Planar graph theory

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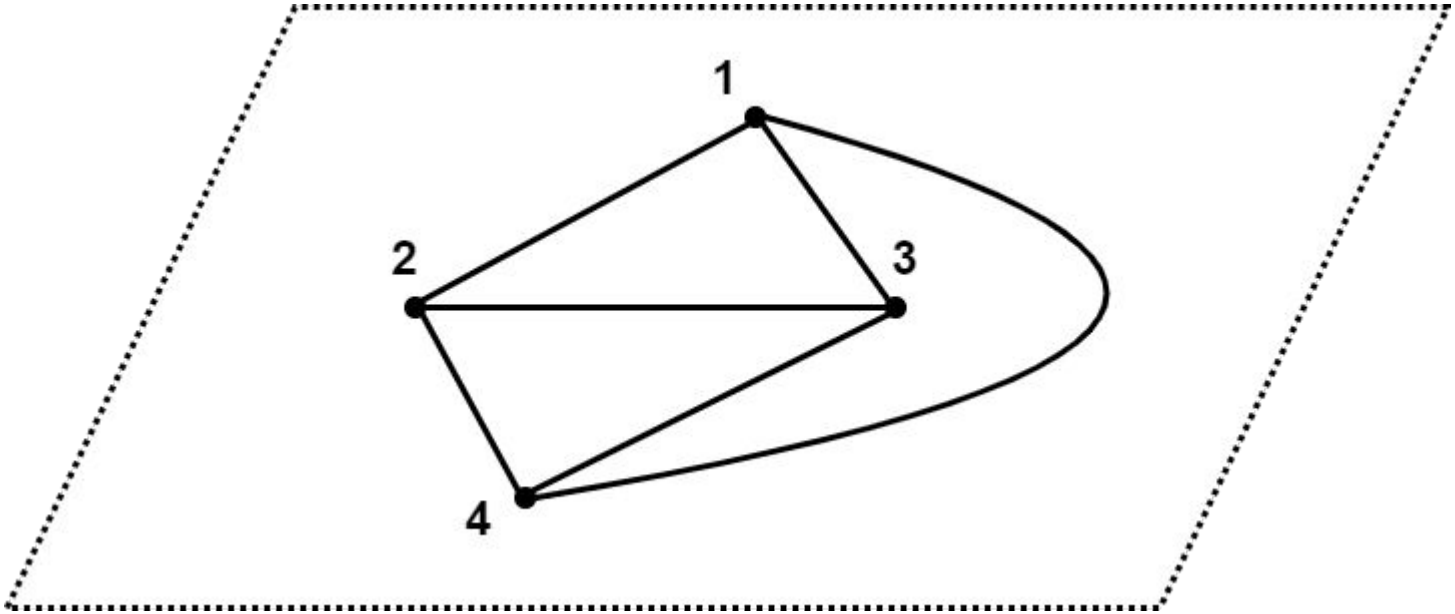
Contents

- Planarity and Duality
- Edge Addition Planarity Testing Algorithm
- Planarity Testing Based on PC-Trees
- Graph Drawing

Embeddability

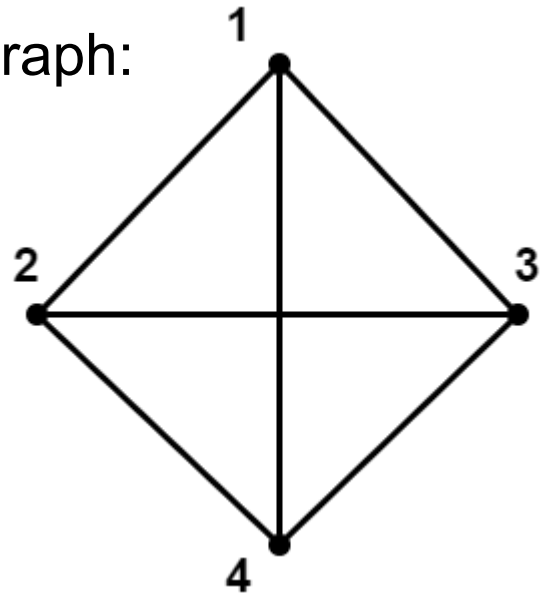
A graph G is said to be embeddable on a surface S if it can be drawn on S so that its edges intersect only at their end vertices.



