Deep Pushdown Automata

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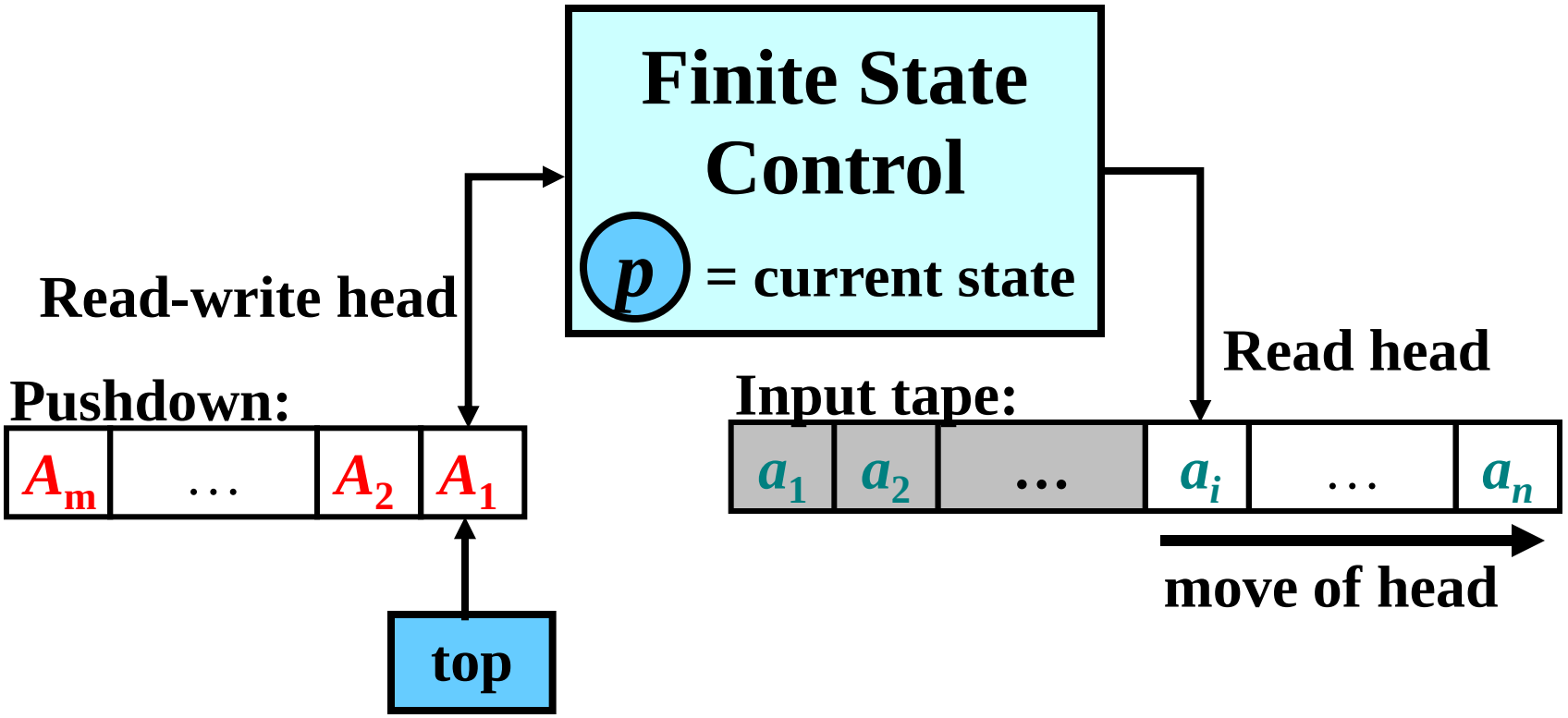
Based on

Meduna, A.: Deep Pushdown Automata,

Acta Informatica, 2006

Pushdown Automaton (PDA)



Inspiration

- conversion of CFG to PDA that acts as a general top-down parser
  - if pd top = **input** symbol, **pop**
  - if pd top = **non-input** symbol, **expand**

PDA as a general Top-Down Parser

- Configuration:

$$(state, input, pd)$$

- **Pop:**

$$(q, ax, a\alpha) \underset{p}{\Rightarrow} (q, x, \alpha)$$

- **Expansion:**

$$(q, x, A\alpha) \underset{e}{\Rightarrow} (p, x, \beta\alpha)$$

by rule $qA \rightarrow p\beta$

Final state

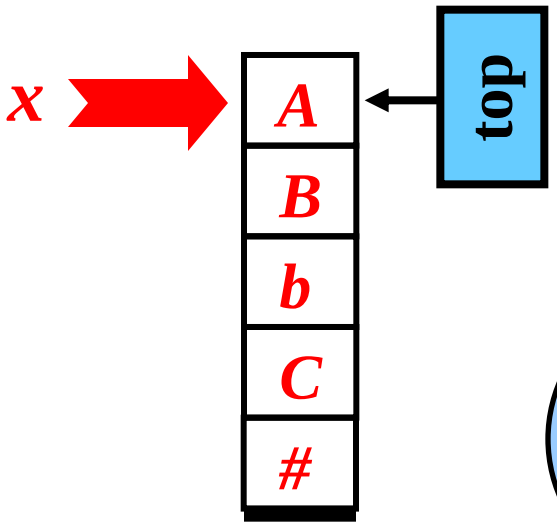
- Acceptance: $(s, x, S) \Rightarrow^{*} (f, \varepsilon, \varepsilon)$

