On Proof Techniques in Jumping Models

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Motivation

- In formal language theory, it is a common task to prove that a certain language can or cannot be accepted by the model in question.
- Student courses (IFJ, etc.) show only basic, well-known techniques (pumping lemmas, etc.).
- This talk shows new techniques used in current research.
- Motivation for further study.

Finite Automata

Finite Automata – Definition (1/2)

Lazy Finite Automaton (LFA)

$$\textit{quintuple} \ \ M = (Q, \Sigma, R, s, F)$$

Q is a finite set of states

Σ is an input alphabet, $Q \cap \Sigma = \emptyset$

R is a finite set of rules: (p, y, q), where $p, q \in Q$, $y \in \Sigma^*$

s is the start state

F is a set of final states

FIT

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Finite Automaton (FA)

If $(p, y, q) \in R$ implies that $|y| \leq 1$.

Finite Automata – Definition (2/2)

Configuration

$$pw$$

p is the state

w is an unprocessed input

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Step/Move

$$pyx \Rightarrow qx$$

if $(p, y, q) \in R$ and $x, y \in \Sigma^*$.
In the standard manner, define $\Rightarrow^+$ and $\Rightarrow^*$.

Configuration

$$pw$$

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Step/Move

$$pyx \Rightarrow qx$$

if $(p, y, q) \in R$ and $x, y \in \Sigma^*$.
In the standard manner, define $\Rightarrow^+$ and $\Rightarrow^*$.

Accepted language

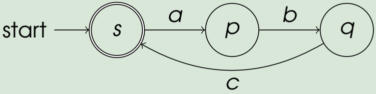
$L(M) = \{w \in \Sigma^* : sw \Rightarrow^* f, \ f \in F\}$

Example automaton

$$M = (\{s, p, q\}, \{a, b, c\}, R, s, \{s\})$$

where R:

(s, a, p)
(p, b, q)
(q, c, s)



FA – Accepted Languages

Example automaton

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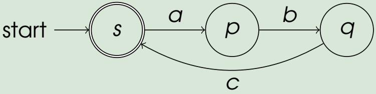

Example input: *abcabc*

$sabcabc \Rightarrow pbcabc \Rightarrow qcabc \Rightarrow sabc \Rightarrow pbc \Rightarrow qc \Rightarrow s$

FA – Accepted Languages

Example automaton

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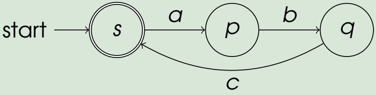

Example input: *abcabc*

$sabcabc \Rightarrow pbcabc \Rightarrow qcabc \Rightarrow sabc \Rightarrow pbc \Rightarrow qc \Rightarrow s$

Resulting language

- FA: $L(M) = \{abc\}^*$

- What about $L = \{a^n b^n : n \geq 0\}$?
- Can we construct an FA that accepts L?
- How to rigorously prove that it is not possible?

- What about $L = \{a^n b^n : n \geq 0\}$?
- Can we construct an FA that accepts L?
- How to rigorously prove that it is not possible?

Pumping lemma for regular languages

Let L be a regular language over Σ. Then there is a constant k, depending on L, such that for each $w \in L$ with $|w| \geq k$ there exist $x, y, z \in \Sigma^*$ such that $w = xyz$ and

1. $|xy| \leq k$,
2. $|y| > 0$,
3. $xy^iz \in L$ for all $i \geq 0$.

- This lemma is necessary but not sufficient.
- There are sufficient lemmas but they are more complicated.

Theorem

There is no FA M such that $L(M) = \{a^n b^n : n \geq 0\}$.

Proof.

By contradiction. Assume that there is a FA M such that $L(M) = \{a^n b^n : n \geq 0\}$. Then, $L(M)$ is a regular language.

Choose $w = a^k b^k$ in $L(M)$. Clearly, $|w| \geq k$.

By the pumping lemma, $w = xyz$ for some $x, y, z \in \Sigma^*$ such that (1) $|xy| \leq k$, (2) $|y| > 0$, and (3) $xy^i z \in L(M)$ for all $i \geq 0$.

