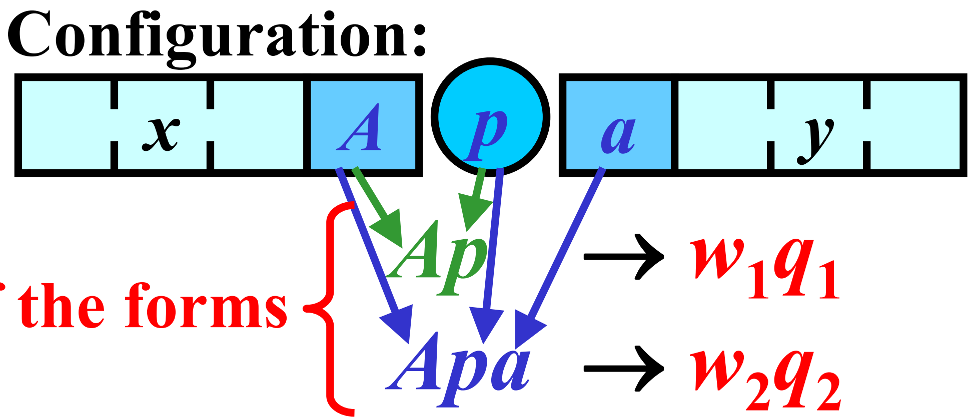
**Illustration:**

**No more that one rule of the forms**



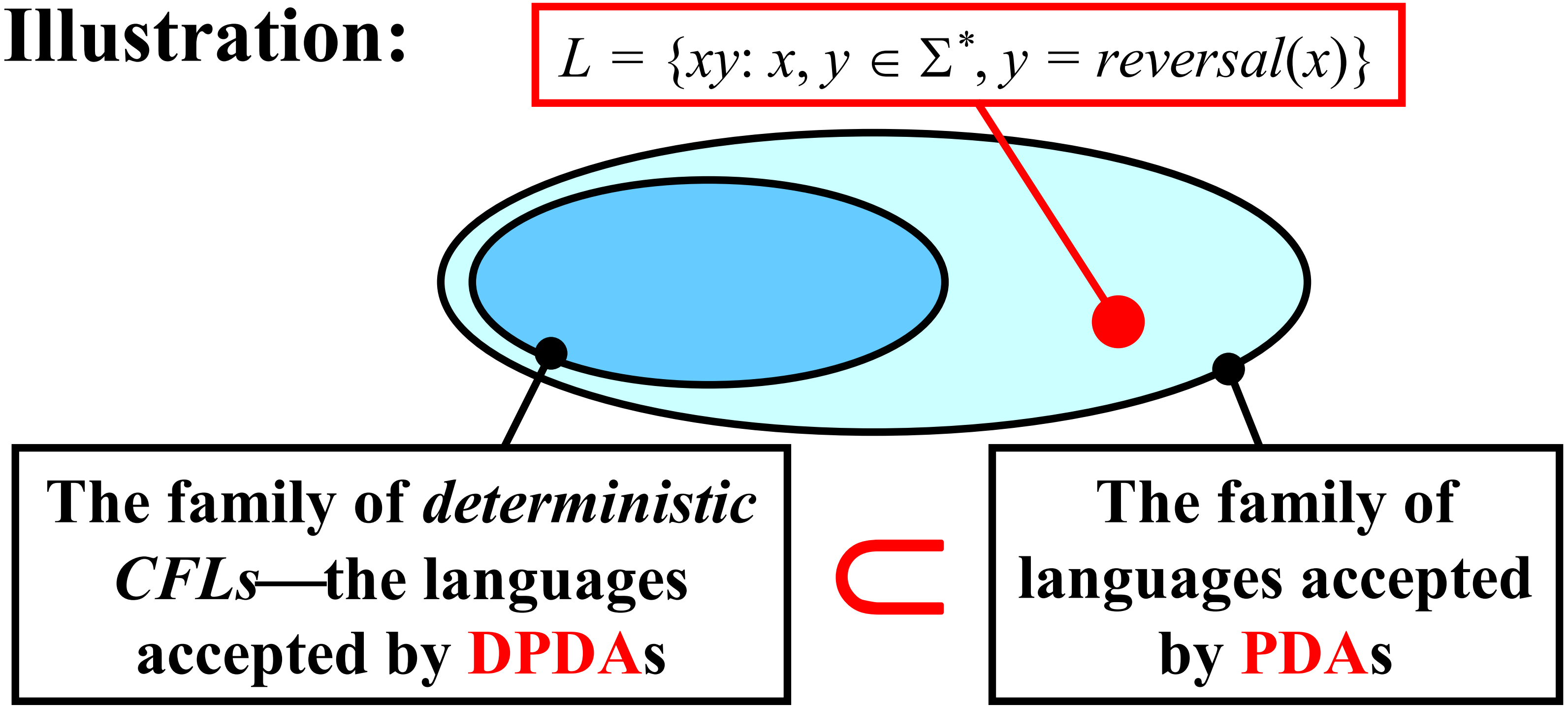
# PDAs are Stronger than DPDAs

**Theorem:** There exists no DPDA $M_{f\varepsilon}$ that accepts

$$L = \{xy: x, y \in \Sigma^*, y = reversal(x)\}$$

**Proof:** See page 431 in [Meduna: Automata and Languages]

**Illustration:**



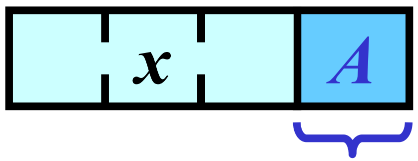
# Extended PDA (EPDA)

**Gist: The pushdown top of an EPDA represents a string rather than a single symbol.**

**Definition:** *An Extended Pushdown automaton* (EPDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where $Q, \Sigma, \Gamma, s, S, F$ are defined as in an PDA and $R$ is a *finite set of rules* of the form: $vpa \rightarrow wq$, where $v, w \in \Gamma^*$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$