Deep Pushdown Automata: Fundamental Modification

- expansion may be performed **deeper in pd**

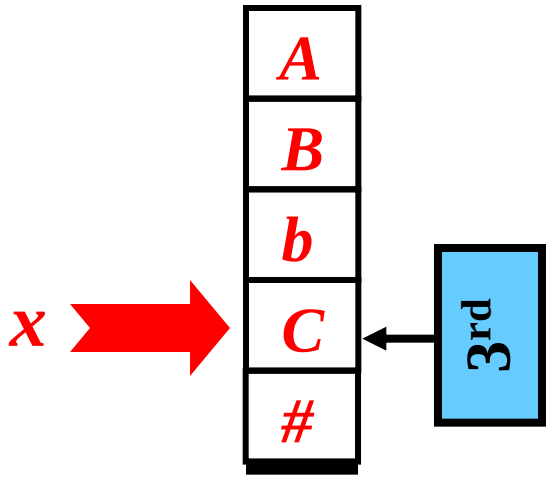
Standard Expansion

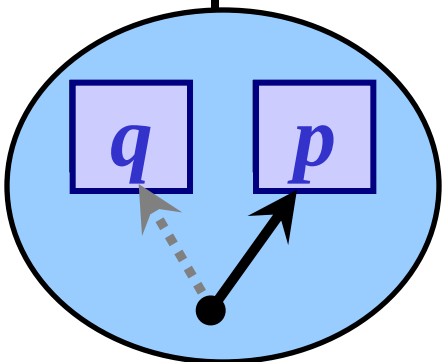
$qA \rightarrow px$



Deep Expansion

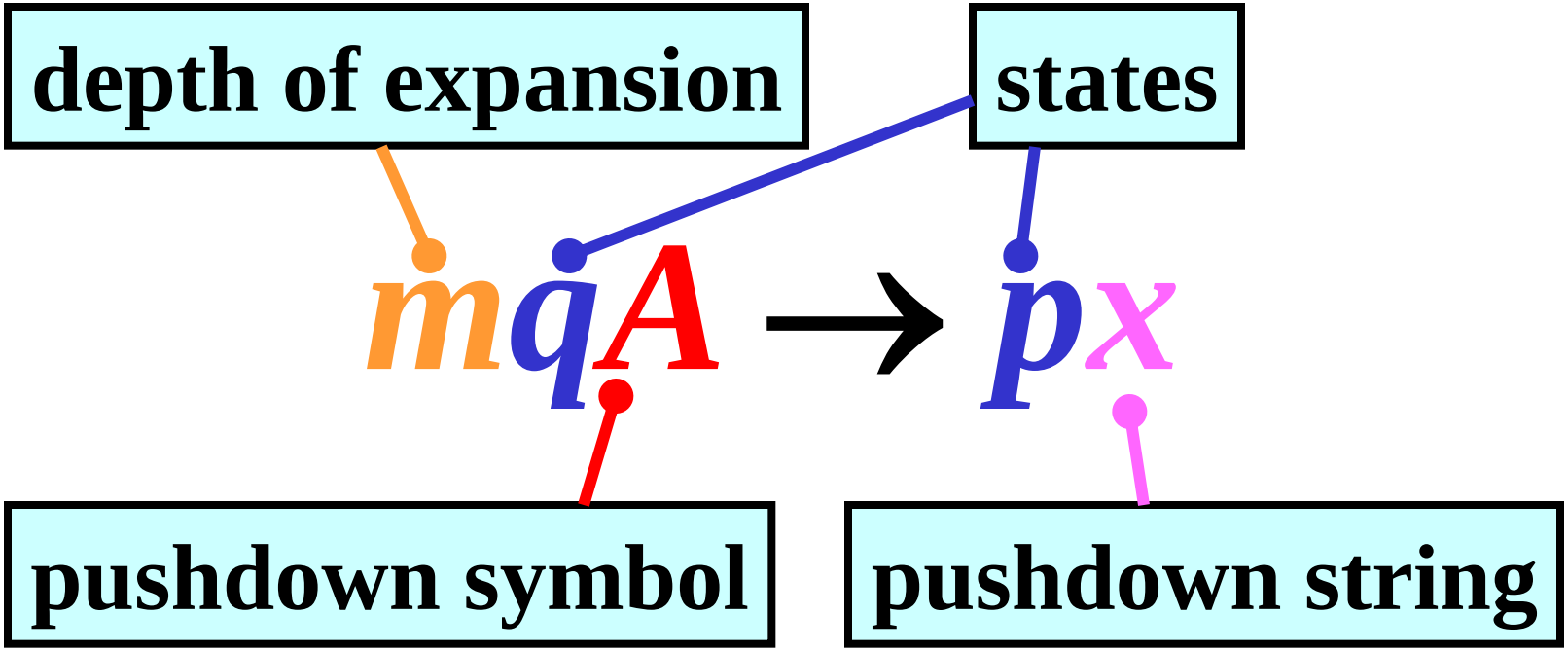
$3qC \rightarrow px$





Deep Pushdown Automata

- Same as Top-Down Parser except *deep expansions*
- *Expansion of depth* m:
  - the mth topmost non-input pd symbol is replaced with a string by rule



Expansion of Depth *m*

- ***Expansion of depth m***:

$$(q, w, uAz) \;{}_{e}\!\Rightarrow (p, w, uvz)$$

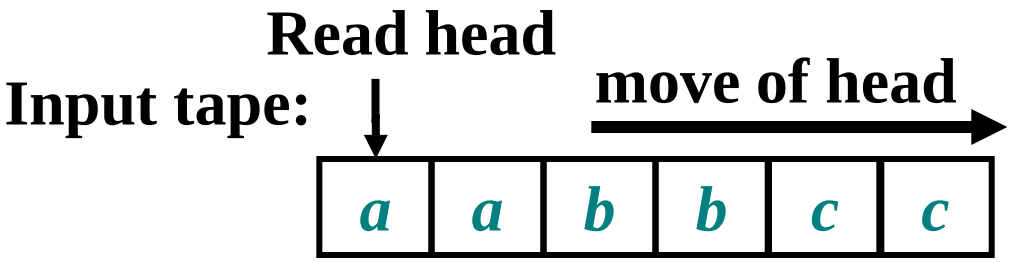
by ***rule of depth m***

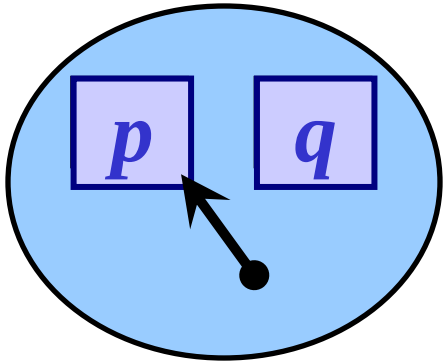
$$[mqA \rightarrow pv],$$

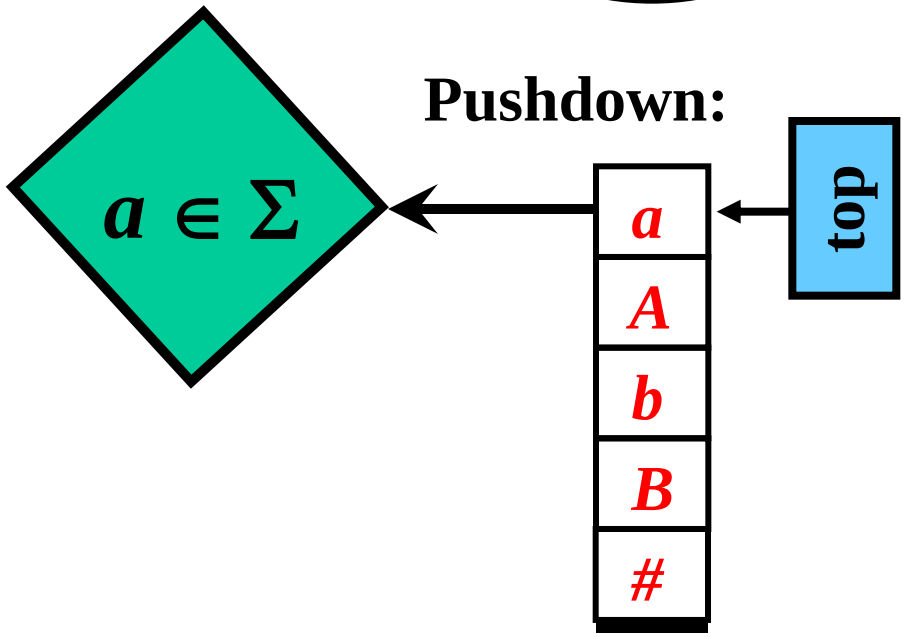
where u contains $m - 1$ non-input symbols