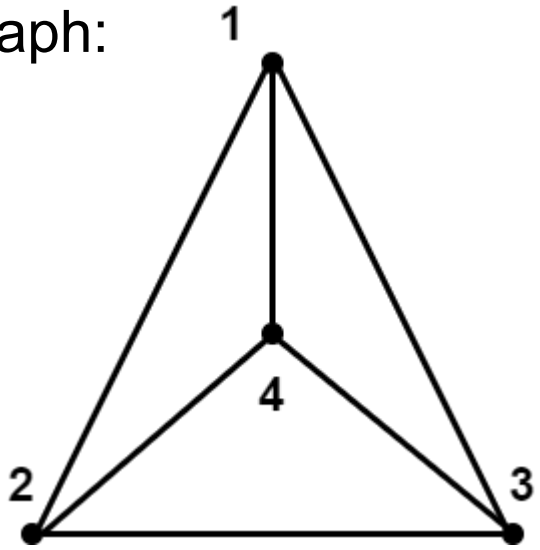
Planarity

A graph is said to be planar if it can be embedded on a plane. Such a drawing of a planar graph G is called a planar embedding of G or plane graph.

Planar graph:

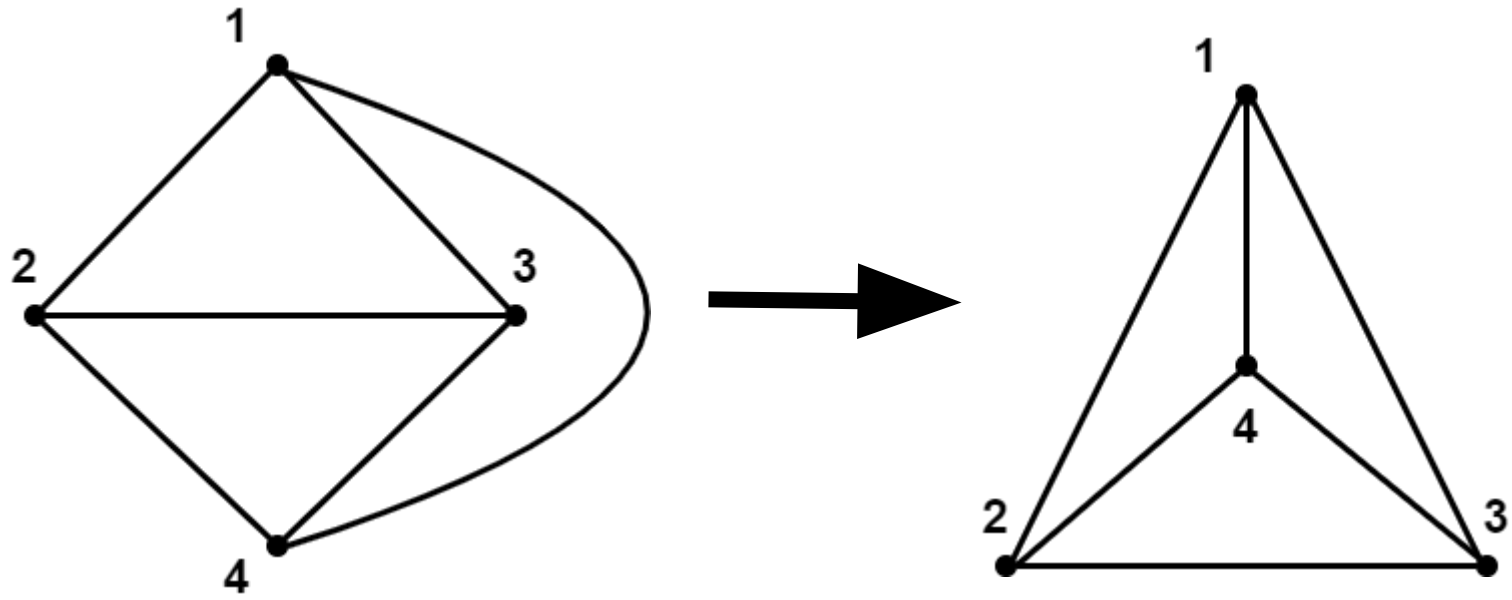


Plane graph:



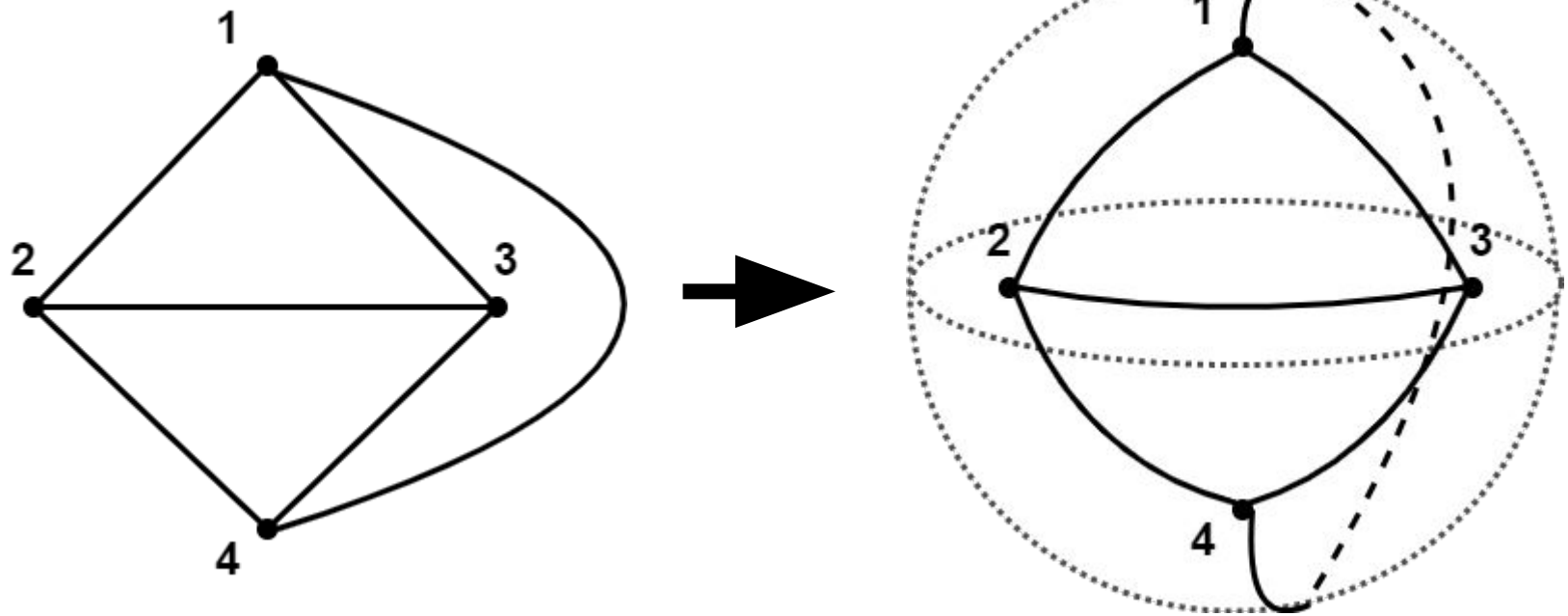
Straight lines theorem:

For every simple planar graph there exists a planar embedding in which all the edges of the graph can be drawn as straight line segments.



Sphere embedding theorem:

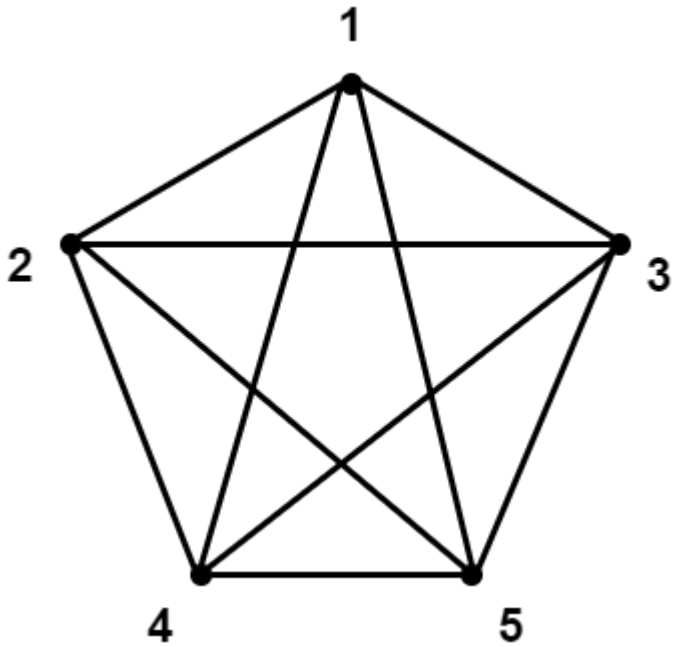
A graph G is embeddable on a plane if and only if G is embeddable on a sphere and vice versa.