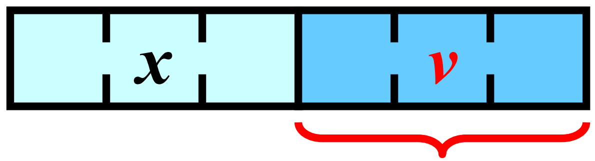
## Illustration:

**Pushdown of PDA:**




**PDA has a single symbols as the pushdown top**

**Pushdown of EPDA:**




**EPDA has a string as the pushdown top**

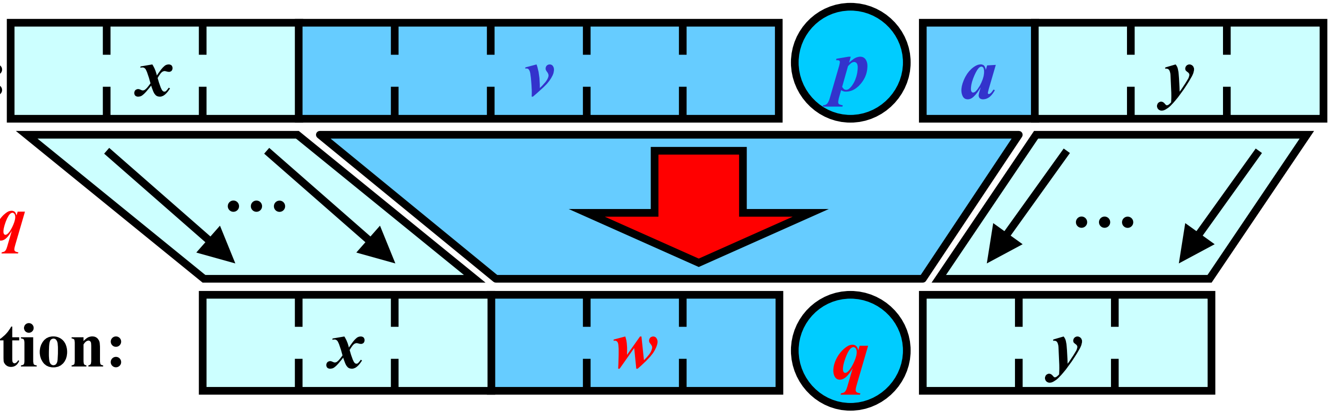
# Move in EPDA

**Definition:** Let $\boldsymbol{xvpay}$ and $\boldsymbol{xwqy}$ be two configurations of an EPDA, $M$, where $\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{w} \in \Gamma^*$, $\boldsymbol{p}, \boldsymbol{q} \in Q$, $\boldsymbol{a} \in \Sigma \cup \{\varepsilon\}$, and $\boldsymbol{y} \in \Sigma^*$. Let $\boldsymbol{r} = \boldsymbol{vpa} \rightarrow \boldsymbol{wq} \in R$ be a rule. Then, $M$ makes a *move* from $\boldsymbol{xvpay}$ to $\boldsymbol{xwqy}$ according to $\boldsymbol{r}$, written as $\boldsymbol{xvpay} \vdash \boldsymbol{xwqy}$ $[\boldsymbol{r}]$ or $\boldsymbol{xvpay} \vdash \boldsymbol{xwqy}$.

**Configuration:**

**Rule:** $\boldsymbol{vpa} \rightarrow \boldsymbol{wq}$

**New configuration:**



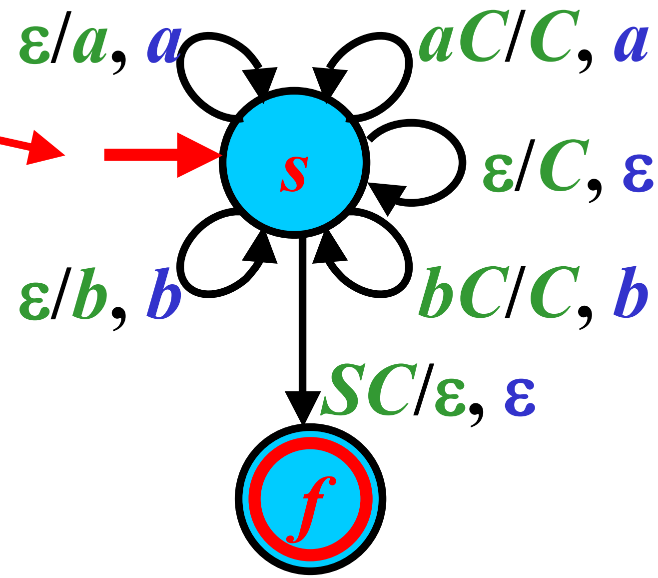

**Note:** $\vdash^n$, $\vdash^+$, $\vdash^*$, $L(M)_f$, $L(M)_\varepsilon$, and $L(M)_{f\varepsilon}$ are defined analogically to the corresponding definitions for PDA.

# EPDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, f\}$;
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, b, S, C\}$;
- $R = \{ \quad sa \rightarrow as,$
  $sb \rightarrow bs,$
  $s \rightarrow Cs,$
  $aCsa \rightarrow Cs,$
  $bCsb \rightarrow Cs,$
  $SCs \rightarrow f \}$
- $F = \{f\}$




**Question:** $abba \in L_{f\varepsilon}(M)$?

$Ssabba \vdash Sasbba \vdash Sabsba$
$\vdash SabCsba \vdash SaCsa$
$\vdash SCs \vdash f$

**Answer:** YES

**Note:** $L(M)_f = L(M)_\varepsilon = L(M)_{f\varepsilon} = \{xy: x, y \in \Sigma^*, y = \text{reversal}(x)\}$

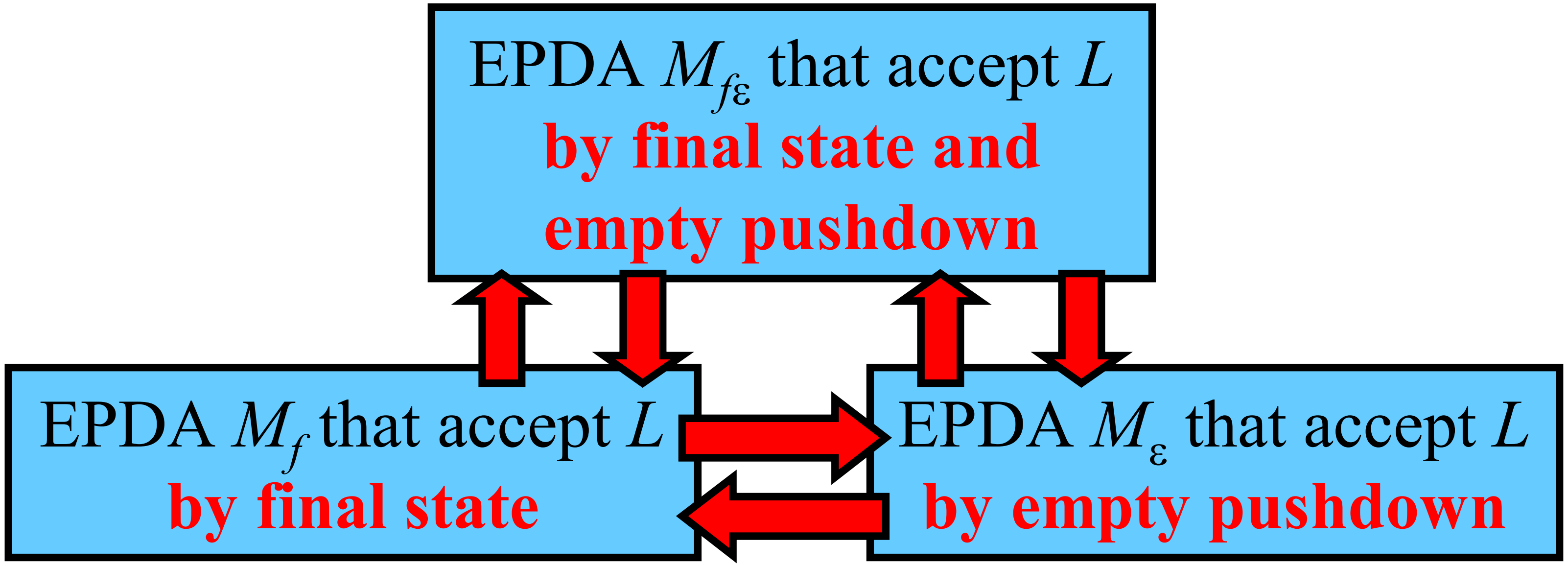
# Three Types of Acceptance: Equivalence

**Theorem:**

- $L = L(M_f)_f$ for an EPDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for an EPDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
- $L = L(M_f)_f$ for an EPDA $M_f \Leftrightarrow L = L(M_\varepsilon)_\varepsilon$ for an EPDA $M_\varepsilon$