Pop: Illustration

Move: $(p, aabbcc, aAbB\#) \underset{p}{\Rightarrow} (p, abbcc, AbB\#)$

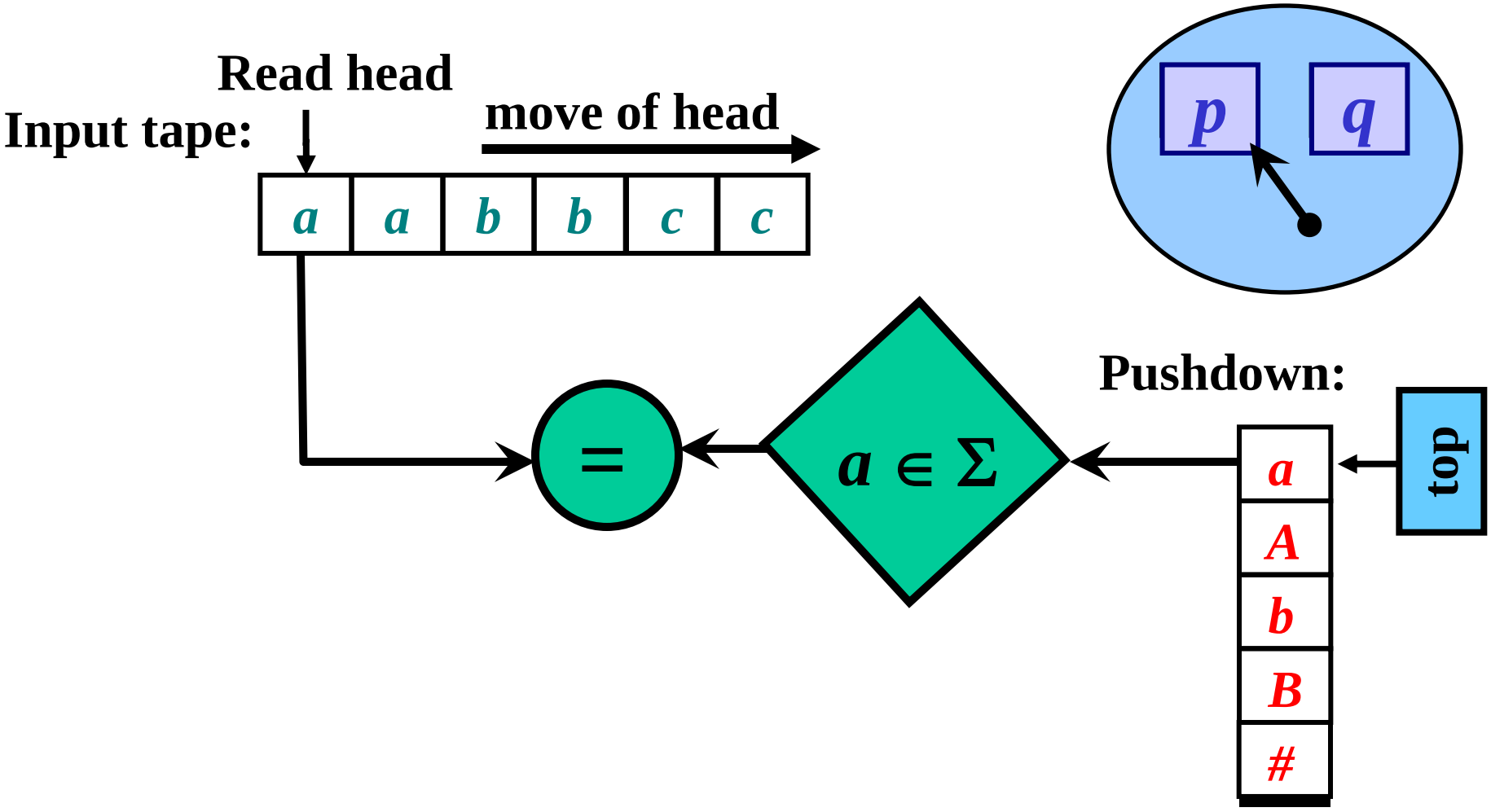






Pop: Illustration

Move: $(p, aabbcc, aAbB\#) \underset{p}{\Rightarrow} (p, abbcc, AbB\#)$



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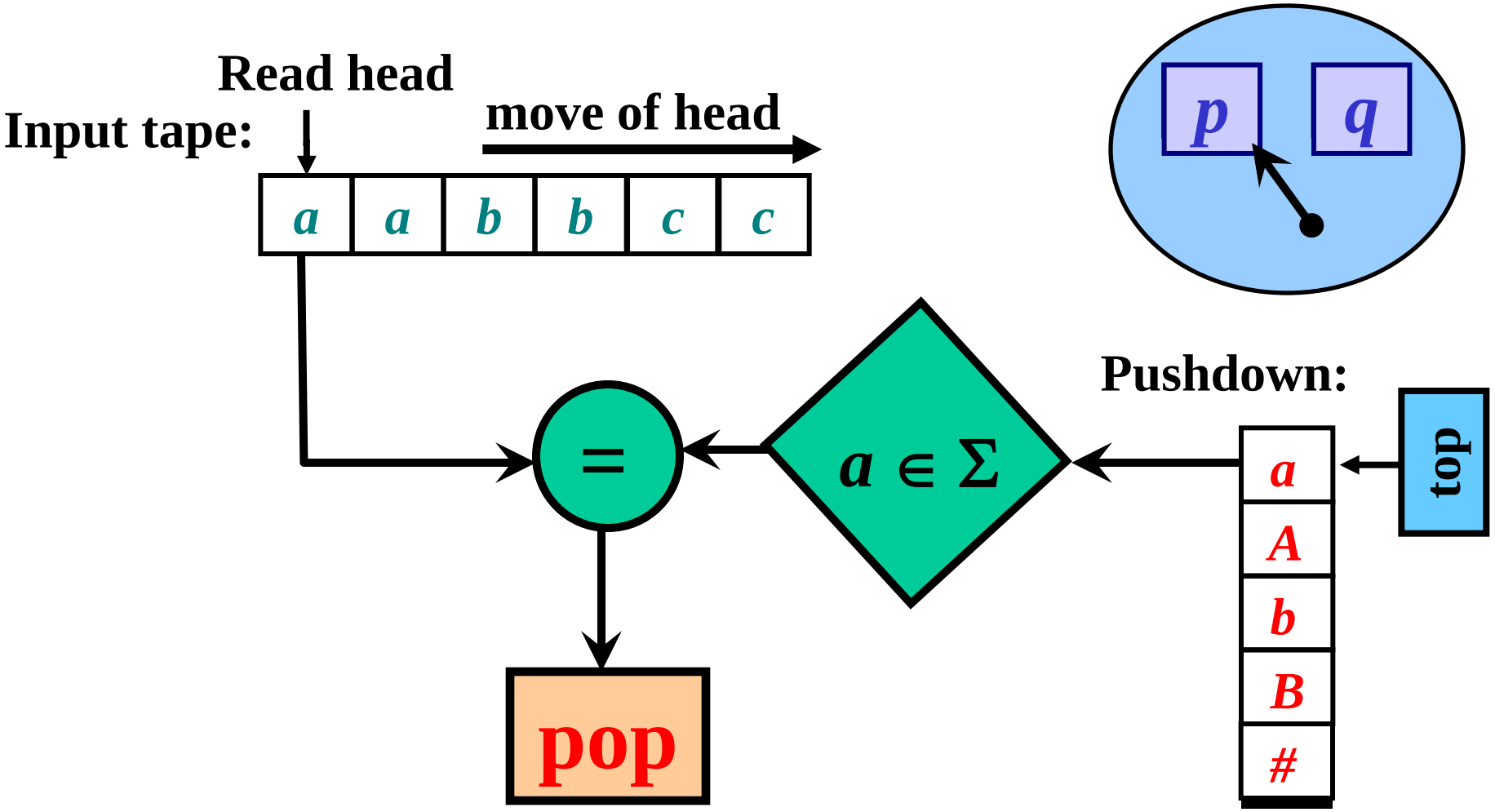

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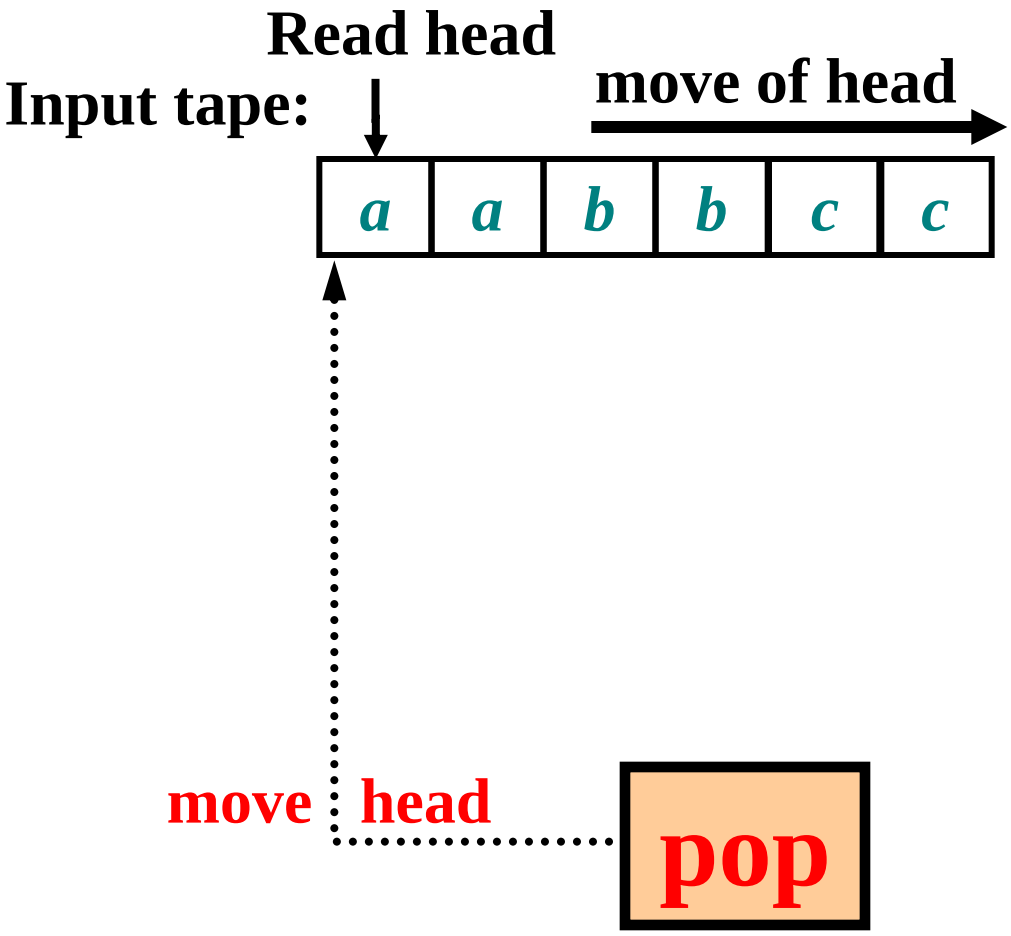
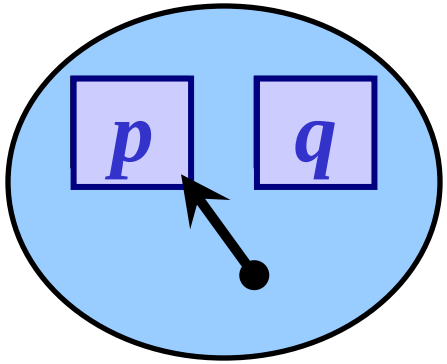
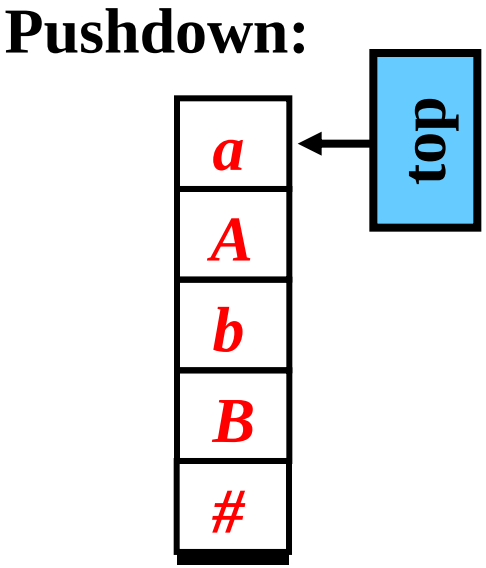
Move: $(p, aabbcc, aAbB\#) \underset{p}{\Rightarrow} (p, abbcc, AbB\#)$