By (1) and (2), we have $y = a^m$, $1 \leq m \leq k$.
But $xy^0 z = xz = a^{k-m} b^k \notin L(M)$. Thus, (3) does not hold.
Therefore, there is no FA M such that $L(M) = \{a^n b^n : n \geq 0\}$. $\square$

Jumping Finite Automata

Based on

- Alexander Meduna and Petr Zemek
  Jumping Finite Automata
  Int. J. Found. Comput. Sci. 23(7):1555–1578 (2012)

- Alexander Meduna and Petr Zemek
  Regulated Grammars and Automata
  Springer (2014)

General Jumping Finite Automaton (GJFA)

$$\textit{quintuple} \;\; M = (Q, \Sigma, R, s, F)$$

Q, Σ, R, s, F are defined as in LFA.

If $(p, y, q) \in R$ implies that $|y| \leq 1$, then M is a jumping finite automaton (JFA).

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upv where $u, v \in \Sigma^*$ and $p \in Q$.

Jumping Finite Automata – Definition

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Jump

$$xpyz \curvearrowright x'qz'$$

if $x, z, x', z' \in \Sigma^*$ such that $xz = x'z'$ and $(p, y, q) \in R$; $\curvearrowright^+$, $\curvearrowright^*$.

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Accepted language

$L(M) = \{uv : u, v \in \Sigma^*, \; usv \curvearrowright^* f, \; f \in F\}$

Example automaton

$$M = (\{s, p, q\}, \{a, b, c\}, R, s, \{s\})$$

where R:

(s, a, p)
(p, b, q)
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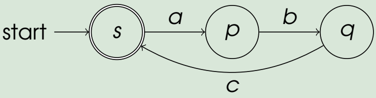

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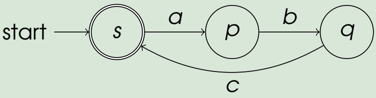

Example input: *abbacc*

$abbsacc \curvearrowright abpbcc \curvearrowright abqcc \curvearrowright sabc \curvearrowright pbc \curvearrowright qc \curvearrowright s$

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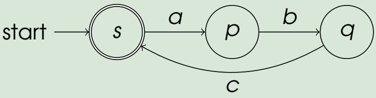

Example input: *abbacc*

$abbsacc \curvearrowright abpbcc \curvearrowright abqcc \curvearrowright sabc \curvearrowright pbc \curvearrowright qc \curvearrowright s$

Resulting language

- FA: $L(M) = \{abc\}^*$
- JFA: $L(M) = \{w : w \in \{a, b, c\}^*, \ |w|_a = |w|_b = |w|_c\}$

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Theorem

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Theorem

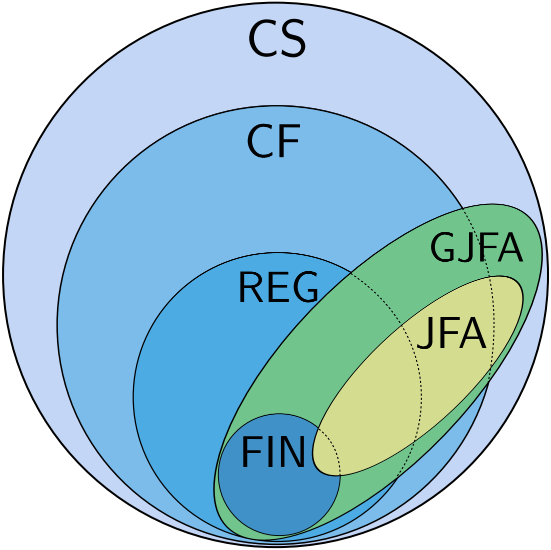
There is no GJFA M such that $L(M) = \{a\}^*\{b\}^*$.

Proof.

By contradiction. Let $K = \{a\}^*\{b\}^*$. Assume that there is a GJFA $M = (Q, \Sigma, R, s, F)$ such that $L(M) = K$.

