Regulated Pushdown Automata

Fundamental References

- **Meduna Alexander, Kolář Dušan:**
Regulated Pushdown Automata, ***Acta Cybernetica*, Vol. 2000, No. 4, p. 653-664**

- **Meduna Alexander, Kolář Dušan:**
One-Turn Regulated Pushdown Automata and Their Reduction, *Fundamenta Informatica*, Vol. 2002, No. 16, p. 399-405

Inspiration: Regulated Grammars

- **Grammar *G*:**

  1. $S \rightarrow AC$
  2. $A \rightarrow aAb$
  3. $A \rightarrow ab$
  4. $C \rightarrow Cc$
  5. $C \rightarrow c$

- $\Xi = \{1\}\{24\}^*\{35\}$

Regulated Grammars 1/2

- **Grammar G:**

$1.\ S \to AC$
$2.\ A \to aAb$
$3.\ A \to ab$
$4.\ C \to Cc$
$5.\ C \to c$

$\Xi = \{1\}\{24\}^*\{35\}$

- **Without Ξ, G generates $aabbccc$:**

$$
\begin{aligned}
S &\Rightarrow AC && [1]\\
&\Rightarrow aAbC && [2]\\
&\Rightarrow aAbCc && [4]\\
&\Rightarrow aabbCc && [3]\\
&\Rightarrow aabbCcc && [4]\\
&\Rightarrow aabbccc && [5]
\end{aligned}
$$

$$\boxed{L(G) = \{a^n b^n c^m : n, m \geq 1\}}$$

Regulated Grammars 2/2

- with Ξ, G does not generate ***aabbccc***, because

$$\mathbf{124345} \notin \Xi = \{\mathbf{1}\}\{\mathbf{24}\}^*\{\mathbf{35}\}$$

- with Ξ, G generates ***aabbcc***:

$$\begin{array}{lll} \boldsymbol{S} & \Rightarrow \boldsymbol{AC} & [1] \\ & \Rightarrow \boldsymbol{aAbC} & [2] \\ & \Rightarrow \boldsymbol{aAbCc} & [4] \\ & \Rightarrow \boldsymbol{aabbCc} & [3] \\ & \Rightarrow \boldsymbol{aabbcc} & [5] \end{array}$$

and $\mathbf{12435} \in \Xi$

$$\boxed{\boldsymbol{L(G, \Xi) = \{a^n b^n c^n : n \geq 1\}}}$$

PDA: Notation

- **A PDA is based on a finite set of rules of the form:**

pushdown symbol

states

$$Aqa \rightarrow xp$$

input symbol or ε

pushdown string

New Concept: Regulated PDAs

- **PDA *M*:**

  1. $Ssa \rightarrow Sas$
  2. $asa \rightarrow aas$
  3. $asb \rightarrow q$
  4. $aqb \rightarrow q$
  5. $Sqc \rightarrow Sq$
  6. $Sqc \rightarrow f$

- $\Xi = \{12^m34^n5^n6 : m, n \geq 0\}$

Regulated PDAs 1/2

- **PDA M:**

1. $Ssa \rightarrow Sas$

2. $asa \rightarrow aas$

3. $asb \rightarrow q$

4. $aqb \rightarrow q$

5. $Sqc \rightarrow Sq$

6. $Sqc \rightarrow f$

$\Xi = \{12^m34^n5^n6: m, n \geq 0\}$

- **Without Ξ, M accepts $aabbccc$:**

$$
\begin{aligned}
& Ssaabbccc & \\
\Rightarrow\ & Sasabbccc & [1] \\
\Rightarrow\ & Saasbbccc & [2] \\
\Rightarrow\ & Saqbccc & [3] \\
\Rightarrow\ & Sqccc & [4] \\
\Rightarrow\ & Sqcc & [5] \\
\Rightarrow\ & Sqc & [5] \\
\Rightarrow\ & f & [6]
\end{aligned}
$$

$$L(M) = \{a^nb^nc^m: n, m \geq 1\}$$

Regulated PDAs 2/2

- with Ξ, M does not accept ***aabbcccc*** because

$$\mathbf{1234556} \notin \Xi = \{\mathbf{12}^m\mathbf{34}^n\mathbf{5}^n\mathbf{6}: m, n \geq 0\}$$

- with Ξ, M accepts ***aabbcc***:

$$
\begin{aligned}
Ssaabbcc &\Rightarrow Sasabbcc & [1]\\
&\Rightarrow Saasbbcc & [2]\\
&\Rightarrow Saqbcc & [3]\\
&\Rightarrow Sqcc & [4]\\
&\Rightarrow Sqc & [5]\\
&\Rightarrow f & [6]
\end{aligned}
$$

and $\mathbf{123456} \in \Xi$

$$\boxed{L(M, \Xi) = \{a^n b^n c^n : n \geq 1\}}$$

Gist: Regulated PDAs

- Consider a pushdown automaton, M, and control language, Ξ.
- M accepts a string, x, if and only if Ξ contains a control string according to which M makes a sequence of moves so it reaches a final configuration after reading x.