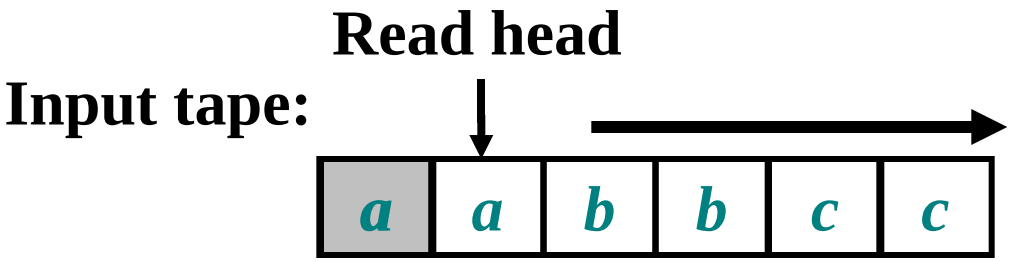


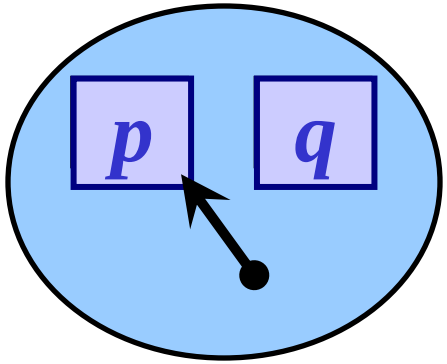


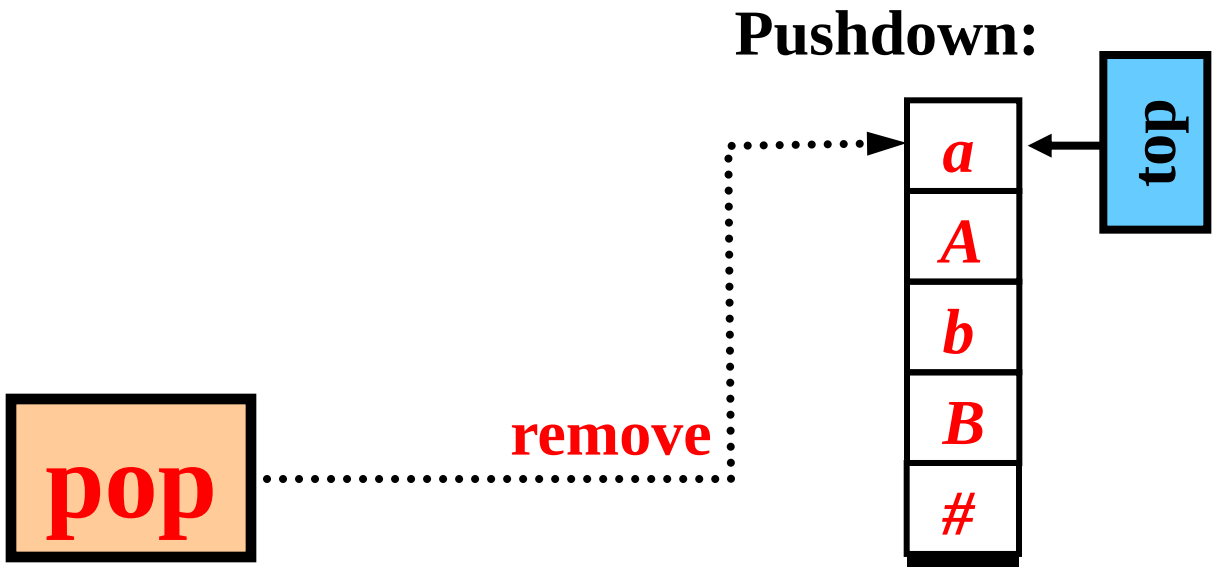


Pop: Illustration

Move: $(p, aabbcc, aAbB\#) \underset{p}{\Rightarrow} (p, abbcc, AbB\#)$





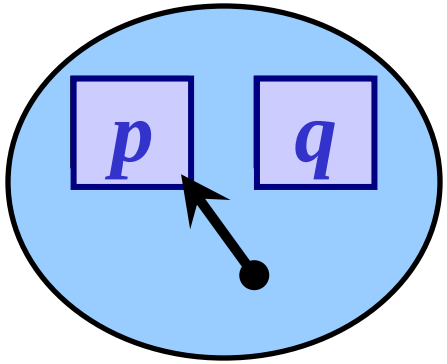


Pop: Illustration

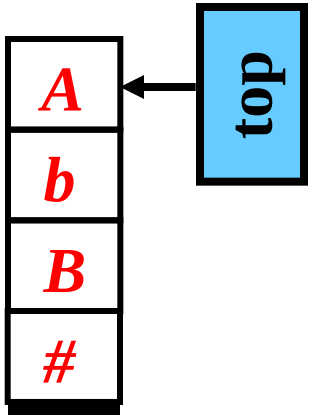
Move: $(p, aabbcc, aAbB\#) \underset{p}{\Rightarrow} (p, abbcc, AbB\#)$

Input tape:

Read head

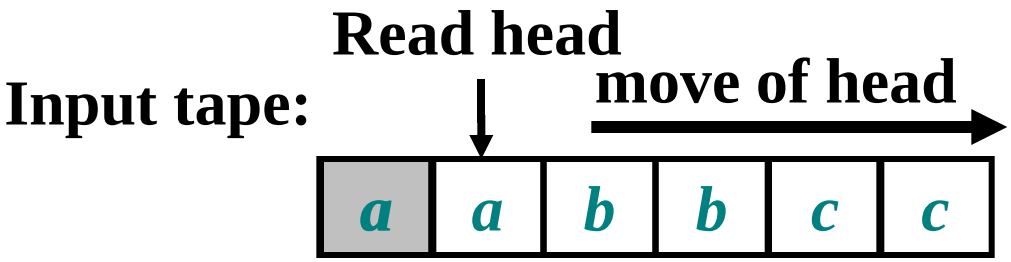


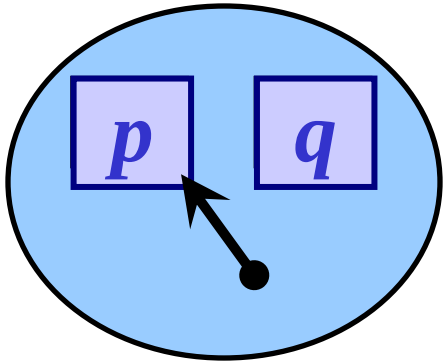
Pushdown:

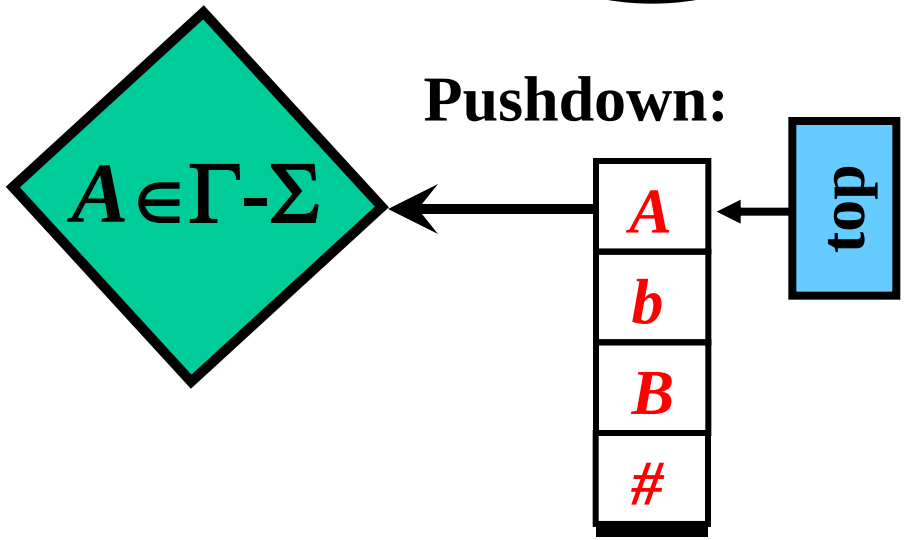


Deep Expansion: Illustration

Move: $(p, abbcc, AbB\#) \Rightarrow_e (q, abbcc, AbBc\#)$ $[2pB \rightarrow qBc]$



