

# #-Rewriting Systems and An Infinite Hierarchy Resulting from Them

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Based upon

**Křivka, Z., Meduna, A., Schönecker, R.:**

Generation of Languages by Rewriting Systems that Resemble  
Automata,

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# Contents

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- 4. Open Problem Areas**

# #-Rewriting Systems in Formal Language Theory

- Language-defining models
- Pure rewriting systems
- Between automata and grammars:  
have states but generate languages

# Concept

**#-Rewriting System** is based on the rules of the form

$$p_{\textcolor{brown}{m}} \# \rightarrow q \textcolor{magenta}{x}_0 \# \textcolor{magenta}{x}_1 \dots \# \textcolor{magenta}{x}_n$$

by which the system makes a computational  
step  $\Rightarrow$  as

$$\begin{array}{c}
 \textcolor{brown}{m}^{\text{th}} \# \\
 \downarrow \\
 (\textcolor{teal}{p}, \dots \# \textcolor{teal}{y}_{m-1} \textcolor{red}{\#} \textcolor{teal}{y}_m \# \textcolor{teal}{y}_{m+1} \dots) \Rightarrow \\
 (\textcolor{blue}{q}, \dots \# \textcolor{teal}{y}_{m-1} \textcolor{magenta}{x}_0 \# \textcolor{magenta}{x}_1 \dots \# \textcolor{magenta}{x}_n \textcolor{teal}{y}_m \# \textcolor{teal}{y}_{m+1} \dots)
 \end{array}$$

## Definition 1/2

**#-Rewriting System (#RS)** is a quadruple

$$H = (Q, \Sigma, s, R), \text{ where}$$

- $Q$ —finite set of *states*,
- $\Sigma$ —*alphabet*,  $\# \in \Sigma$  is called a *bounder*,
- $s \in Q$ —*start state*,
- $R$ —finite set of *rules* of the form

$$\mathbf{p}_{\color{brown}m}\# \rightarrow \mathbf{q}\mathbf{x}$$

where  $\mathbf{p}, \mathbf{q} \in Q$ ,  $\color{brown}m$  is a positive integer,  $\mathbf{x} \in \Sigma^*$ .

## Definition 2/2

*Configuration:*  $(q, \textcolor{magenta}{x})$ ,  $q \in Q, \textcolor{magenta}{x} \in \Sigma^*$

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*Computational step:*

$(p, \textcolor{magenta}{u}\#\textcolor{magenta}{v}) \Rightarrow (q, \textcolor{magenta}{u}x\textcolor{magenta}{v})$  [ $p_{\textcolor{brown}{m}}\# \rightarrow qx \in R$ ],

where the number of  $\#$ s in  $\textcolor{magenta}{u}$  is  $\textcolor{brown}{m} - 1$ ,

$p, q \in Q, \textcolor{magenta}{u}, \textcolor{magenta}{x}, \textcolor{magenta}{v} \in \Sigma^*$ .

*Generated language:*

$L(H) = \{w \in (\Sigma - \#)^*: (s, \#) \Rightarrow^* (q, w) \text{ in } H, q \in Q\}$ .

## Example: #RS

#RS H:

H generates *aabbcc*:

[1].  $s \ 1\# \rightarrow p \ ##$

[2].  $p \ 1\# \rightarrow q \ a\#b$

[3].  $q \ 2\# \rightarrow p \ #c$

[4].  $p \ 1\# \rightarrow f \ ab$

[5].  $f \ 1\# \rightarrow f \ c$

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

$$\begin{array}{c} (\textcolor{blue}{s}, \#) \\ \Rightarrow \end{array}$$

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ \#\#$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ \#c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

$$\begin{array}{ccc} (\textcolor{blue}{s}, \textcolor{orange}{\#}) & & [1] \\ \Rightarrow & & \end{array}$$

## Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

$$\begin{aligned} & (s, \#) \\ \Rightarrow & (p, \##) \quad [1] \\ \Rightarrow & \end{aligned}$$

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ \#\#$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ \#c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

$$\begin{aligned} & (s, \#) \\ \Rightarrow & (p, \#\#) \quad [1] \\ \Rightarrow & \quad [2] \end{aligned}$$

## Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

$$\begin{aligned} & (s, \#) \\ \Rightarrow & (p, \##) \quad [1] \\ \Rightarrow & (q, a\#b\#) \quad [2] \\ \Rightarrow & \end{aligned}$$

