Symbolic Dynamical Systems

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19. 10. 2007

Subshifts

Bi-infinite words $A^{\mathbb{Z}} = \{x; x = \ldots x_{-1}x_0x_1 \ldots, \forall i : x_i \in A\}$

Distance For $x, y \in A^{\mathbb{Z}}$:

$$d(x, y) = 2^{-e(x,y)}$$

$$e(x, y) = max\{n \geq 0; \forall i, -n \geq i \geq n : x_i = y_i\}$$

Shift For $x \in A^{\mathbb{Z}}$: $\sigma(x) = y; \forall n : y_n = x_{n+1}$

Subshift $S \subseteq A^{\mathbb{Z}}$, which is

- Topologically closed with respect to d
- Shift-invariant: $\sigma(S) = S$

Subshift Examples

Full shift $S = A^{\mathbb{Z}}$

Golden mean $S = \{x \in \{\mathrm{a}, \mathrm{b}\}^{\mathbb{Z}}, x \text{ does not contain ``bb''}\}$

Edge shift For $G = (V, E)$: $S_G \subseteq E^{\mathbb{Z}}$: bi-infinite paths in G

Sofic shift For $A = (Q, E)$: bi-infinite words "recognized" by a finite automaton (Q, E, q_0, q_F)

Subshift Factors

Subshift factors $F_S = \{x_i \dots x_j; x \in S, i \leq j\} \cup \{\epsilon\}$

Properties:

- $\forall S : F = F_S$ is both:

  factorial $uvw \in F \implies v \in F$

  extendable $\forall v \in F \exists a, b \in A : avb \in F$

- Let $F \subseteq A^*$ be factorial and extendable. Then

  $$S = \{x \in A^{\mathbb{Z}}; \forall i \leq j : x_i \dots x_j \in F\}$$

  is a subshift, and $F_S = F$.

- S is a sofic shift iff F_S is recognizable by a finite automaton.

Recurrence

One-sided shift $S(x) = \{y \in A^{\mathbb{N}}; F_y \subseteq F_x\}$

Recurrence x is recurrent if any substring of x has infinite occurrences in x.

Theorem: x is recurrent iff $S(x)$ is *irreducible*, i. e. $\forall x, y \in F_S \exists u : xuy \in F_S$.

Uniform recurrence x is uniformly recurrent if $\forall n > 0 \exists m > n$: any substring of x with length n appears in any substring of x with length m.

Theorem: Let $x \in A^{\mathbb{N}}$. Then the following conditions are equivalent:

- x is uniformly recurrent.
- $S(x)$ is minimal, i. e. $\forall T \subseteq S : T = \emptyset \vee T = S$.

Subshifts and Automata

Local automaton is a finite automaton for which the current state is determined by a bounded number of past and future labels.

Transitive automaton is a finite automaton with a strongly-connected state graph.

Properties:

- A *transitive* finite automaton is local iff there is at most one infinite path with any given label.
- A *deterministic* finite automaton is local iff its current state is determined by a bounded number of past labels.

Shift of Finite Type A subshift made of all infinite words that avoid a finite set of subwords.

Theorem: A subshift is of finite type iff it is recognized by a local automaton.

Subshifts and Automata

- Any sofic system can be recognized by a DFA.
- A sofic system is irreducible iff it is recognized by a transitive DFA.

Synchronizing word $x \in F_S$ is synchronizing if

$$\forall u, v : ux, xv \in F_S \implies uxv \in F_S.$$

Word rank For $x \in A^*$, rank of x is $|\{q.x | q \in Q\}|$

- In a non-empty irreducible sofic system, all words of minimal non-zero rank are synchronizing.

Theorem: Any irreducible sofic system has a unique minimal automaton, constructed to recognize $\{y; xy \in F_S\}$ for some synchronizing word x.

Morphisms

Morphism $f : S \to T$, f is continuous and $f\sigma = \sigma f$

k-Local $f : S \to T$ is k-local iff $\exists \bar{f} : A^k \to B, m \in \mathbb{Z}$:
$\forall x, y = f(x)$: $y_{n+m} = \bar{f}(x_{n-(k-1)} \dots x_{n-1}x_n)$

Theorem: $f : S \to T$ is a morphism iff $\exists k \geq 0$: f is k-local

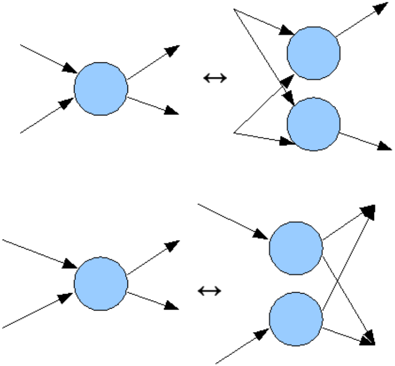
Isomorphism f: a bijective morphism

Properties:

- Any subshift isomorphic to a shift of finite type is of finite type.
- Any subshift isomorphic to a sofic shift is a sofic shift.

Morphisms

Theorem: Any isomorphism between shifts of finite type can be obtained as a composition of splits and merges performed on the automaton.



Thank you

Questions?