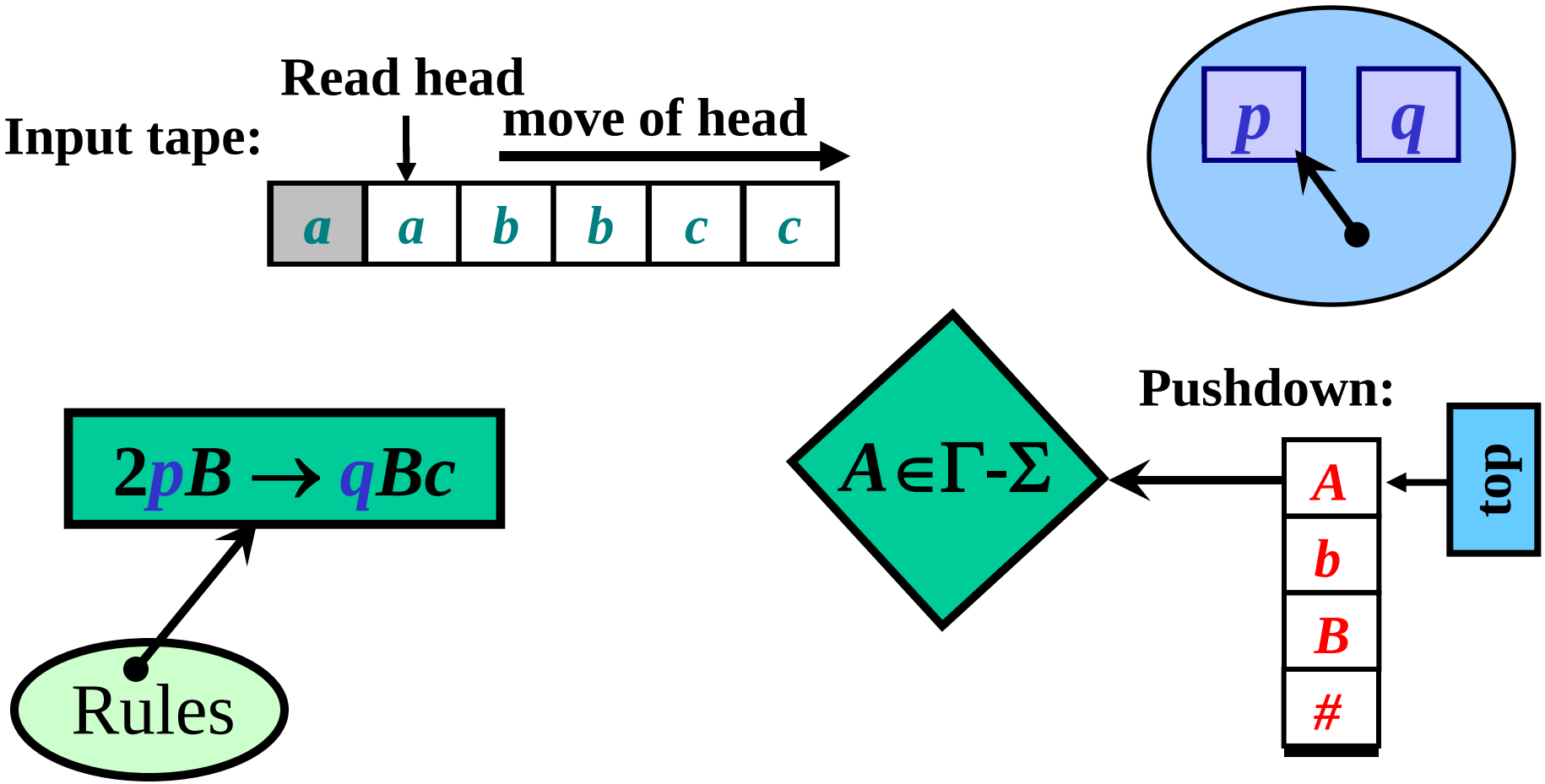




Rules

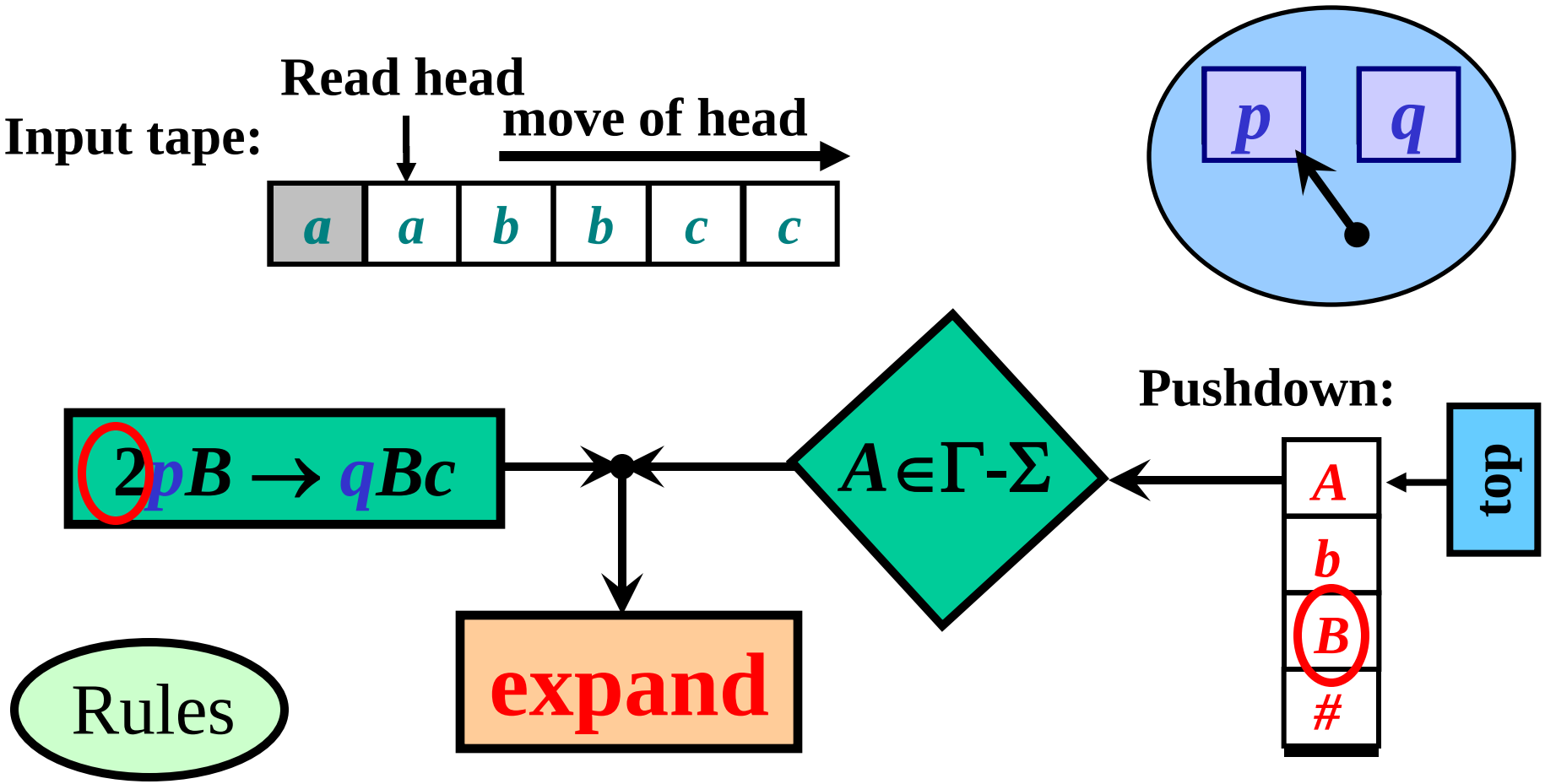
Deep Expansion: Illustration

Move: $(p, abbcc, AbB\#) \underset{e}{\Rightarrow} (q, abbcc, AbBc\#)$ $[2pB \rightarrow qBc]$



Deep Expansion: Illustration

Move: $(p, abbcc, AbB\#) \Rightarrow_{e} (q, abbcc, AbBc\#)$ $[2pB \rightarrow qBc]$



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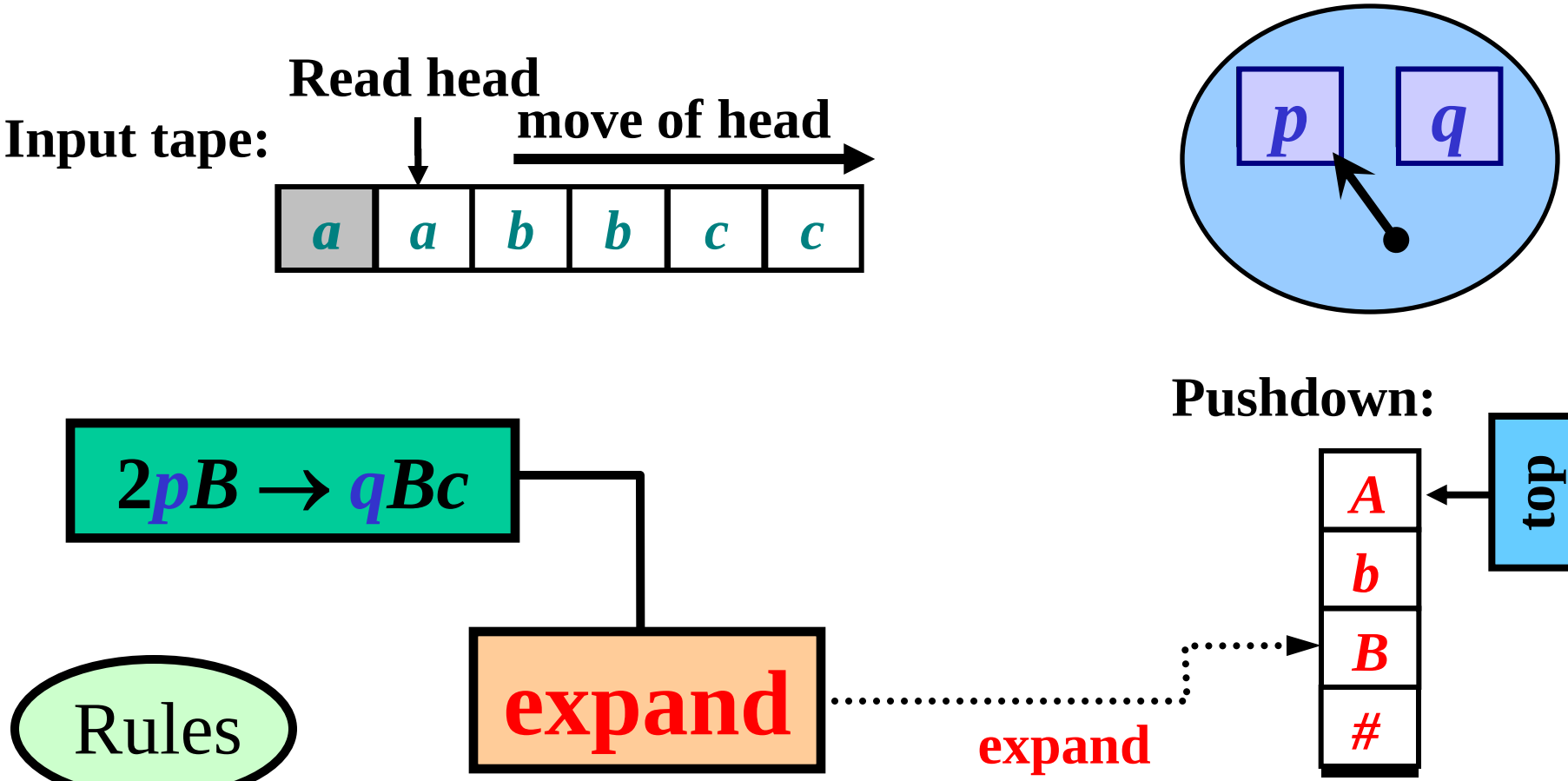

Deep Expansion: Illustration

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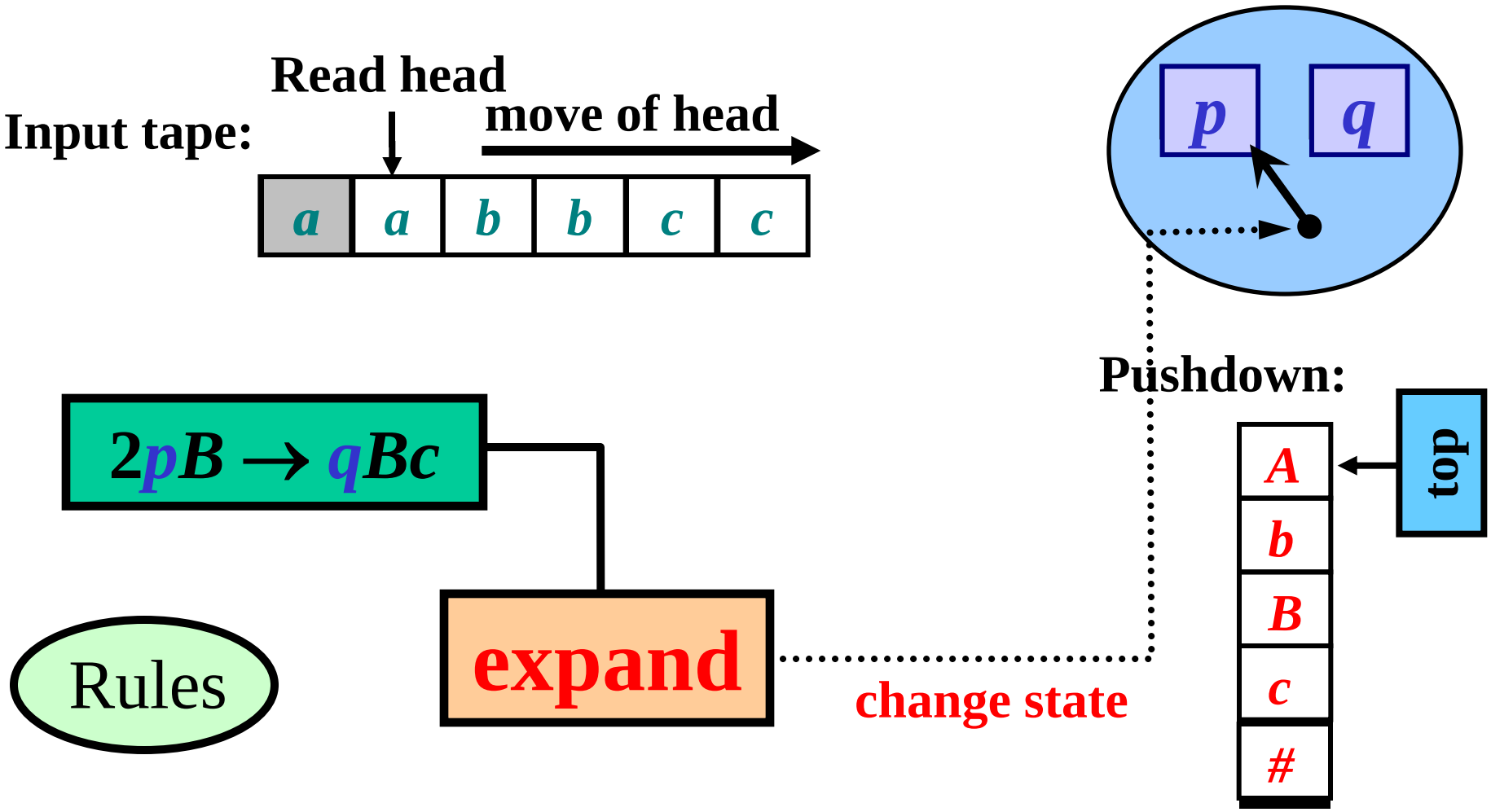