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

- $(s, \#)$
- $\Rightarrow (p, \##)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow$  [3]

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

- $(s, \#)$
- $\Rightarrow (p, \##)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\#b\#c)$  [3]
- $\Rightarrow$

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

- $(s, \#)$
- $\Rightarrow (p, \##)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\#b\#c)$  [3]
- $\Rightarrow$  [4]

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

- $(s, \#)$
- $\Rightarrow (p, \##)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\#b\#c)$  [3]
- $\Rightarrow (f, aabb\#c)$  [4]
- $\Rightarrow$

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

- ( $s$ , #)
- $\Rightarrow (p, ##)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\#b\#c)$  [3]
- $\Rightarrow (f, aabb\#c)$  [4]
- $\Rightarrow$  [5]

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates *aabbcc*:

- $(s, \#)$
- $\Rightarrow (p, \##)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\#b\#c)$  [3]
- $\Rightarrow (f, aabb\#c)$  [4]
- $\Rightarrow (f, aabbcc)$  [5]

# Example: #RS

#RS H:

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

H generates  $aabbcc$ :

- ( $s$ , #)
- $\Rightarrow (p, ##)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\#b\#c)$  [3]
- $\Rightarrow (f, aabb\#c)$  [4]
- $\Rightarrow (f, aabbcc)$  [5]

$$L(H) = \{a^n b^n c^n : n \geq 1\}$$

# Finite index of $\#RS$

#-Rewriting systems of *index k*:

⇒ over configurations with  $k$  or fewer #s

$\#RS_k$  – the language family generated by  
#RSs of index  $k$

Example: Index  $k = 2$ :

1.  $(p, a\#a\#b) \Rightarrow (q, aa\#aa\#b)$  [ $p_1\# \rightarrow qa\#a \in R$ ]  
**OK**

2.  $(p, a\#a\#b) \not\Rightarrow (q, a\#aa\#\#bb)$  [ $p_2\# \rightarrow qa\#\#b \in R$ ]  
**INCORRECT**

# Example: $\#RS$ of finite index

$\#RS\ H:$

- [1].  $s \ 1\# \rightarrow p \ \#\#$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ \#c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

$H$  generates  $aabbcc$ :

- $(s, \#)$
- $\Rightarrow (p, \#\#)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\#b\#c)$  [3]
- $\Rightarrow (f, aabb\#c)$  [4]
- $\Rightarrow (f, aabbcc)$  [5]

$H$  is of index 2.

$$L(H) = \{a^n b^n c^n : n \geq 1\} \in \#RS_2$$

# Main Result: An Infinite Hierarchy

**Theorem:**  $\#RS_k \subset \#RS_{k+1}$ , for all  $k \geq 1$ .

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**Proof:**

makes use of programmed grammars (*PG*) of index  $k$

# Proof: Programmed Grammars

**Programmed Grammar (PG)** is a modification of context-free grammar based on the rules of the form:

$$\textcolor{orange}{r}: A \rightarrow \textcolor{magenta}{x}, W_{\textcolor{orange}{r}}$$

- $\textcolor{orange}{r}: A \rightarrow \textcolor{magenta}{x}$  is a context-free rule labeled by  $\textcolor{orange}{r}$ ,
- $W_{\textcolor{orange}{r}}$ —finite set of rule labels

**Derivation step ( $\Rightarrow$ ):**

after the application of rule  $\textcolor{orange}{r}$ ,  
a rule from  $W_{\textcolor{orange}{r}}$  has to be applied

# Proof: Finite index of $PG$

Programmed grammars of *index k*:

- $\Rightarrow$  over sentential forms with  $k$  or fewer occurrences of nonterminals.