Let $n = \max\{|y| : (p, y, q) \in R\}$ and $w = a^n b$.

When accepting w, a rule $(p, a^i b, q) \in R$, $0 \leq i < n$, has to be used. However, then M also accepts from the configuration $a^i bsa^{n-i}$ or $sa^i ba^{n-i}$. This implies that $a^i ba^{n-i} \in L(M)$. But that is a contradiction with the assumption that $L(M) = K$. Therefore, there is no GJFA M such that $L(M) = \{a\}^*\{b\}^*$. $\square$


CS
CF
GJFA
REG
JFA
FIN

Jumping $5' \to 3'$ Watson-Crick Finite Automata

Based on

- Radim Kocman, Benedek Nagy, Zbyněk Křivka, and Alexander Meduna
  A Jumping $5' \to 3'$ Watson-Crick Finite Automata Model
  Proceedings of NCMA 2018

- Radim Kocman, Zbyněk Křivka, Alexander Meduna, and Benedek Nagy
  A Jumping $5' \to 3'$ Watson-Crick Finite Automata Model
  Acta Informatica (in review)

Watson-Crick Finite Automata (WKA)

- biology-inspired model (the core model is similar to FA)
- work with the Watson-Crick tape (double-stranded tape, resembles DNA, the elements of the strands are pairwise complements of each other)
- uses two heads (one for each strand of the tape)

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$5' \to 3'$ Watson-Crick Finite Automata

- the heads read in the biochemical $5' \to 3'$ direction
- that is physically/mathematically in opposite directions

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Sensing $5' \rightarrow 3'$ Watson-Crick Finite Automata

- the heads sense that they are meeting
- the processing of the input ends if for all pairs of the sequence one of the letters is read
- the tape notation is usually simplified: $\left[\begin{smallmatrix} A \\ T \end{smallmatrix}\right]$ as $a, \ldots$

Combined model

- the combination of GJFA and sensing $5' \rightarrow 3'$ WKA
- two heads as in sensing $5' \rightarrow 3'$ WKA
- each head can traverse the whole input in its direction
- all pairs of symbols are read only once

Expectations

- better accepting power than the non-combined models
- ability to model languages with some crossed agreements

Jumping $5' \to 3'$ WK Automaton

quintuple $M = (V, Q, q_0, F, \delta)$

V (Σ), Q, q_0 (s), F as in LFA, $V \cap \{\#\} = \emptyset$,

$\delta \colon (Q \times V^* \times V^* \times D) \to 2^Q$ (finite),
$D = \{\oplus, \ominus\}$ indicates the mutual position of heads.

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Configuration

$$(q, s, w_1, w_2, w_3)$$

- q is the state
- s is the position of heads
- w_1 is the unprocessed input before the first head
- w_2 is the unprocessed input between the heads
- w_3 is the unprocessed input after the second head

Steps

Let $x, y, u, v, w_2 \in V^*$ and $w_1, w_3 \in (V \cup \{\#\})^*$.

1. $\oplus$-reading: $(q, \oplus, w_1, xw_2y, w_3) \curvearrowright (q', s, w_1\{\#\}^{|x|}, w_2, \{\#\}^{|y|}w_3)$, where $q' \in \delta(q, x, y, \oplus)$, and s is either $\oplus$ if $|w_2| > 0$ or $\ominus$.
2. $\ominus$-reading: $(q, \ominus, w_1y, \varepsilon, xw_3) \curvearrowright (q', \ominus, w_1, \varepsilon, w_3)$, where $q' \in \delta(q, x, y, \ominus)$.
3. $\oplus$-jumping: $(q, \oplus, w_1, uw_2v, w_3) \curvearrowright (q, s, w_1u, w_2, vw_3)$, where s is either $\oplus$ if $|w_2| > 0$ or $\ominus$.
4. $\ominus$-jumping: $(q, \ominus, w_1\{\#\}^*, \varepsilon, \{\#\}^*w_3) \curvearrowright (q, \ominus, w_1, \varepsilon, w_3)$.