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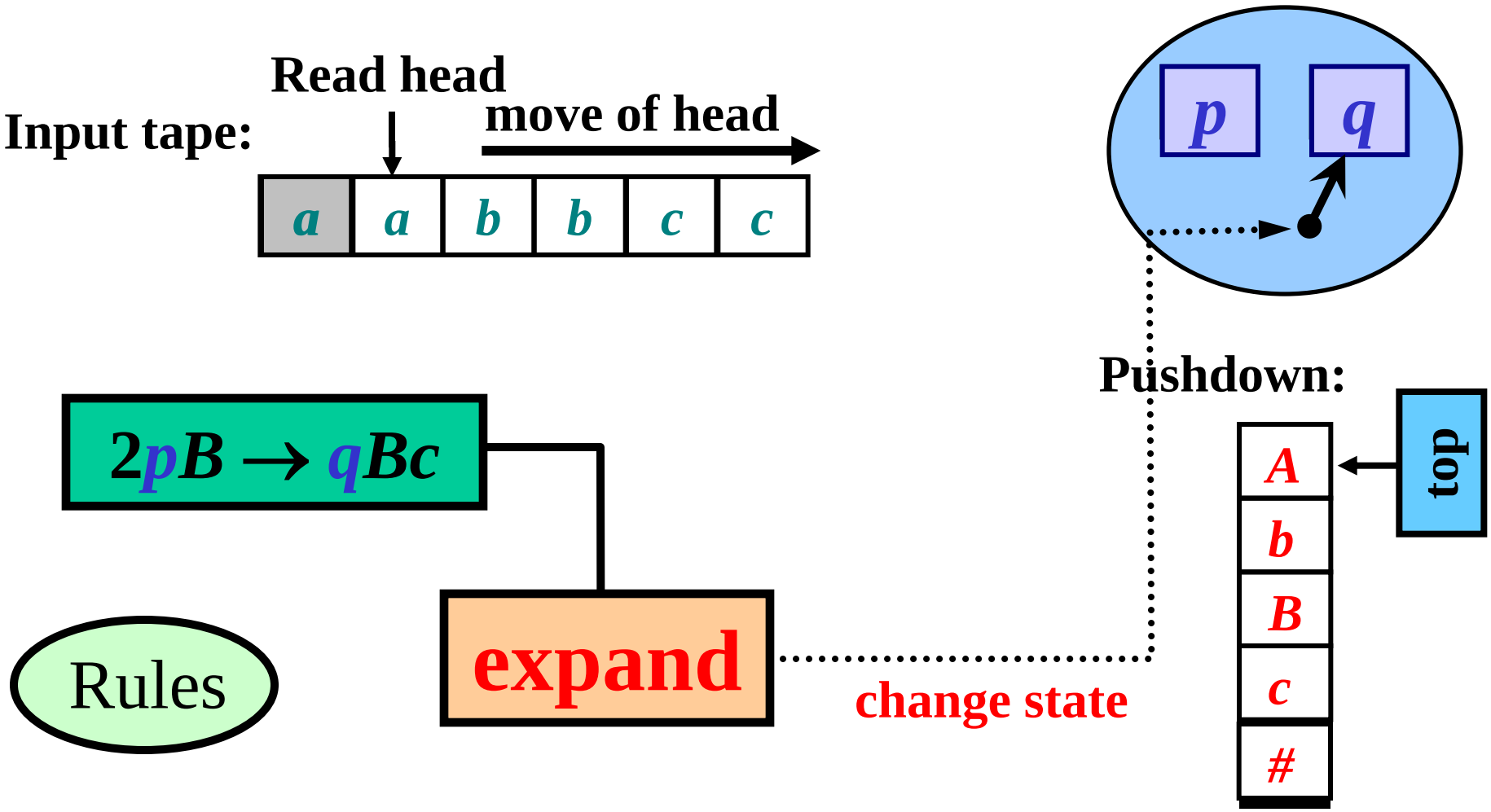
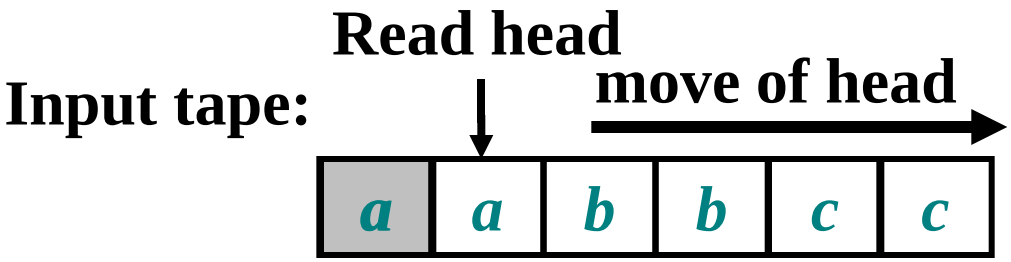
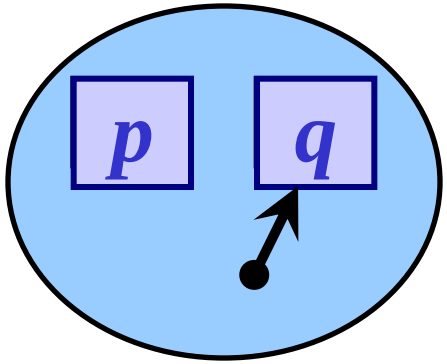
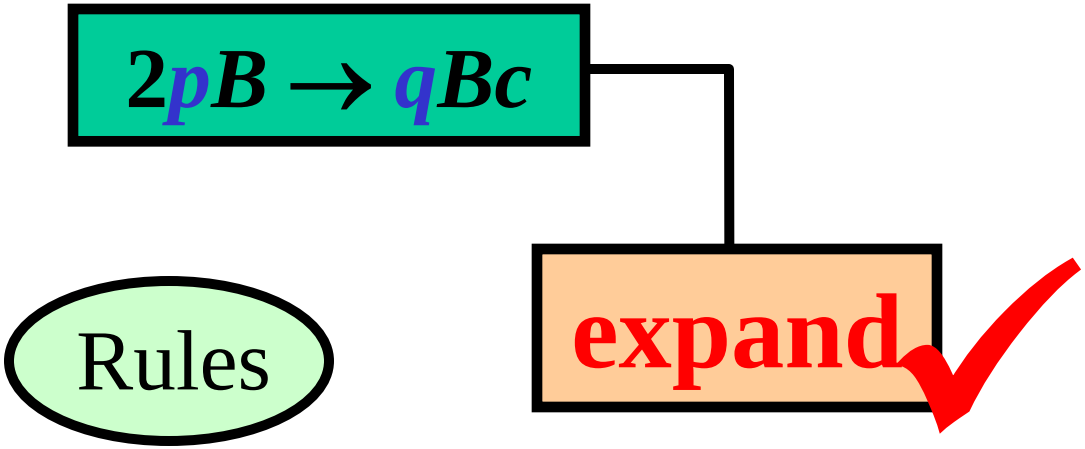