Definition: Regulated PDA 1/4

A pushdown automaton is a 7-tuple

$$M = (Q, \Sigma, \Omega, R, s, S, F), \text{ where}$$

- Q is a *finite set of states*,
- Σ is an *input alphabet*,
- Ω is a *pushdown alphabet*,
- R is a *finite set of rules* of the form:

$$Apa \rightarrow wq, \text{ where}$$

$$A \in \Omega, p,q \in Q, a \in \Sigma \cup \{\varepsilon\}, w \in \Omega^*$$

- $s \in Q$ is the *start state*
- $S \in \Omega$ is the *start symbol*
- $F \subseteq Q$ is a set of *final states*

Definition: Regulated PDA 2/4

- Let Ψ be an alphabet of *rule labels*. Let every rule $Apa \rightarrow wq$ be labeled with a unique $\rho \in \Psi$ as

$$\rho.\ Apa \rightarrow wq.$$

- A configuration of M, χ, is any string from $\Omega^* Q \Sigma^*$

- For every $x \in \Omega^*$, $y \in \Sigma^*$, and $\rho.\ Apa \rightarrow wq \in R$, M makes a move from configuration $xApay$ to configuration $xwqy$ according to ρ, written as

$$xApay \Rightarrow xwqy\ [\rho]$$

Definition: Regulated PDA 3/4

- Let χ be any configuration of *M*. *M* makes *zero moves* from χ to χ *according to* ε, written as

$$\chi \Rightarrow^0 \chi\ [\varepsilon]$$

- Let there exist a sequence of configurations $\chi_0, \chi_1, ..., \chi_n$ for some $n \geq 1$ such that $\chi_{i-1} \Rightarrow \chi_i\ [\rho_i]$, where $\rho_i \in \Psi$, for $i = 1,...,n$, then *M* makes *n moves from* χ_0 *to* χ_n *according to* $[\rho_1 \ldots \rho_n]$, written as

$$\chi_0 \Rightarrow^n \chi_n\ [\rho_1 ... \rho_n]$$

Definition: Regulated PDA 3/4

- If for some $n \geq 0$, $\chi_0 \Rightarrow^n \chi_n\ [\rho_1 \ldots \rho_n]$, we write

$$\chi_0 \Rightarrow^* \chi_n\ [\rho_1 \ldots \rho_n]$$

- Let Ξ be a *control language* over Ψ, that is, $\Xi \subseteq \Psi^*$. With Ξ, M accepts its language, $L(M, \Xi)$, as

$$L(M, \Xi) = \{w : w \in \Sigma^*, Ssw \Rightarrow^* f\ [\sigma], \sigma \in \Xi\}$$

Language Families

- ***LIN*** - the family of linear languages
- ***CF*** - the family of context-free languages
- ***RE*** - the family of recursively enumerable languages

- ***RPD(REG)*** - the family of languages accepted by PDAs regulated by regular languages
- ***RPD(LIN)*** - the family of languages accepted by PDAs regulated by linear languages

Theorem 1 and its Proof 1/2

$\boldsymbol{RPD(REG) = CF}$

Proof:

I. $\boldsymbol{CF \subseteq RPD(REG)}$ is clear.

II. $\boldsymbol{RPD(REG) \subseteq CF}$**:**

- Let $L = L(M, \Xi)$,

PDA **Regular language**

- Let $\Xi = L(G)$, G - regular grammar based on rules: $\boldsymbol{A \rightarrow aB}$, $\boldsymbol{A \rightarrow a}$

Theorem 1 and its Proof 2/2

Transform *M* regulated by Ξ to a *PDA N* as follows:

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1) for every $a.Cqb \rightarrow xp$ from M and every $A \rightarrow aB$ from G, add $C\langle qA\rangle b \rightarrow x\langle pB\rangle$ to N

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2) for every $a.Cqb \rightarrow xp$ from M and every $A \rightarrow a$ from G, add $C\langle qA\rangle b \rightarrow x\langle pf\rangle$ to N

New symbol

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3) The set of final states in N:

$\{\langle pf\rangle: p$ is a final state in $M\}$

Theorem 2

$$\boldsymbol{RPD(LIN) = RE}$$

Proof:

- See [**Meduna Alexander, Kolář Dušan: Regulated Pushdown Automata**, ***Acta Cybernetica***, **Vol. 2000, No. 4, p. 653-664**]