---

**Note:** There exist these conversion:

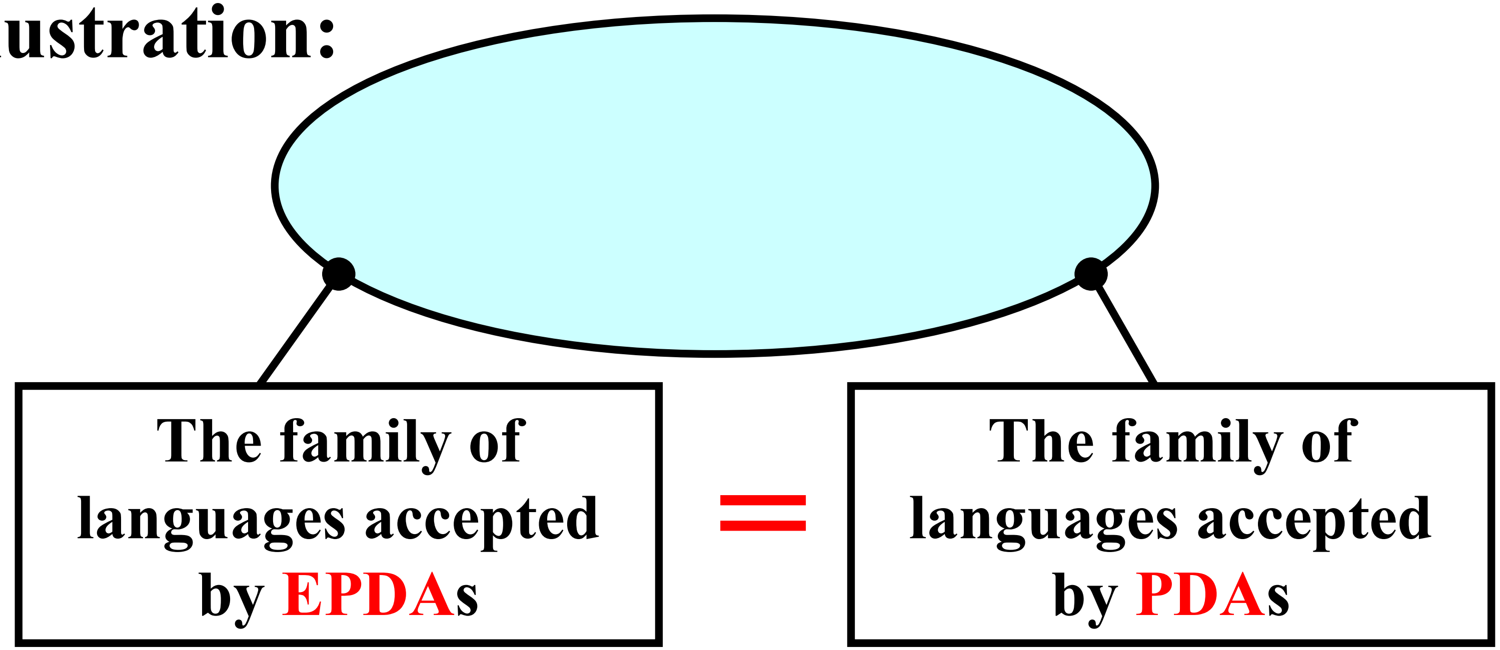


# EPDAs and PDAs are Equivalent

**Theorem:** For every EPDA $M$, there is a PDA $M'$, and $L(M)_f = L(M')_f$.

**Proof:** See page 419 in [Meduna: Automata and Languages]

**Illustration:**

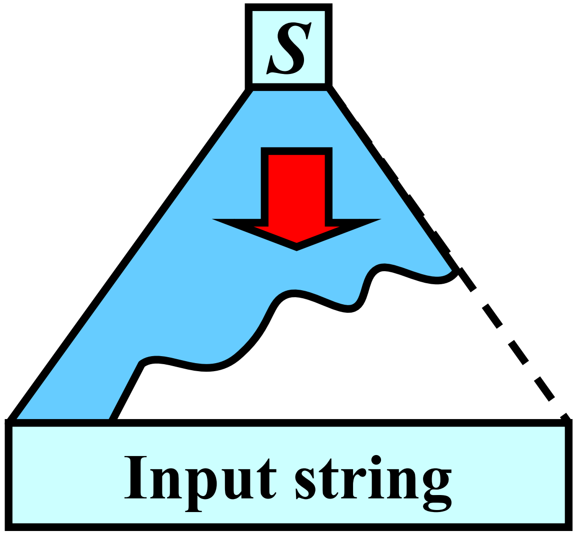


# EPDAs and PDAs as Parsing Models for CFGs

**Gist: An EPDA or a PDA can simulate the construction of a derivation tree for a CFG**

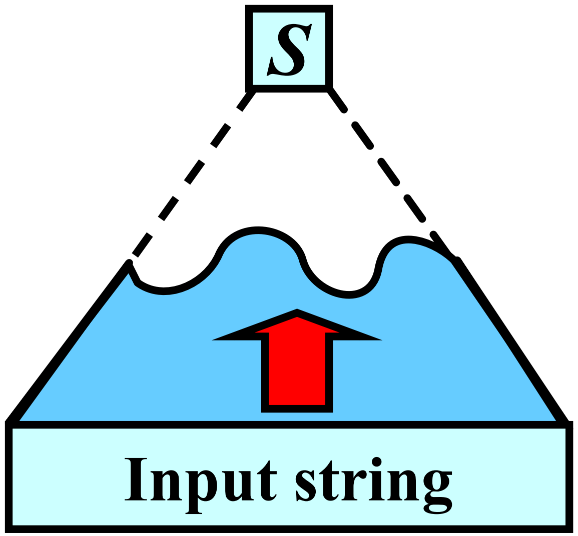
- **Two basic approaches:**

**1) Top-Down Parsing**




**From *S* towards the input string**

**2) Bottom-Up Parsing**

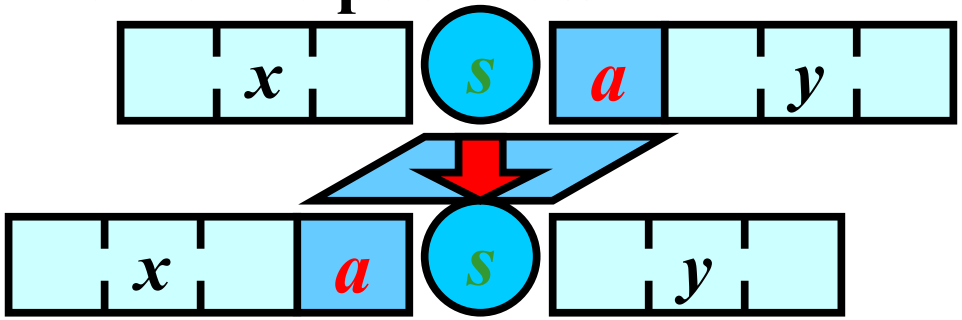



**From the input string towards *S***

# EPDAs as Models of Bottom-Up Parsers 1/2

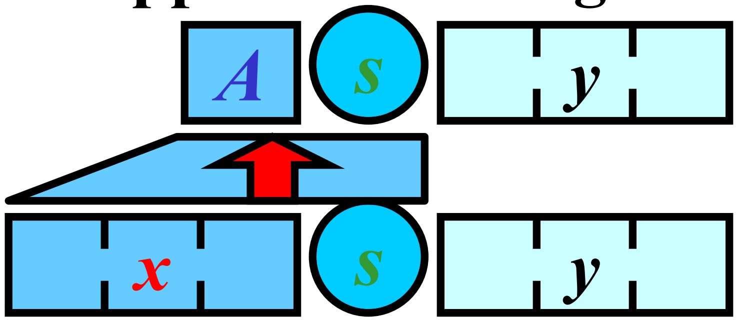
**Gist: An EPDA *M* underlies a bottom-up parser**

1) ***M* contains *shift* rules that copy the input symbols onto the pushdown:**




for every $a \in \Sigma$:
add $sa \rightarrow as$ to $R$;

2) ***M* contains *reduction* rules that simulate the application of a grammatical rule in reverse:**

