

# Context-Free Grammars

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# Context-Free Grammar

## Context-Free Grammar

$$G = (N, T, P, S)$$

$N$  alphabet of nonterminals

$T$  alphabet of terminals

$P$  finite set of productions of the form

$$A \rightarrow x$$

with  $A \in N$  and  $x \in (N \cup T)^*$

$S$  the start symbol,  $S \in N$

# Proper Context-Free Grammar

## Useful Symbol

A symbol  $X \in N \cup T$  is useful if

**1**  $S \Rightarrow^* uXv$

**2**  $X \Rightarrow^* y$

for some  $u, v \in (N \cup T)^*$  and  $y \in T^*$

## Proper Context-Free Grammar

A context-free grammar  $G = (N, T, P, S)$  is proper if

**1**  $N \cup T$  contains only useful symbols

**2**  $G$  is  $\varepsilon$ -free

**3**  $G$  is unit-free

# Properties of Proper Context-Free Grammars

## Theorem

*For every context-free language  $L$ , there is a proper context-free grammar  $G$  such that*

$$L - \{\varepsilon\} = L(G)$$

## Claim

If  $G = (N, T, P, S)$  is proper, then for every  $A \in N$

$$S \Rightarrow^* uAy \Rightarrow^* uwy$$

with  $u, w, y \in T^*$

# Weak Pumping Lemma

## Weak Pumping Lemma

Let  $L$  be an infinite context-free language. Then,  $L$  contains a string  $z = uvwxy$  such that

**1**  $uv^iwx^iy \in L$  for every  $i \geq 0$

**2**  $|vx| \geq 1$

# Weak Pumping Lemma – Proof

Let  $G$  be a proper context-free grammar such that  $L = L(G)$

**1** By contradiction: assume that no derivation in  $G$  contains two identical nonterminals. Then,  $L(G)$  is finite – a contradiction.

**2** Thus, there is

$$S \Rightarrow^* u' Ay' \Rightarrow^+ u' v' Ax' y' \Rightarrow^* u' v' wx' y'$$

in  $G$ , where  $u', v', x', y' \in (N \cup T)^*$ ,  $A \in N$ ,  $w \in T^*$ ,  $|v'x'| \geq 1$ . As  $G$  is proper,

$$u' \Rightarrow^* u, v' \Rightarrow^* v, x' \Rightarrow^* x, \text{ and } y' \Rightarrow^* y$$

for some  $u, v, x, y \in T^*$ ,  $|vx| \geq 1$ . Therefore,

$$S \Rightarrow^* uAy \Rightarrow^+ uvAxy \Rightarrow^* uvwxy.$$

Thus,  $uv^iwx^iy \in L$  for every  $i \geq 0$ .

□

# Weak Pumping Lemma – Example

## Example

Consider  $L = \{a^n b^n c^n : n \geq 0\}$ . By weak pumping lemma,  $L$  contains  $z = uvwxy$  such that  $|vx| \geq 1$  and  $uv^i wx^i y \in L$  for every  $i \geq 0$ .

**1** Let  $v$  or  $x$  be in

$$\{a\}^+ \{b\}^+ \cup \{b\}^+ \{c\}^+ \cup \{a\}^+ \{b\}^+ \{c\}^+.$$

Then,  $uvvwxxy \notin L$  – contradiction.

**2** Let  $v$  or  $x$  be in

$$\{a\}^+ \cup \{b\}^+ \cup \{c\}^+.$$

Then,  $uwy \notin L$  – contradiction. □

# Pumping Lemma

## Pumping Lemma

Let  $L$  be a context-free language. Then, there is  $k \geq 1$  such that for every  $z \in L$  with  $|z| \geq k$ ,

$$z = uvwxy$$

so that

- 1**  $vx \neq \varepsilon$
- 2**  $|vwx| \leq k$
- 3**  $uv^mwx^my \in L$  for all  $m \geq 0$ .



# Pumping Lemma – Example

## Example

Consider  $L = \{a^{n^2} : n \geq 1\}$ . Set  $z = a^{k^2}$ , where  $k$  is the pumping lemma constant. As  $k^2 \geq k$ ,  $|z| \geq k$ . Express  $z$  as

$$z = uvwxy.$$

By pumping lemma,  $uv^2wx^2y \in L$ . Observe that  $|vx| \leq k$ , so

$$\begin{aligned} k^2 = |uvwxy| &< |uv^2wx^2y| = |uvwxy| + |vx| \leq \\ &k^2 + k < k^2 + 2k + 1 = (k+1)^2. \end{aligned}$$

As  $k^2 < |uv^2wx^2y| < (k+1)^2$ ,  $uv^2wx^2y \notin L$  – contradiction.  $L$  is not a context-free language.  $\square$

# Homework Assignment

- 1** Establish a pumping lemma for regular languages (based on regular grammars). Use this lemma to prove that some context-free languages are not regular.
- 2** By using this lemma, demonstrate that a computer program that decides whether a positive integer  $n$  is prime cannot be based on any finite automaton.

# Bibliography



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