

# Scattered Context Grammars

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# Scattered Context Grammar

## Scattered Context Grammar

$$G = (V, T, P, S)$$

$V$  is a finite alphabet

$T$  is a set of terminals,  $T \subset V$

$S$  is the start symbol,  $S \in V - T$

$P$  is a finite set of productions of the form

$$(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n),$$

where  $A_1, \dots, A_n \in V - T$ ,  $x_1, \dots, x_n \in V^*$

## Propagating Scattered Context Grammar

- each  $(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \in V^+$

# Derivation Step

## Derivation Step

For  $(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n) \in P$  and

$$u = u_1 A_1 \dots u_n A_n u_{n+1}$$

$$v = u_1 x_1 \dots u_n x_n u_{n+1}$$

we write  $u \Rightarrow v [(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)]$

## Generated Language

$$L(G) = \{x \in T^* : S \Rightarrow^* x\}$$

## Generative Power

- $\mathcal{L}(SC) = \mathcal{L}(RE)$
- $\mathcal{L}(CF) \subset \mathcal{L}(PSC) \subseteq \mathcal{L}(CS)$

## Example

Propagating scattered context grammar

$$G = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P, S)$$

with

$$P = \{(S) \rightarrow (ABC), \\ (A, B, C) \rightarrow (aA, bB, cC), \\ (A, B, C) \rightarrow (a, b, c)\}$$

Example of derivation

$$S \Rightarrow ABC \Rightarrow aAbBcC \Rightarrow aaAbbBccC \Rightarrow aaabbbccc$$

Generated language

$$L(G) = \{a^n b^n c^n : n \geq 1\}$$

# Example II

## Example

Propagating scattered context grammar

$$G = (\{S, W, X, Y, Z, A, a\}, \{a\}, P, S),$$

where

$$P = \{1 : (S) \rightarrow (a), \\ 2 : (S) \rightarrow (aa), \\ 3 : (S) \rightarrow (WAXY), \\ 4 : (W, A, X, Y) \rightarrow (a, W, X, AAY), \\ 5 : (W, X, Y) \rightarrow (a, W, AXY), \\ 6 : (W, X, Y) \rightarrow (Z, Z, a), \\ 7 : (Z, A, Z) \rightarrow (Z, a, Z), \\ 8 : (Z, Z) \rightarrow (a, a)\}$$

$$L = \{a^{2^n} : n \geq 0\}$$

$$\begin{aligned} S &\Rightarrow WAXY && [3] \\ &\Rightarrow aWX A^2 Y && [4] \\ &\Rightarrow a^2 WAA^2 XY && [5] \\ &\Rightarrow a^3 WAA X A^2 Y && [4] \\ &\Rightarrow a^4 WAX A^4 Y && [4] \\ &\Rightarrow a^5 WX A^6 Y && [4] \\ &\Rightarrow a^6 WA^7 XY && [5] \\ &\Rightarrow a^6 ZA^7 Z a && [6] \\ &\Rightarrow^7 a^6 Z a^7 Z a && [7^7] \\ &\Rightarrow a^{16} && [8] \end{aligned}$$

## Production length

$$\blacksquare \text{len}((A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)) = |A_1 \dots A_n| = n$$

## Definitions

nonterminal complexity is the number of nonterminals in  $G$

degree of context-sensitivity  $\text{dcs}(G)$  is the number of context-sensitive productions in  $G$

maximum context sensitivity  $\text{mcs}(G)$  is the greatest number in

$$\{\text{len}(p_i) - 1 : 1 \leq i \leq |P|\}$$

overall context sensitivity  $\text{ocs}(G)$  is the sum of all members in

$$\{\text{len}(p_i) - 1 : 1 \leq i \leq |P|\}$$

# Reduction – Results I

## Lemma

There exists a scattered context grammar  $G$  such that  $G$  defines a *non-context-free* language and  $\text{dcs}(G) = \text{mcs}(G) = \text{ocs}(G) = 1$ .