$P_k$  – the language family defined by programmed grammars of index  $k$

## Example: $PG$

$PG\ G:$

- 1:  $S \rightarrow ABC, \{2, 5\}$
- 2:  $A \rightarrow aA, \{3\}$
- 3:  $B \rightarrow bB, \{4\}$
- 4:  $C \rightarrow cC, \{2, 5\}$
- 5:  $A \rightarrow a, \{6\}$
- 6:  $B \rightarrow b, \{7\}$
- 7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

## Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

2:  $A \rightarrow aA, \{3\}$

3:  $B \rightarrow bB, \{4\}$

4:  $C \rightarrow cC, \{2, 5\}$

5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$S$

$\Rightarrow$

## Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

2:  $A \rightarrow aA, \{3\}$

3:  $B \rightarrow bB, \{4\}$

4:  $C \rightarrow cC, \{2, 5\}$

5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{array}{c} S \\ \Rightarrow \\ [1] \end{array}$$

## Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

2:  $A \rightarrow aA, \{3\}$

3:  $B \rightarrow bB, \{4\}$

4:  $C \rightarrow cC, \{2, 5\}$

5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{aligned} S \\ \Rightarrow ABC & [1] \\ \Rightarrow \end{aligned}$$

## Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

2:  $A \rightarrow aA, \{3\}$

3:  $B \rightarrow bB, \{4\}$

4:  $C \rightarrow cC, \{2, 5\}$

5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{array}{ccc} S & & \\ \Rightarrow ABC & [1] & \\ \Rightarrow & [2] & \\ \Rightarrow & & \end{array}$$

## Example: $PG$

$PG\ G:$

$$1: \textcolor{red}{S} \rightarrow ABC, \{\textcolor{blue}{2}, 5\}$$

$$2: \textcolor{red}{A} \rightarrow aA, \{\textcolor{blue}{3}\}$$

$$3: \textcolor{orange}{B} \rightarrow bB, \{\textcolor{blue}{4}\}$$

$$4: \textcolor{red}{C} \rightarrow cC, \{\textcolor{blue}{2}, 5\}$$

$$5: \textcolor{red}{A} \rightarrow a, \{\textcolor{blue}{6}\}$$

$$6: \textcolor{red}{B} \rightarrow b, \{\textcolor{blue}{7}\}$$

$$7: \textcolor{red}{C} \rightarrow c, \emptyset$$

$G$  generates  $aabbcc$ :

$$\begin{aligned} & \textcolor{red}{S} \\ & \Rightarrow \textcolor{red}{ABC} & [1] \\ & \Rightarrow aA\textcolor{orange}{B}C & [2] \\ & \Rightarrow \end{aligned}$$

# Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

2:  $A \rightarrow aA, \{3\}$

3:  $B \rightarrow bB, \{4\}$

4:  $C \rightarrow cC, \{2, 5\}$

5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{aligned} S \\ \Rightarrow ABC & [1] \\ \Rightarrow aA\textcolor{red}{BC} & [2] \\ \Rightarrow & [3] \end{aligned}$$

# Example: $PG$

$PG\ G:$

$$1: \textcolor{red}{S} \rightarrow ABC, \{\textcolor{blue}{2}, 5\}$$

$$2: \textcolor{red}{A} \rightarrow aA, \{\textcolor{blue}{3}\}$$

$$3: \textcolor{red}{B} \rightarrow bB, \{\textcolor{blue}{4}\}$$

$$4: \textcolor{red}{C} \rightarrow cC, \{\textcolor{blue}{2}, 5\}$$

$$5: \textcolor{red}{A} \rightarrow a, \{\textcolor{blue}{6}\}$$

$$6: \textcolor{red}{B} \rightarrow b, \{\textcolor{blue}{7}\}$$

$$7: \textcolor{red}{C} \rightarrow c, \emptyset$$

$G$  generates  $aabbcc$ :

$$\begin{aligned} & \textcolor{red}{S} \\ & \Rightarrow \textcolor{red}{ABC} & [1] \\ & \Rightarrow aA\textcolor{red}{B}C & [2] \\ & \Rightarrow aAbB\textcolor{orange}{C} & [3] \\ & \Rightarrow \end{aligned}$$

# Example: $PG$

$PG\ G:$

$$1: \textcolor{red}{S} \rightarrow ABC, \{\textcolor{blue}{2}, 5\}$$

$$2: A \rightarrow aA, \{\textcolor{blue}{3}\}$$

$$3: \textcolor{red}{B} \rightarrow bB, \{\textcolor{blue}{4}\}$$

$$4: C \rightarrow cC, \{\textcolor{orange}{2}, \textcolor{blue}{5}\}$$

$$5: A \rightarrow a, \{\textcolor{blue}{6}\}$$

$$6: \textcolor{red}{B} \rightarrow b, \{\textcolor{blue}{7}\}$$

$$7: \textcolor{red}{C} \rightarrow c, \emptyset$$

$G$  generates  $aabbcc$ :

$$\begin{aligned} & \textcolor{red}{S} \\ & \Rightarrow ABC & [1] \\ & \Rightarrow aA\textcolor{red}{B}C & [2] \\ & \Rightarrow aAbB\textcolor{red}{C} & [3] \\ & \Rightarrow & [4] \end{aligned}$$

# Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

2:  $A \rightarrow aA, \{3\}$

3:  $B \rightarrow bB, \{4\}$

4:  $C \rightarrow cC, \{2, 5\}$

5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{aligned}
 & S \\
 \Rightarrow & ABC & [1] \\
 \Rightarrow & aA\textcolor{red}{BC} & [2] \\
 \Rightarrow & aAb\textcolor{black}{B}\textcolor{red}{C} & [3] \\
 \Rightarrow & a\textcolor{orange}{A}bBcC & [4] \\
 \Rightarrow &
 \end{aligned}$$

# Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

2:  $A \rightarrow aA, \{3\}$

3:  $B \rightarrow bB, \{4\}$

4:  $C \rightarrow cC, \{2, 5\}$

5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{aligned}
 & S \\
 \Rightarrow & ABC & [1] \\
 \Rightarrow & aA\mathbf{BC} & [2] \\
 \Rightarrow & aAb\mathbf{BC} & [3] \\
 \Rightarrow & aAbBcC & [4] \\
 \Rightarrow & 
 \end{aligned}$$

# Example: $PG$

$PG\ G:$

- 1:  $S \rightarrow ABC, \{2, 5\}$
- 2:  $A \rightarrow aA, \{3\}$
- 3:  $B \rightarrow bB, \{4\}$
- 4:  $C \rightarrow cC, \{2, 5\}$
- 5:  $A \rightarrow a, \{6\}$
- 6:  $B \rightarrow b, \{7\}$
- 7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{aligned}
 & S \\
 \Rightarrow & ABC & [1] \\
 \Rightarrow & aA\textcolor{red}{BC} & [2] \\
 \Rightarrow & aAb\textcolor{black}{B}\textcolor{red}{C} & [3] \\
 \Rightarrow & aAbBcC & [4] \\
 \Rightarrow & aab\textcolor{brown}{B}cC & [5] \\
 \Rightarrow &
 \end{aligned}$$

# Example: $PG$

$PG\ G:$

- 1:  $S \rightarrow ABC, \{2, 5\}$
- 2:  $A \rightarrow aA, \{3\}$
- 3:  $B \rightarrow bB, \{4\}$
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- 7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{aligned}
 & S \\
 \Rightarrow & ABC & [1] \\
 \Rightarrow & aA\mathbf{BC} & [2] \\
 \Rightarrow & aAb\mathbf{BC} & [3] \\
 \Rightarrow & aAbBcC & [4] \\
 \Rightarrow & aab\mathbf{BcC} & [5] \\
 \Rightarrow & 
 \end{aligned}$$

$[6]$

# Example: $PG$

$PG\ G:$

- 1:  $S \rightarrow ABC, \{2, 5\}$
- 2:  $A \rightarrow aA, \{3\}$
- 3:  $B \rightarrow bB, \{4\}$
- 4:  $C \rightarrow cC, \{2, 5\}$
- 5:  $A \rightarrow a, \{6\}$
- 6:  $B \rightarrow b, \{7\}$
- 7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{aligned}
 & S \\
 \Rightarrow & ABC & [1] \\
 \Rightarrow & aA\textcolor{red}{BC} & [2] \\
 \Rightarrow & aAb\textcolor{black}{B}\textcolor{red}{C} & [3] \\
 \Rightarrow & aAbBcC & [4] \\
 \Rightarrow & aab\textcolor{red}{B}cC & [5] \\
 \Rightarrow & aabb\textcolor{orange}{c}\textcolor{black}{C} & [6] \\
 \Rightarrow &
 \end{aligned}$$

# Example: $PG$

$PG\ G:$

- 1:  $S \rightarrow ABC, \{2, 5\}$
- 2:  $A \rightarrow aA, \{3\}$
- 3:  $B \rightarrow bB, \{4\}$
- 4:  $C \rightarrow cC, \{2, 5\}$
- 5:  $A \rightarrow a, \{6\}$
- 6:  $B \rightarrow b, \{7\}$
- 7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