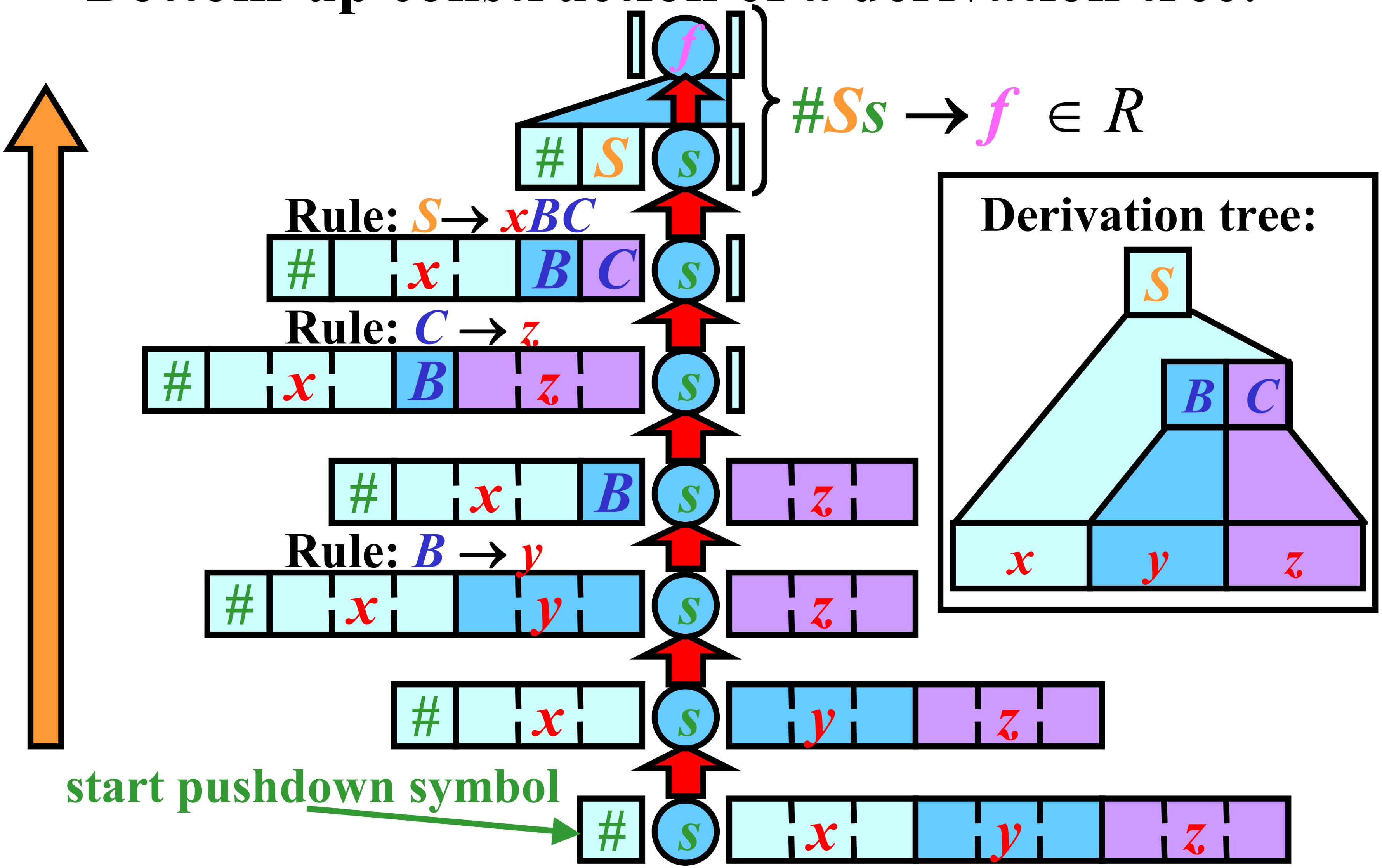



for every $A \rightarrow x \in P$ in $G$:
add $xs \rightarrow As$ to $R$;

3) ***M* also contains the rule $\#Ss \rightarrow f$ that takes *M* to a final state $f$**

# EPDAs as Models of Bottom-Up Parsers 2/2

**Bottom-up construction of a derivation tree:**



# Algorithm: From CFG to EPDA

- **Input:** CFG $G = (N, T, P, S)$
- **Output:** EPDA $M = (Q, \Sigma, \Gamma, R, s, \#, F)$; $L(G) = L(M)_f$
- **Method:**
- $Q := \{s, f\}$;
- $\Sigma := T$;
- $\Gamma := N \cup T \cup \{\#\}$;
- Construction of $R$:
  - for every $a \in \Sigma$, add $sa \rightarrow as$ to $R$;
  - for every $A \rightarrow x \in P$, add $xs \rightarrow As$ to $R$;
  - add $\#Ss \rightarrow f$ to $R$;
- $F := \{f\}$;

# From CFG to EPDA: Example 1/2

- $G = (N, T, P, S)$, where:

$N = \{S\}$, $T = \{(, )\}$, $P = \{S \rightarrow (S), S \rightarrow ()\}$

**Objective:** An EPDA $M$ such that $L(G) = L(M)_f$

---

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$ where:

$Q = \{s, f\}$; $\Sigma = T = \{(, )\}$; $\Gamma = N \cup T \cup \{\#\} = \{S, (, ), \#\}$

**"("** $\in$ ***T***    **")"** $\in$ ***T***    $S \rightarrow (S) \in P$    $S \rightarrow () \in P$

$R = \{s( \rightarrow (s,\ s) \rightarrow )s,\ (S)s \rightarrow Ss,\ ()s \rightarrow Ss,\ \#Ss \rightarrow f\}$