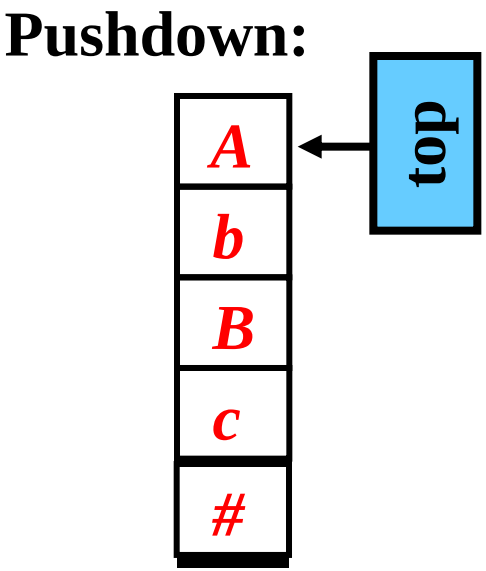
Deep Expansion: Illustration

Move: $(p, abbcc, AbB\#) \underset{e}{\Rightarrow} (q, abbcc, AbBc\#)$ $[2pB \rightarrow qBc]$









Example: Deep PDA

Deep PDA *M*:

[1]. $1sS \rightarrow qAB$

[2]. $1qA \rightarrow paAb$

[3]. $1qA \rightarrow fab$

[4]. $2pB \rightarrow qBc$

[5]. $1fB \rightarrow fc$

***M* accepts *aabbcc*:**

$(s, aabbcc, S\#)$

$\underset{e}{\Rightarrow} (q, aabbcc, AB\#)$ [1]

$\underset{e}{\Rightarrow} (p, aabbcc, aAbB\#)$ [2]

$\underset{p}{\Rightarrow} (p, abbcc, AbB\#)$

$\underset{e}{\Rightarrow} (q, abbcc, AbBc\#)$ [4]

$\underset{e}{\Rightarrow} (f, abbcc, abbBc\#)$ [3]

$\underset{p}{\Rightarrow} (f, bbcc, bbBc\#)$

$\underset{p}{\Rightarrow}^{2} (f, cc, Bc\#)$

$\underset{e}{\Rightarrow} (f, cc, cc\#)$ [5]

$\underset{p}{\Rightarrow} (f, c, c\#)$

$\underset{p}{\Rightarrow} (f, \varepsilon, \#)$

$$L(M) = \{a^n b^n c^n : n \geq 1\} \in PD_2$$

Definition 1/3

A deep pushdown automaton is a 7-tuple

$M = (Q, \Sigma, \Gamma, R, s, S, F)$, where

- Q – *states*,
- $\Sigma \subseteq \Gamma$ – *input alphabet*,
- Γ – *pushdown alphabet*, *bottom symbol* $\# \in \Gamma - \Sigma$
- R – *finite set of rules* of the form

$$mqA \rightarrow pw \quad \text{or} \quad mq\# \rightarrow pv\#$$

- $s \in Q$ – *start state*
- $S \in \Gamma$ – *start pushdown symbol*
- $F \subseteq Q$ – *final states*

Definition 2/3

- if an input symbol is on pd top, ***M* pops** the pd as

$$(q, au, az) \underset{p}{\Rightarrow} (q, u, z), \quad a \in \Sigma$$

- no explicit rule needed in R

- if a non-input symbol is on pd top, ***M* expands** the pd as

$$(q, w, uAz) \underset{e}{\Rightarrow} (p, w, uvz) \; [mqA \rightarrow pv],$$

where u contains $m - 1$ non-input symbols

Definition 3/3

- ***M*** is ***of depth n***, denotated by ${}_{n}\boldsymbol{M}$, if n is the minimal positive integer such that each of M's rules is of depth n or less.

- Language accepted by ${}_{n}M$, $L({}_{n}M)$, is defined as

$$L({}_{n}M) = \{\boldsymbol{w} \in \Sigma^{*}: (\boldsymbol{s}, \boldsymbol{w}, \boldsymbol{S}\#) \Rightarrow^{*} (\boldsymbol{f}, \boldsymbol{\varepsilon}, \#) \text{ in } {}_{n}M \text{ with } \boldsymbol{f} \in F\}.$$

Main Result and its Proof

- PD_n – the language family defined by DeepPDAs of depth n.

Theorem: $PD_n \subset PD_{n+1}$, for all $n \geq 1$.

Proof (Sketch):

- State grammars (Kasai, 1970) are needed in the proof
- State grammar is a modification of CFG based on rules of the form

$$(q, A) \rightarrow (p, v)$$

Proof 1/6: State Grammar

- ***State grammar*** $G = (V, W, T, P, S)$
  - V – total alphabet, W – states, $T \subseteq V$ – terminals,
  - P – set of rules of the form $(q, A) \to (p, v)$
  - $S \in (V - T)$ – start symbol,

- ***Configuration*** – (q, x)
- ***Derivation step***:

$$(q, uAz) \Rightarrow (p, uvz)\ [(q, A) \to (p, v)]$$

and for every nonterminal B in u, P contains no rule with (q, B) on the left-hand side

Proof 2/6: *n*-limited Step

- ***n-limited derivation step***:
  each derivation step within the first ***n*** non-terminals

$$(q, uAz) \underset{n}{\Rightarrow} (p, uvz) \text{ and}$$

uA has ***n*** or fewer non-terminals

- ***n-limited state language:***

$$L(G, n) = \{w \in T^* : (q, S) \underset{n}{\Rightarrow}^* (p, w)\}$$

- ST_n – the family of *n*-limited state languages

Proof 3/6: Example

State Grammar G:

[1]. $(1, S) \rightarrow (2, AC)$

[2]. $(2, A) \rightarrow (3, aAb)$

[3]. $(2, A) \rightarrow (4, ab)$

[4]. $(3, C) \rightarrow (2, Cc)$

[5]. $(4, C) \rightarrow (4, c)$

$W = \{1, 2, 3, 4\}$

G generates $aabbcc$:

$(S, 1) \Rightarrow (AC, 2)$ [1]

$\Rightarrow (aAbC, 3)$ [2]

$\Rightarrow (aAbCc, 2)$ [4]

$\Rightarrow (aabbCc, 4)$ [3]

$\Rightarrow (aabbcc, 4)$ [5]

$$L(G) = \{a^n b^n c^n : n \geq 1\} \in ST_2$$

Proof 4/6: $\boldsymbol{PD_n \subseteq ST_n}$, $n \geq 1$

- G simulates the application of $\boldsymbol{ipA \rightarrow qy} \in R$:
  - make a left-to-right scan of the pd until the ***i***th occurence of a non-terminal
  - if $X_i = \boldsymbol{A}$, then replace $\boldsymbol{A}$ with $\boldsymbol{y}$ and return to the beginning of the sentential form
  - rightmost symbol is always a special $\boldsymbol{a'}$, and G completes the simulation by changing $\boldsymbol{a'}$ to $\boldsymbol{a}$

Proof 5/6: $\boldsymbol{ST_n \subseteq PD_n}$, $n \geq 1$

- ${}_nM$ simulates the n-limited derivations of G in pd:
  - always records the first $\boldsymbol{n}$ non-terminals from the current sentential form of G in its state
  - fewer than $\boldsymbol{n}$ non-terminals are extended by #s
  - reads the string, empties pd, enters $\$ \in F$

Proof 6/6: $PD_n \subset PD_{n+1}$, $n \geq 1$

1) As $PD_n \subseteq ST_n$ and $ST_n \subseteq PD_n$ for all $n \geq 1$, $ST_n = PD_n$.

2) Kasai (1970): $ST_n \subset ST_{n+1}$, for all $n \geq 1$.

For all $n \geq 1$, $PD_n = ST_n \subset ST_{n+1} = PD_{n+1}$

Q. E. D.

Open Problem Areas

- Determinism
- Rules of form $\boldsymbol{mqA \rightarrow p\varepsilon}$

Discussion and End

- Any questions, please?