- $S$   
 $\Rightarrow ABC$  [1]
- $\Rightarrow aABC$  [2]
- $\Rightarrow aAbBC$  [3]
- $\Rightarrow aAbBcC$  [4]
- $\Rightarrow aabBcC$  [5]
- $\Rightarrow aabbcC$  [6]
- $\Rightarrow aabbcc$  [7]

## Example: $PG$

$PG\ G:$

- 1:  $S \rightarrow ABC, \{2, 5\}$
- 2:  $A \rightarrow aA, \{3\}$
- 3:  $B \rightarrow bB, \{4\}$
- 4:  $C \rightarrow cC, \{2, 5\}$
- 5:  $A \rightarrow a, \{6\}$
- 6:  $B \rightarrow b, \{7\}$
- 7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

- $S$   
 $\Rightarrow ABC$  [1]
- $\Rightarrow aA**$\Rightarrow aAbBC$  [3]**$
- $\Rightarrow aAbBcC$  [4]
- $\Rightarrow aabBcC$  [5]
- $\Rightarrow aabbC$  [6]
- $\Rightarrow aabbcc$  [7]

$$L(G) = \{a^n b^n c^n : n \geq 1\} \in P_3$$

Proof:  $P_k = \#RS_k, k \geq 1$

$P_k \subseteq \#RS_k$ :

Let  $G$  be a  $PG$  of index  $k$ . Construct a  $\#RS$   $H$  of index  $k$ , so  $H$  simulates derivation step

$a\underline{A}bBc \Rightarrow_G a\textcolor{magenta}{dXY}bBc [p: A \rightarrow \textcolor{magenta}{dXY}, \{q, o\}] \Rightarrow_G \dots [\textcolor{orange}{q}]$

as

$$(\langle \underline{A}B, p \rangle, a\#b\#c) \Rightarrow_H (\langle XYB, q \rangle, a\textcolor{magenta}{d}\#\#b\#c)$$

$$[\langle \underline{A}B, p \rangle \xrightarrow{1\#} \langle XYB, q \rangle \textcolor{magenta}{d}\#\#]$$

# Proof: $\#RS_k = P_k$ , $k \geq 1$

$\#RS_k \subseteq P_k$ :

Let  $H$  be a  $\#RS$  of index  $k$ . Construct a  $PG$   $G$  of index  $k$ , so  $G$  simulates a computational step

$$(\mathbf{p}, a\#\mathbf{b}\#\mathbf{c}) \Rightarrow_H (\mathbf{q}, aa\#\mathbf{b}\#\mathbf{b}\#\mathbf{c}) [\mathbf{p}_1 \# \rightarrow \mathbf{q} \ a\#\mathbf{b}\#]$$

as

$$a\langle \mathbf{p}, 1, 2 \rangle b\langle \mathbf{p}, 2, 2 \rangle c$$

- 1) Renumbering:  $\Rightarrow_G a\langle q'', 1, 3 \rangle b\langle p, 2, 2 \rangle c$   
 $\Rightarrow_G a\langle q'', 1, 3 \rangle b\langle q', 3, 3 \rangle c$
- 2) Rewriting:  $\Rightarrow_G aa\langle q', 1, 3 \rangle b\langle q', 2, 3 \rangle b\langle q', 3, 3 \rangle c$
- 3) Finalization:  $\Rightarrow_G aa\langle q, 1, 3 \rangle b\langle q', 2, 3 \rangle b\langle q', 3, 3 \rangle c$   
 $\Rightarrow_G aa\langle q, 1, 3 \rangle b\langle q, 2, 3 \rangle b\langle q', 3, 3 \rangle c$   
 $\Rightarrow_G aa\langle q, 1, 3 \rangle b\langle q, 2, 3 \rangle b\langle q, 3, 3 \rangle c$

Proof:  $\#RS_k \subset \#RS_{k+1}$ ,  $k \geq 1$

Recall that:

- $P_k \subset P_{k+1}$ , for all  $k \geq 1$
- 

As  $P_k = \#RS_k$ , for all  $k \geq 1$ , we have

Theorem:  $\#RS_k \subset \#RS_{k+1}$ , for all  $k \geq 1$ .

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# Future Investigation

- Determinism
- Unlimited index
- Other variants:
  - Right-linear
  - Context-sensitive
  - Parallel

# Discussion