**shift rules**    **reduction rules**

$F = \{f\}$

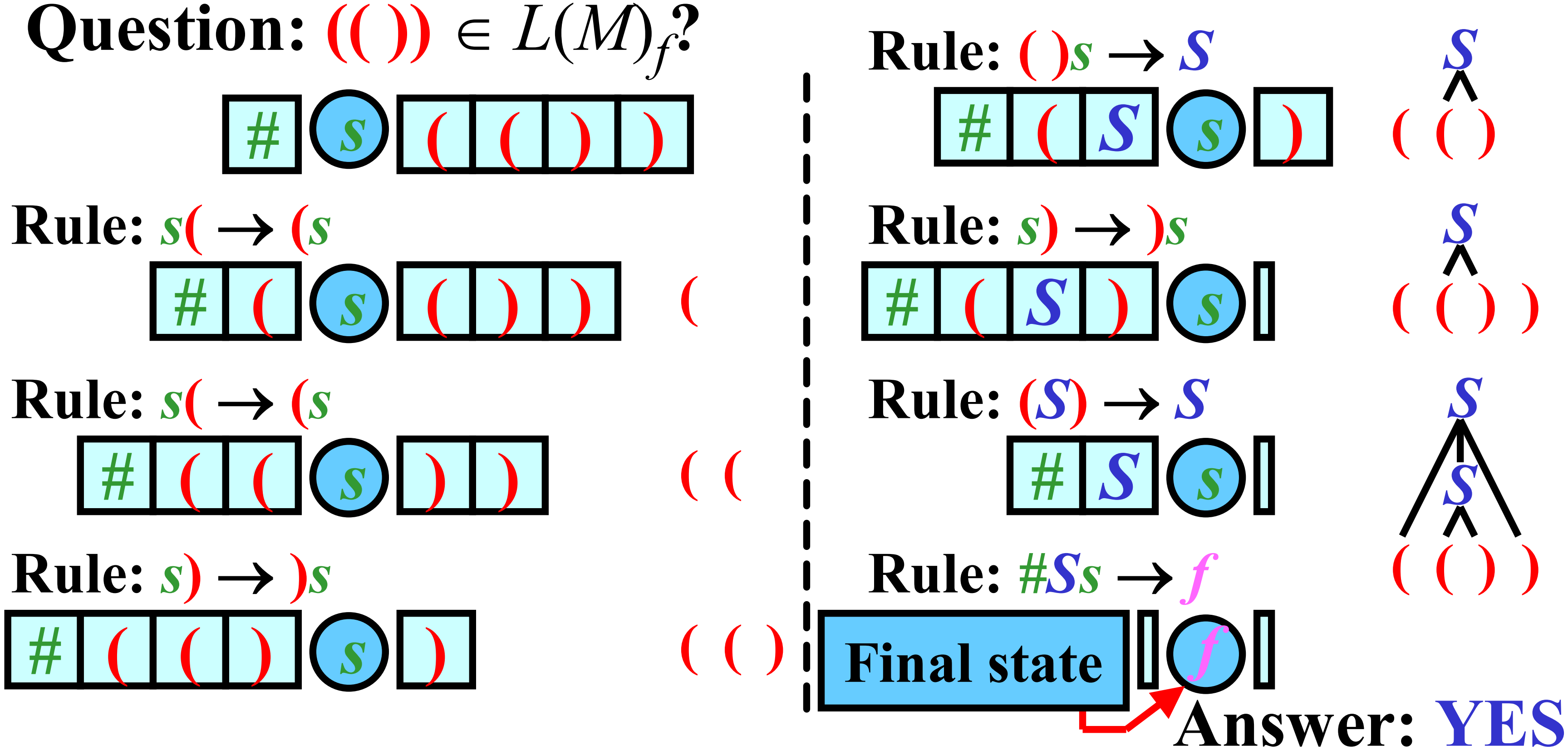
# From CFG to EPDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$, where:

$Q = \{s, f\}$, $\Sigma = T = \{(, )\}$, $\Gamma = \{(, ), S, \#\}$, $F = \{f\}$

$R = \{s( \rightarrow (s,\ s) \rightarrow )s,\ (S)s \rightarrow Ss,\ ()s \rightarrow Ss,\ \#Ss \rightarrow f\}$

**Question:** $(())\in L(M)_f$?

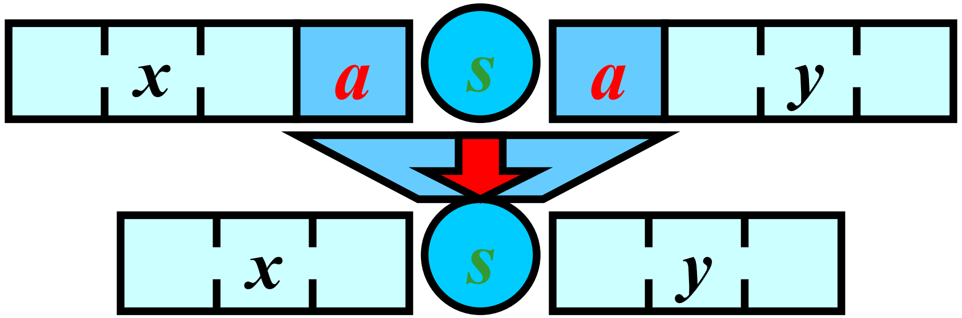


# PDAs as Models of Top-Down Parsers 1/2

**Gist: An PDA *M* underlies a top-down parser**

---

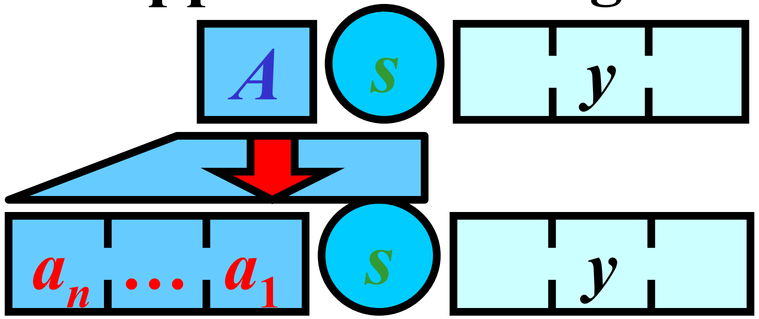
**1) *M* contains *popping* rules that pops the top symbol from the pushdown and reads the input symbol if both coincide:**




for every $a \in \Sigma$:
add $asa \rightarrow s$ to $R$;