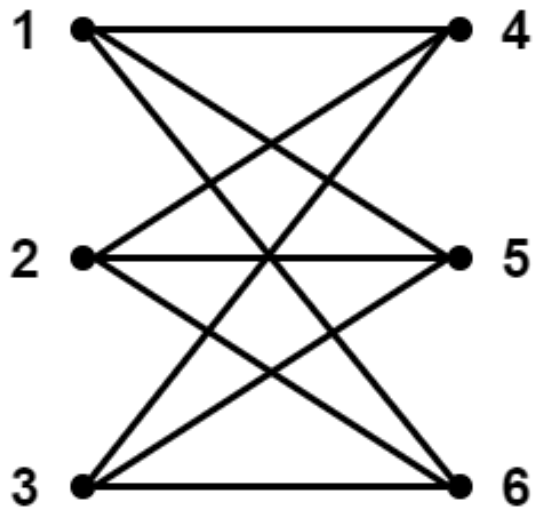
Kuratowski's graphs

There exist two basic nonplanar graphs called Kuratowski's graphs. One of these is K_5, the complete graph on five vertices, and the other is the complete bipartite graph $K_{3,3}$.

K_5:

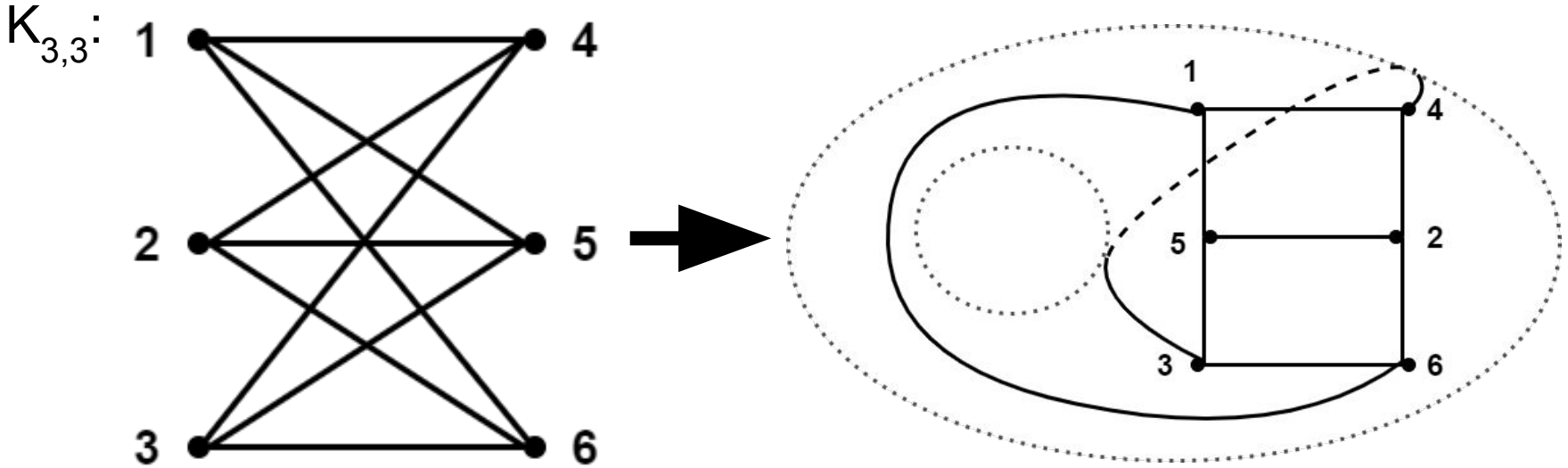


$K_{3,3}$:



Toroidal Embedding

The first and simplest non-spherical surface is the torus, obtained from the sphere by poking a hole through the center. It is possible to embed the $K_{3,3}$ into the torus.