In the standard manner, define $\curvearrowright^+$ and $\curvearrowright^*$.

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In the standard manner, define $\curvearrowright^+$ and $\curvearrowright^*$.

Accepted language

$L(M) = \{w \in V^* : (q_0, \oplus, \varepsilon, w, \varepsilon) \curvearrowright^* (q_f, \ominus, \varepsilon, \varepsilon, \varepsilon),\ q_f \in F\}$

Example automaton

$$M = (\{a, b\}, \{s\}, s, \{s\}, \delta)$$

where δ:

$$\delta(s, a, b, \oplus) = \{s\}$$
$$\delta(s, a, b, \ominus) = \{s\}$$

Example automaton

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Example input: *aaabbb*

$$
\begin{array}{ll}
(s, \oplus, \varepsilon, aaabbb, \varepsilon) \curvearrowright & \oplus\text{-reading} \\
(s, \oplus, \#, aabb, \#) \curvearrowright & \oplus\text{-reading} \\
(s, \oplus, \#\#, ab, \#\#) \curvearrowright & \oplus\text{-reading} \\
(s, \ominus, \#\#\#, \varepsilon, \#\#\#) \curvearrowright & \ominus\text{-jumping} \\
(s, \ominus, \varepsilon, \varepsilon, \varepsilon) &
\end{array}
$$

Example automaton

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Example input: *baabba*

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(s, \oplus, \varepsilon, baabba, \varepsilon) \curvearrowright & \oplus\text{-jumping} \\
(s, \oplus, b, aabb, a) \curvearrowright & \oplus\text{-reading} \\
(s, \oplus, b\#, ab, \#a) \curvearrowright & \oplus\text{-reading} \\
(s, \ominus, b\#\#, \varepsilon, \#\#a) \curvearrowright & \ominus\text{-jumping} \\
(s, \ominus, b, \varepsilon, a) \curvearrowright & \ominus\text{-reading} \\
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\end{array}$$

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$(s, \oplus, b, aabb, a) \curvearrowright$	$\oplus$-reading
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$(s, \ominus, b\#\#, \varepsilon, \#\#a) \curvearrowright$	$\ominus$-jumping
$(s, \ominus, b, \varepsilon, a) \curvearrowright$	$\ominus$-reading
$(s, \ominus, \varepsilon, \varepsilon, \varepsilon)$	

Resulting language

$$L(M) = \{w : w \in \{a, b\}^*,\ |w|_a = |w|_b\}$$

- What happens if we remove $\delta(s, a, b, \ominus) = \{s\}$ from M?
  $\rightarrow L(M) = \{a^n b^n : n \geq 0\}$

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- And if we use only $\delta(s, a, \varepsilon, \oplus) = \{s\}$ and $\delta(s, \varepsilon, b, \oplus) = \{s\}$?
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  $\rightarrow L(M) = \{a\}^*\{b\}^*$

- **REG** $\subset$ **JWK**
- **LIN** $\subset$ **JWK**
- $\{w_1 w_2 : w_1 \in \{a, b\}^*,\ w_2 \in \{c, d\}^*,\ |w_1|_a = |w_2|_c,\ |w_1|_b = |w_2|_d\} \in$ **JWK** which is a non-context-free language
- **JWK** $\subset$ **CS**

Theorem

There is no jumping $5' \to 3'$ WK automaton M such that $L(M) = \{a^n b^n c^n : n \geq 0\}$.

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- How to rigorously prove it?

JWKFA – Undefinable Languages

FIT

Theorem

There is no jumping $5' \to 3'$ *WK automaton M such that* $L(M) = \{a^n b^n c^n : n \geq 0\}$.

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- How to rigorously prove it?
  - The automaton can guarantee the order of symbols in certain cases. We cannot use the JFA technique. ☹

JWKFA – Undefinable Languages

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  - The symbols can be mixed so it is not easy to derive a meaningful pumping lemma. ☹
  - We need a different proof technique:
    $\to$ introducing the new debt lemma.