---

**2) *M* contains *expansion* rules that simulate the application of a grammatical rule:**

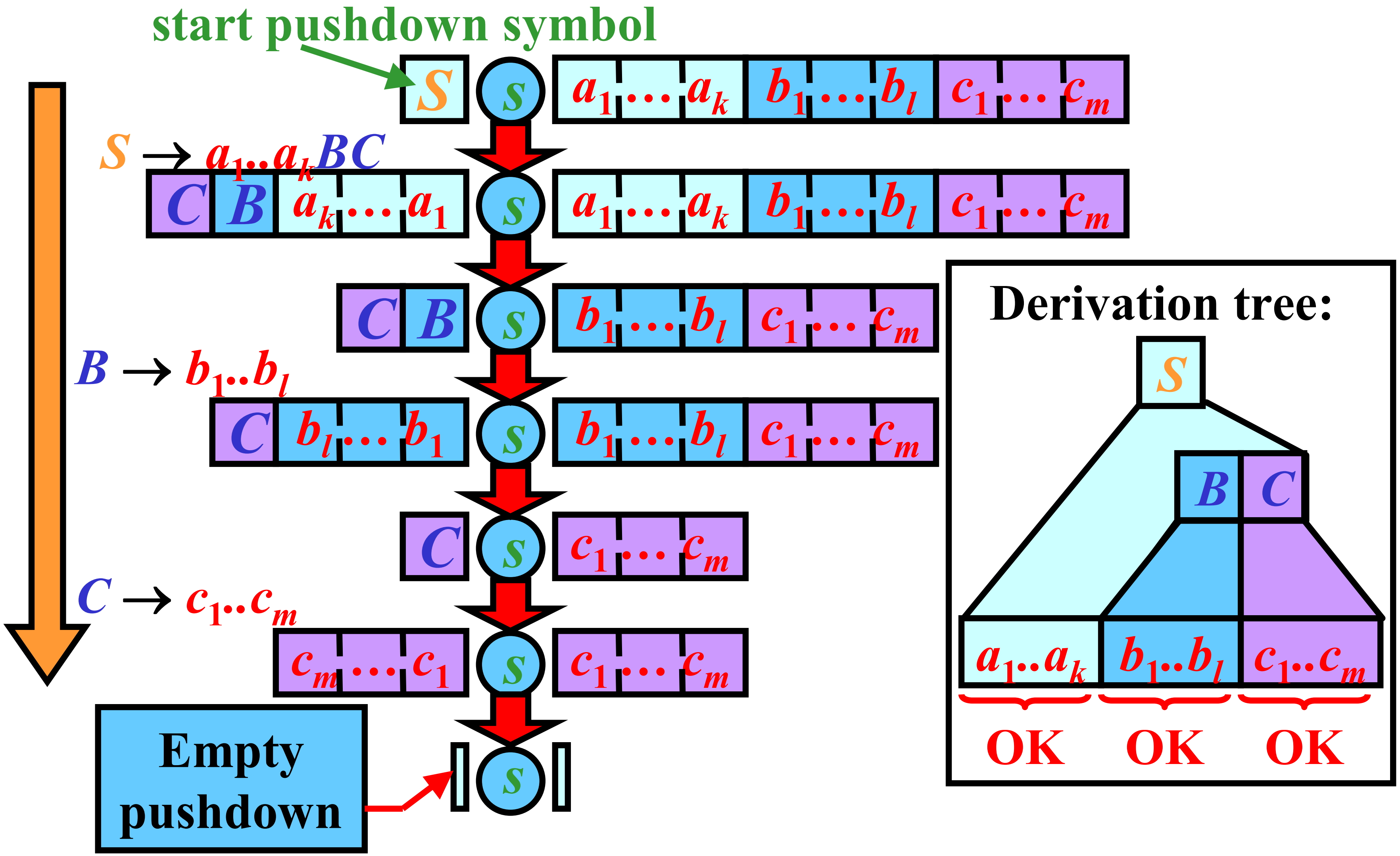



for every $A \rightarrow a_1 \dots a_n \in P$ in $G$,
add $As \rightarrow \underbrace{a_n \dots a_1}s$ to $R$;

**= reversal($a_1 \dots a_n$)**

# PDAs as Models of Top-Down Parsers 2/2

**Top-down construction of a derivation tree:**



# Algorithm: From CFG to PDA

- **Input:** CFG $G = (N, T, P, S)$
- **Output:** PDA $M = (Q, \Sigma, \Gamma, R, s, S, F)$; $L(G) = L(M)_\varepsilon$

---

- **Method:**
- $Q := \{s\}$;
- $\Sigma := T$;
- $\Gamma := N \cup T$;
- Construction of $R$:
  - for every $a \in \Sigma$, add $asa \rightarrow s$ to $R$;
  - for every $A \rightarrow x \in P$, add $As \rightarrow ys$ to $R$, where $y$ = reversal($x$);
- $F := \emptyset$;

# From CFG to PDA: Example 1/2

- $G = (N, T, P, S)$, where:

$N = \{S\}$, $T = \{(, )\}$, $P = \{S \rightarrow (S), S \rightarrow ()\}$

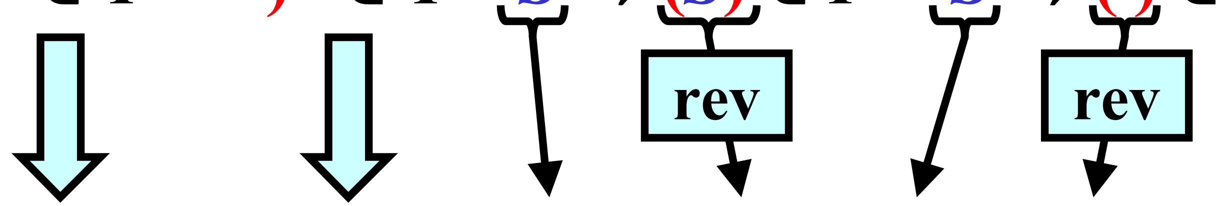
**Objective:** An PDA $M$ such that $L(G) = L(M)_\varepsilon$