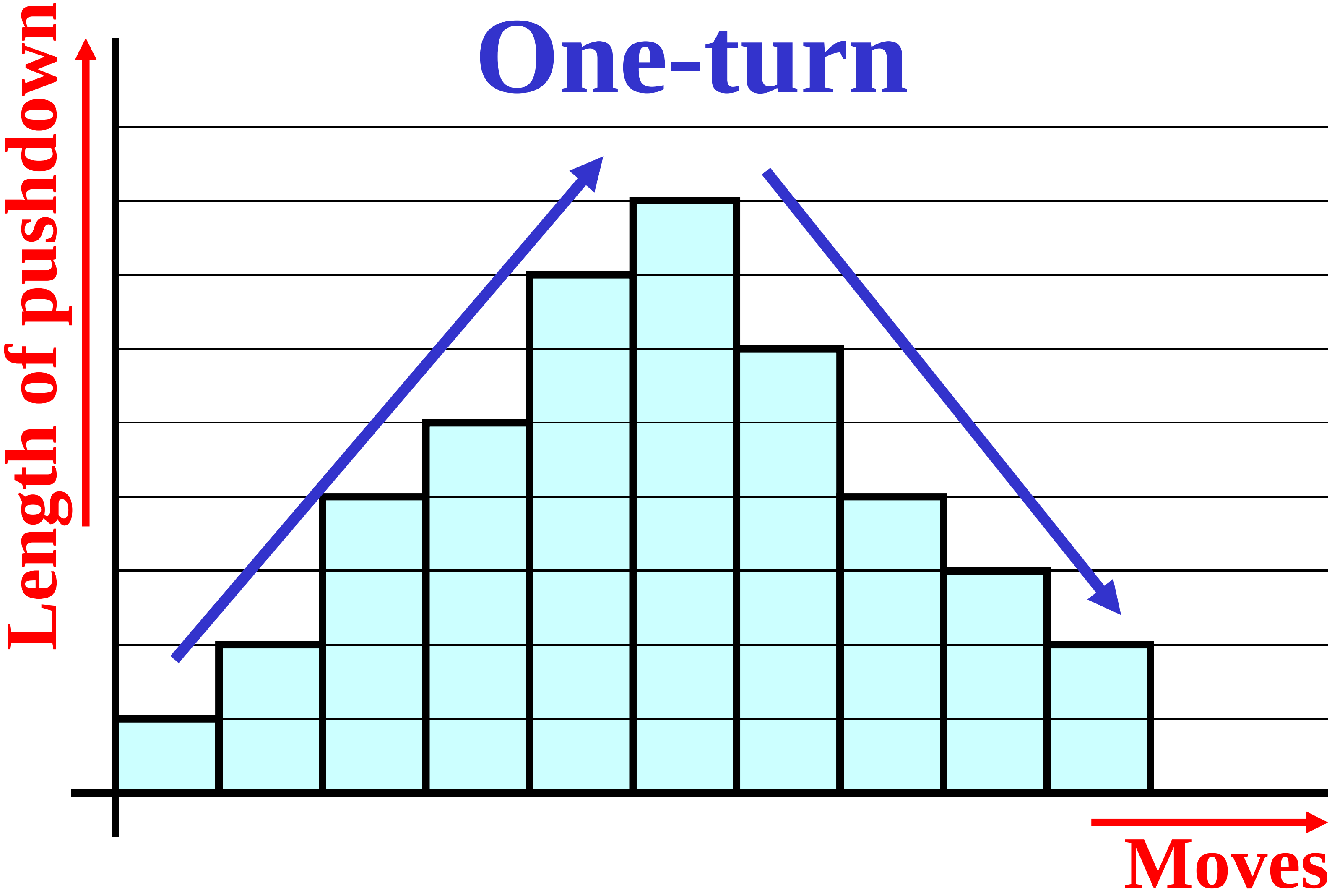
Simplification of RPDAs 1/2

I. consider two consecutive moves made by a pushdown automaton, *M*.

If during the first move *M* does not shorten its pushdown and during the second move it does, then *M* makes ***a turn*** during the second move.

- A pushdown automaton is *one-turn* if it makes no more than one turn during any computation starting from an initial configuration.

One-Turn PDA: Illustration



Simplification of RPDAs 2/2

II. During a move, an *atomic* regulated PDA changes a state and, in addition, performs exactly one of the following actions:

1. pushes a symbol onto the pushdown
2. pops a symbol from the pushdown
3. reads an input symbol

Theorem 3

• Every $L \in$ *RE* is accepted by an atomic one-turn PDA regulated by Ξ, where $\Xi \in$ *LIN*.

Proof:

• See [**Meduna Alexander, Kolář Dušan: One-Turn Regulated Pushdown Automata and Their Reduction, *Fundamenta Informatica*,Vol. 2002, No. 16, p. 399-405**]