Parikh Vector

The Parikh vector associated to a string $x \in V^*$ with respect to the alphabet $V = \{a_1, a_2, \ldots, a_n\}$ is
$\Psi_V(x) = (|x|_{a_1}, |x|_{a_2}, \ldots, |x|_{a_n})$.

For $L \subseteq V^*$ we define $\Psi_V(L) = \{\Psi_V(x) : x \in L\}$.

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Example strings

$$
\begin{array}{lll}
V = \{a, b, c\}, & x = abbccc & \Rightarrow \Psi_V(x) = (1, 2, 3) \\
V = \{a, b, c, d\}, & x = abbccc & \Rightarrow \Psi_V(x) = (1, 2, 3, 0) \\
V = \{a, b, c, d\}, & x = cbabcc & \Rightarrow \Psi_V(x) = (1, 2, 3, 0) \\
V = \{a, b, c, d\}, & x = \varepsilon & \Rightarrow \Psi_V(x) = (0, 0, 0, 0)
\end{array}
$$

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For $L \subseteq V^*$ we define $\Psi_V(L) = \{\Psi_V(x) : x \in L\}$.

Example language

Let $V = \{a, b, c\}$ and $L = \{a^n b^n c^n : n \geq 0\}$. Then, $\Psi_V(L) = \{$

$$
\begin{array}{lcl}
x = \varepsilon & \Rightarrow & \Psi_V(x) = (0,0,0) \\
x = abc & \Rightarrow & \Psi_V(x) = (1,1,1) \\
x = aabbcc & \Rightarrow & \Psi_V(x) = (2,2,2) \\
x = aaabbbccc & \Rightarrow & \Psi_V(x) = (3,3,3) \\
\ldots & &
\end{array}
$$

$\} = \{(0,0,0), (1,1,1), (2,2,2), (3,3,3), \ldots\} = \{(n,n,n) : n \geq 0\}$.

Definition

Let $M = (V, Q, q_0, F, \delta)$ be a jumping $5' \rightarrow 3'$ WK automaton, where $V = \{a_1, \dots, a_n\}$. Following the computation of M on an input $w \in V^*$, let $o = (o_1, \dots, o_n)$ be the Parikh vector built by the processed (read) symbols from w: At first, for the starting configuration, set $o = \Psi_V(\varepsilon)$. For the following configurations, whenever M makes a $\oplus/\ominus$-reading step from some q to q' according to $q' \in \delta(q, u, v, s)$, set $o = o + \Psi_V(uv)$. Using the Parikh mapping of $L(M)$, we define $\Delta(o) = \{\sum_{i=1}^{n}(m_i - o_i) : (m_1, \dots, m_n) \in \Psi_V(L(M)),\ m_i \geq o_i,\ 1 \leq i \leq n\} \cup \{\infty\}$. Finally, we define the debt of the current configuration of M as $\min \Delta(o)$.

Definition

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1. We are counting the processed symbols in the Parikh Vector $o = (o_1, \ldots, o_n)$.
2. The debt of the current configuration of M is the minimum number of symbols that we need to add to o so that it matches some Parikh vector from $\Psi_V(L(M))$.

Example automaton

Let $V = \{a, b, c\}$. Assume that there is a jumping $5' \to 3'$ WK automaton $M = (V, Q, q_0, F, \delta)$ such that $L(M) = \{a^n b^n c^n : n \geq 0\}$.

Therefore, $\Psi_V(L(M)) = \{(n, n, n) : n \geq 0\}$.