---

$M = (Q, \Sigma, \Gamma, R, s, S, F)$ where:

$Q = \{s\}$; $\Sigma = T = \{(, )\}$; $\Gamma = N \cup T = \{S, (, )\}$

“(” $\in T$ “)” $\in T$ $S \rightarrow (S) \in P$ $S \rightarrow () \in P$




$R = \{(s( \rightarrow s,\ )s) \rightarrow s,\ Ss \rightarrow )S(s,\ Ss \rightarrow )(s\}$

**popping rules** **expansion rules**

$F = \emptyset$

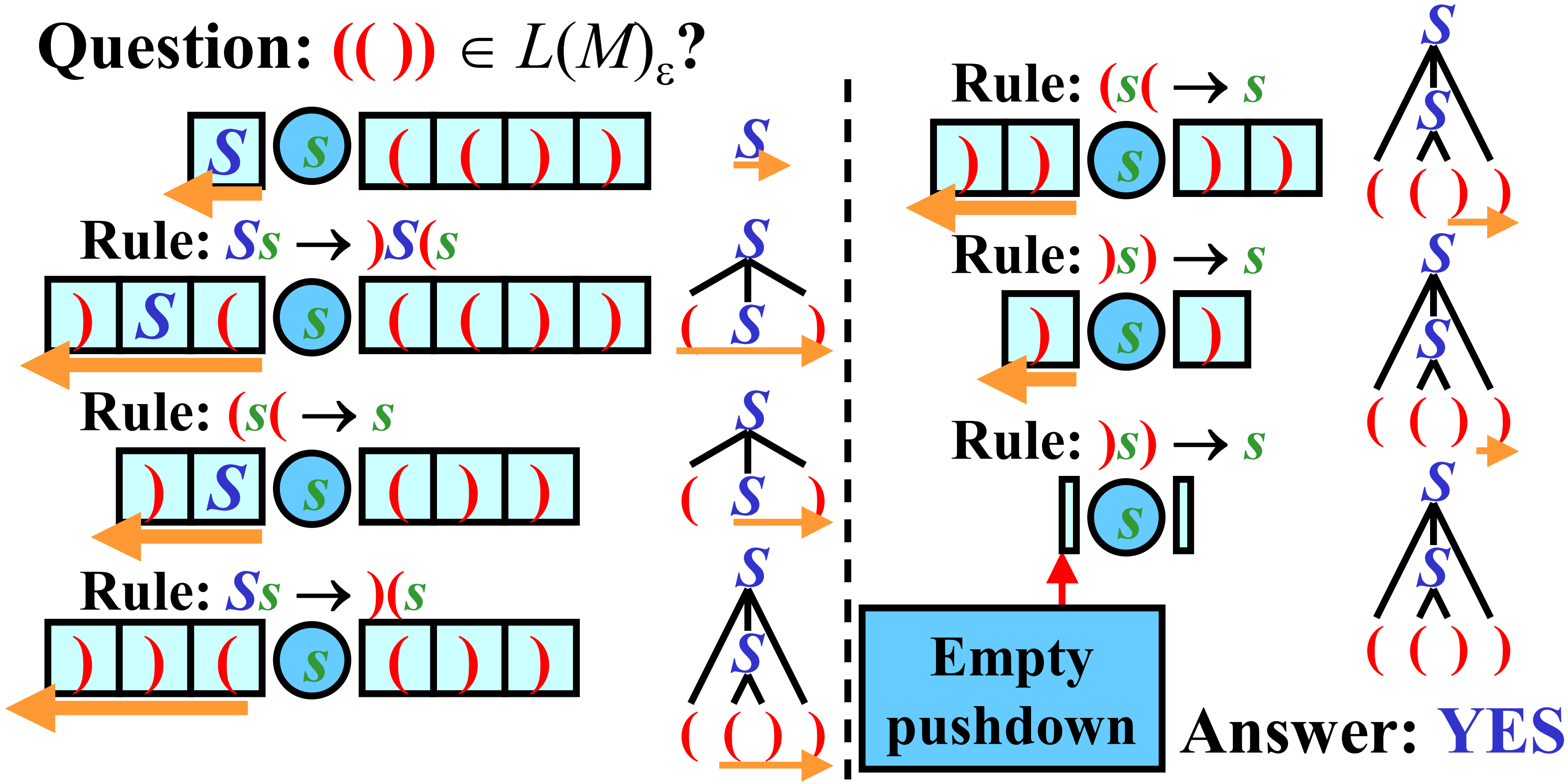
# From CFG to PDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, S, F)$, where:

$Q = \{s\}$, $\Sigma = T = \{(, )\}$, $\Gamma = \{(, ), S\}$, $F = \emptyset$

$P = \{ (s( \rightarrow s, \quad )s) \rightarrow s, \quad Ss \rightarrow )S(s, \quad Ss \rightarrow )(s \}$

**Question:** $(())\in L(M)_\varepsilon$?



# Models for Context-free Languages

**Theorem:** For every CFG $G$, there is an PDA $M$ such that $L(G) = L(M)_\varepsilon$.

**Proof**: See the previous algorithm.

---

**Theorem:** For every PDA $M$, there is a CFG $G$ such that $L(M)_\varepsilon = L(G)$.

**Proof:** See page 486 in [Meduna: Automata and Languages]

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**Conclusion:** The fundamental models for context-free languages are

1) **Context-free grammars** 2) **Pushdown automata**