## Proof

Consider a scattered context grammar

$$G = (\{S, A, B, C, D\}, \{a, b, c\}, P, S)$$

with

$$P = \{(S) \rightarrow (AC), \\ (A) \rightarrow (aAbB), \\ (A) \rightarrow (\varepsilon), \\ (C) \rightarrow (cCD), \\ (C) \rightarrow (\varepsilon), \\ (B, D) \rightarrow (\varepsilon, \varepsilon)\}$$

$$L(G) = \{a^n b^n c^n : n \geq 0\}$$

$$\text{dcs}(G) = \text{mcs}(G) = \text{ocs}(G) = 1$$

## Reduction – Results II

### Theorem

There are *context-sensitive* languages which *cannot* be described by a scattered context grammar  $G = (V, T, P, S)$  satisfying  $|V - T| = 1$ .

### Theorem

Every recursively enumerable language is generated by a scattered context grammar  $G = (V, T, P, S)$  satisfying

$$|V - T| = 2, \text{ dcs}(G) = \infty, \text{ mcs}(G) = \infty, \text{ ocs}(G) = \infty.$$

### Theorem

Every recursively enumerable language is generated by a scattered context grammar  $G = (V, T, P, S)$  satisfying

$$|V - T| = 5, \text{ dcs}(G) = 2, \text{ mcs}(G) = 3, \text{ ocs}(G) = 6.$$



### Theorem

*Every recursively enumerable language is generated by a scattered context grammar  $G = (V, T, P, S)$  satisfying*

$$|V - T| = 8, \text{ dcs}(G) = 6, \text{ mcs}(G) = 1, \text{ ocs}(G) = 6.$$

### Theorem

*Every recursively enumerable language is generated by a scattered context grammar  $G = (V, T, P, S)$  satisfying*

$$|V - T| = 4, \text{ dcs}(G) = 4, \text{ mcs}(G) = 5, \text{ ocs}(G) = 20.$$

## Context-Free and Context-Sensitive Productions

For a scattered context production  $p$ , if  $\text{len}(p)$

$= 1$  then the production is **context-free**

$\geq 2$  then the production is **context-sensitive**

## Theorem

Let  $H = (M, T, R, S)$  be a phrase-structure grammar in Kuroda normal form. Then, there exists a scattered context grammar,  $G = (V, T, P, E)$ , that satisfies

- 1  $L(G) = L(H)$ ,
- 2  $|V| = |M| + 5$ ,
- 3  $P$  contains 4 new context-sensitive productions,
- 4  $P$  contains 1 new context-free production.

# Leftmost Derivations

## Leftmost Derivation Step

For  $(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n) \in P$  and

$$u = u_1 A_1 \dots u_n A_n u_{n+1}$$

$$v = u_1 x_1 \dots u_n x_n u_{n+1},$$

where  $A_i \notin \text{alph}(u_i)$  for all  $1 \leq i \leq n$ , we write

$$u \xrightarrow{\text{lm}} v [(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)]$$

## Theorem

*Every context-sensitive language can be generated by a propagating scattered context grammar which uses only leftmost derivations.*

## Extended Propagating Scattered Context Grammar

An **extended propagating scattered context grammar** is a scattered context grammar

$$G = (V, T, P, S)$$

in which every

$$(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n) \in P$$

satisfies  $|x_1 \dots x_n| \geq n$

## Theorem

*Every context-sensitive language can be generated by an extended propagating scattered context grammar.*

# Unordered Scattered Context Grammar

## Unordered Scattered Context Grammar

- scattered context grammar in which the order of context-free productions in a scattered context production is unimportant
- for  $(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n) \in P$ , a permutation  $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ , and

$$u = u_1 A_{\pi(1)} \dots u_n A_{\pi(n)} u_{n+1}$$

$$v = u_1 x_{\pi(1)} \dots u_n x_{\pi(n)} u_{n+1}$$

we write  $u \Rightarrow v [(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)]$

## Generative Power

- $\mathcal{L}(USC) = \mathcal{L}(P, \varepsilon)$
- $\mathcal{L}(PUSC) = \mathcal{L}(P) \subset \mathcal{L}(PSC)$

## Open Problem

Are propagating scattered context grammars powerful enough to characterize all context-sensitive languages?

## Open Problem

Can every recursively enumerable language be described by a scattered context grammar containing only a single context-sensitive production?

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