Example steps

$$
\begin{array}{lll}
(s, \oplus, \varepsilon, aabbcc, \varepsilon) \curvearrowright & o = (0,0,0) & \min \Delta(o) = 0 \\
(?, \oplus, \#, abbcc, \varepsilon) \curvearrowright & o = (1,0,0) & \min \Delta(o) = 2 \\
(?, \oplus, \#a, bbcc, \varepsilon) \curvearrowright & o = (1,0,0) & \min \Delta(o) = 2 \\
(?, \oplus, \#a\#, bc, \#) \curvearrowright & o = (1,1,1) & \min \Delta(o) = 0 \\
(?, \ominus, \#a\#\#, \varepsilon, \#\#) \curvearrowright & o = (1,2,2) & \min \Delta(o) = 1 \\
(?, \ominus, \#a, \varepsilon, \varepsilon) \curvearrowright & o = (1,2,2) & \min \Delta(o) = 1 \\
(?, \ominus, \#, \varepsilon, \varepsilon) \curvearrowright & o = (2,2,2) & \min \Delta(o) = 0 \\
(?, \ominus, \#, \varepsilon, \varepsilon) \curvearrowright & o = (2,2,2) & \min \Delta(o) = 0 \\
(?, \ominus, \varepsilon, \varepsilon, \varepsilon) \curvearrowright & o = (2,2,2) & \min \Delta(o) = 0
\end{array}
$$

Debt lemma

Let L be a language, and let $M = (V, Q, q_0, F, \delta)$ be a jumping $5' \rightarrow 3'$ WK automaton. If $L(M) = L$, M accepts all $w \in L$ using only configurations that have their debt bounded by some constant k for M.

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Example automaton

$$M = (\{a, b\}, \{s\}, s, \{s\}, \delta)$$

where δ:

$$\delta(s, a, b, \oplus) = \{s\}$$
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$$L(M) = \{w : w \in \{a, b\}^*, \ |w|_a = |w|_b\}$$

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$k = 0$ is sufficient ☺

You can go to Bonus for the proof.

Theorem

There is no jumping $5' \rightarrow 3'$ WK automaton M such that $L(M) = \{a^n b^n c^n : n \geq 0\}$.

FIT

Theorem

There is no jumping $5' \to 3'$ WK automaton M such that $L(M) = \{a^n b^n c^n : n \geq 0\}$.

Proof (1/3).

Basic idea. Considering any sufficiently large constant k, we show that M cannot process all symbols of $a^{10k} b^{10k} c^{10k}$ using only configurations that have their debt bounded by k.

Formal proof. (sketch) By contradiction. Let $L = \{a^n b^n c^n : n \geq 0\}$, and let $M = (V, Q, q_0, F, \delta)$ be a jumping $5' \to 3'$ WK automaton such that $L(M) = L$.

Consider any k such that $k > \max\{|uv| : \delta(q, u, v, s) \neq \emptyset,\ u, v \in V^*\}$.

Represent the debt of the configuration as $\langle d_a, d_b, d_c \rangle$.

For all traversed configurations must hold $d_a + d_b + d_c \leq k$.

Let $w = a^{10k} b^{10k} c^{10k}$.

Theorem

There is no jumping $5' \rightarrow 3'$ *WK automaton* M *such that* $L(M) = \{a^n b^n c^n : n \geq 0\}$.

Proof (2/3).

First, we explore the maximum number of symbols that M can read from w before the heads meet. Starting from $(q_0, \oplus, \varepsilon, w, \varepsilon)$ $\langle 0, 0, 0 \rangle$ and until the position $\ominus$ is reached. Consider the optimal reading strategy to process the maximum number of symbols from $a^{10k} b^{10k} c^{10k}$:

1. M processes (with multiple steps) a^k and c^k and reaches $\langle 0, k, 0 \rangle$,
2. M reads l symbols together in one step (balanced number of a's, b's, and c's) while keeping $\langle 0, k, 0 \rangle$, $l < k$,
3. M processes b^{2k} and a^k (or c^k) and reaches $\langle 0, 0, k \rangle$ (or $\langle k, 0, 0 \rangle$).

No further reading is possible; this strategy processed $5k + l$ symbols.

Theorem

There is no jumping $5' \rightarrow 3'$ *WK automaton M such that* $L(M) = \{a^n b^n c^n : n \geq 0\}$.

Proof (3/3).

Second, when the heads meet, $a^{>4k} b^{>4k} c^{>4k}$ has yet to be processed. Consider one of the optimal reading strategies:

1. the heads are between b's and c's,
2. the debt of the current configuration is $\langle 0, k, 0 \rangle$,
3. M processes b^{2k} and c^k and reaches $\langle k, 0, 0 \rangle$.

No further reading is possible; this strategy processed $3k$ symbols.

M is not able to process more than $8k + l$ symbols; but the input contains $30k$ symbols. Consequently, there is no constant k that bounds the debt of configurations of M. $\square$

Theorem

There is no jumping $5' \rightarrow 3'$ WK automaton M such that $L(M) = \{w \in \{a, b, c\}^* : |w|_a = |w|_b = |w|_c\}$.

Proof (1/10).

...

Theorem

There is no jumping $5' \to 3'$ WK automaton M such that $L(M) = \{w \in \{a,b,c\}^* : |w|_a = |w|_b = |w|_c\}$.

Proof (1/10).

. . . **NO**

Proof.

$\Psi_V(\{w \in \{a,b,c\}^* : |w|_a = |w|_b = |w|_c\}) = \Psi_V(\{a^n b^n c^n : n \geq 0\})$

$w = a^{10k} b^{10k} c^{10k}$

Since the debt depends only on o and Ψ_V,
the proof is analogous. $\square$

JWKFA – Language Families (1/2)

- **JWK** is incomparable with **GJFA** and **JFA**.
- **JWK** and **CF** are incomparable.

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Restrictions

N stateless, i.e., with only one state: if $Q = F = \{q_0\}$

F all-final, i.e., with only final states: if $Q = F$

S simple (at most one head moves in a step)

1 1-limited (exactly one letter is being read in a step)

Further variations such as **NS**, **FS**, **N1**, and **F1** WK automata can be identified in a straightforward way by using multiple constraints.

JWKFA – Language Families (2/2)

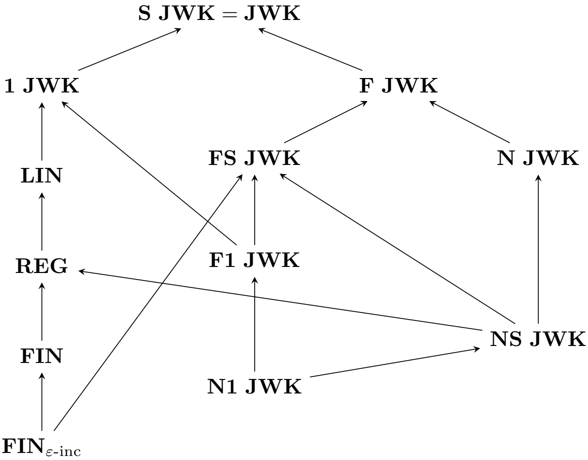

Figure: If there is an arrow from family X to family Y in the figure, then $X \subset Y$. Furthermore, if there is no path (following the arrows) between families X and Y, then X and Y are incomparable.

Conclusion

- The debt lemma was used only in JWKFAs so far.
- It can work in any automaton model that reads at least semi-continuously and where the steps depend only on the current state (not the previous readings, e.g., no stack).
- It can work in FAs.

Welcome at the end of this presentation!

And now Bonus. . .

Definition

Let $M = (V, Q, q_0, F, \delta)$ be a jumping $5' \rightarrow 3'$ WK automaton. Assuming some states $q, q' \in Q$ and a mutual position of heads $s \in \{\oplus, \ominus\}$, we say that q' is reachable from q and s if there exists a configuration (q, s, w_1, w_2, w_3) such that $(q, s, w_1, w_2, w_3) \curvearrowright^* (q', s', w_1', w_2', w_3')$ in M, $s' \in \{\oplus, \ominus\}$, $w_1, w_2, w_3, w_1', w_2', w_3' \in (V \cup \{\#\})^*$.

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Example automaton

$$M = (\{a\}, \{s, p, q\}, s, \{s\}, \delta)$$

where δ:

$$\delta(s, a, \varepsilon, \oplus) = \{p\}$$
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p is not reachable from s and $\ominus$

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Definition

Let $M = (V, Q, q_0, F, \delta)$ be a jumping $5' \rightarrow 3'$ WK automaton. Assuming some states $q, q' \in Q$ and a mutual position of heads $s \in \{\oplus, \ominus\}$, we say that q' is reachable from q and s if there exists a configuration (q, s, w_1, w_2, w_3) such that $(q, s, w_1, w_2, w_3) \curvearrowright^* (q', s', w_1', w_2', w_3')$ in M, $s' \in \{\oplus, \ominus\}$, $w_1, w_2, w_3, w_1', w_2', w_3' \in (V \cup \{\#\})^*$.

Example automaton

$$M = (\{a\}, \{s, p, q\}, s, \{s\}, \delta)$$

where δ:

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p is reachable from s and $\oplus$
p is not reachable from s and $\ominus$
q is reachable from s and $\oplus$
q is reachable from s and $\ominus$

Lemma

Let $M = (V, Q, q_0, F, \delta)$ be a jumping $5' \to 3'$ WK automaton, and let $q \in Q$ and $s \in \{\oplus, \ominus\}$ such that $f \in F$ is reachable from q and s. When $(q_0, \oplus, \varepsilon, w, \varepsilon) \curvearrowright^* (q, s, w_1, w_2, w_3)$ in M, $w \in V^*$, $w_1, w_2, w_3 \in (V \cup \{\#\})^*$, there exists $w' \in L(M)$ such that M starting with w' can reach q and s' ($s' = s$ or $s' = \ominus$) by using the same sequence of $\oplus/\ominus$-reading steps as in $(q_0, \oplus, \varepsilon, w, \varepsilon) \curvearrowright^* (q, s, w_1, w_2, w_3)$ and the rest of w' can be processed with a limited number of steps bounded by some constant k for M.

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1. On a string w with a sequence of steps we reach q and s.
2. A final state is reachable from q and s.
3. There exists some string w' such that we can reach q and s' with the same sequence of steps.
4. We can finish accepting w' with a limited number of additional steps.

Proof.

(idea)

(1) If f is reachable from q and s, there has to exist a sequence of state transitions from $(Q \times \{\oplus, \ominus\})^+$ such that $(p_0, s_0) \cdots (p_n, s_n)$, $p_0 = q$, $s_0 = s'$, $p_n = f$, $s_n = \ominus$, all pairs are unique, ...
This sequence has to be finite and bounded by some constant.

(2) Represent the complete sequence as $(p_0, s_0) \cdots (p_m, s_m)$. At first, for all $i = 0, \dots, m$, set $a_i = \varepsilon$, $b_i = \varepsilon$, $c_i = \varepsilon$, $d_i = \varepsilon$. If $p_{i+1} \in \delta(p_i, u_i, v_i, s_i)$ is used, then if $s_i = \oplus$, set $a_i = u_i$ and $b_i = v_i$, otherwise if $s_i = \ominus$, set $c_i = u_i$ and $d_i = v_i$.

(3) $w' = a_0 \cdots a_m d_m \cdots d_0 c_0 \cdots c_m b_m \cdots b_0 \in L(M)$ □

JWKFA – Why the Debt Lemma Holds (4/4)

Debt lemma

Let L be a language, and let $M = (V, Q, q_0, F, \delta)$ be a jumping $5' \to 3'$ WK automaton. If $L(M) = L$, M accepts all $w \in L$ using only configurations that have their debt bounded by some constant k for M.

Proof.

(idea) By contradiction.
(1) Assume that M does not accept all $w \in L$ exclusively using only configurations that have their debt bounded by some constant k for M, then M can accept some $w \in L$ over a configuration for which the debt cannot be bounded by any k.

(2) Due to previous lemmas, if final state is reachable there is some w' such that $\min \Delta(o)$ must be bounded by some constant.

(3) M cannot accept w over a state q and a mutual position of heads s from which no final state $f \in F$ is reachable.

(4) Consequently, when M accepts w, it must be done over configurations with the debt $\leq k$. But that is a contradiction. □