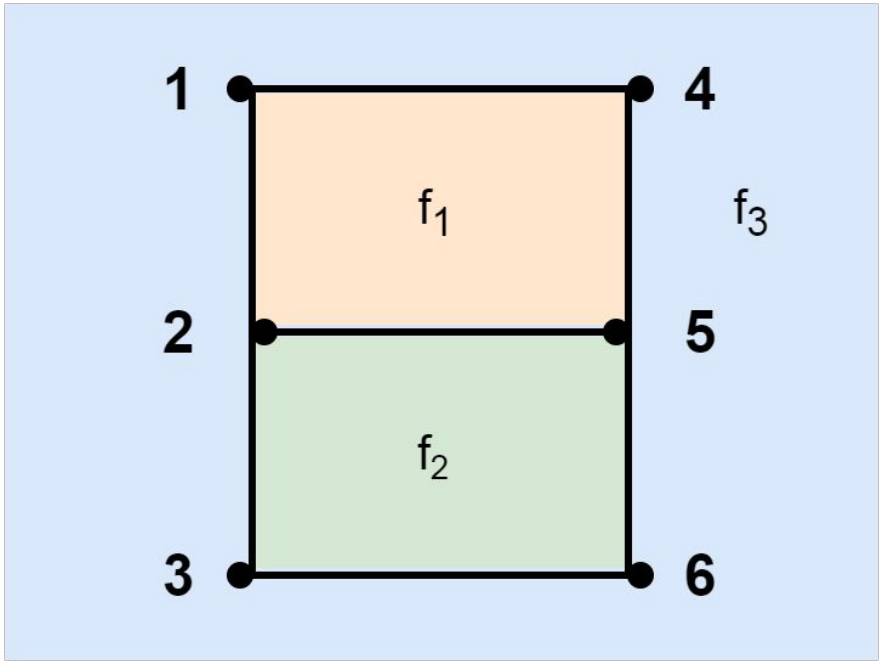
Toroidal Embedding of $K_{3,3}$ puzzle

There are 3 houses and each needs to be connected to water, gas, and electricity with no lines crossing.

[source: Tamás Görbe]

Regions

An embedding of a planar graph on a plane divides the plane into regions. A region is finite if the area it encloses is finite, otherwise it is infinite.



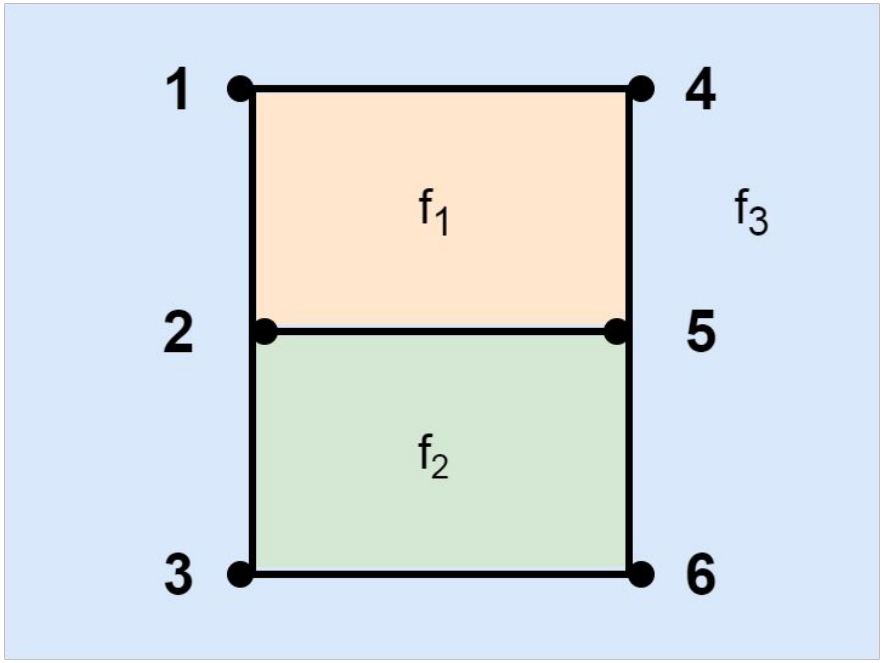
Euler's formula

If a connected planar graph G has *m* edges, *n* vertices, and *r* regions, then: $n - m + r = 2$

- n = 6
- m = 7
- r = 3

$n - m + r = 2$

$6 - 7 + 3 = 2$



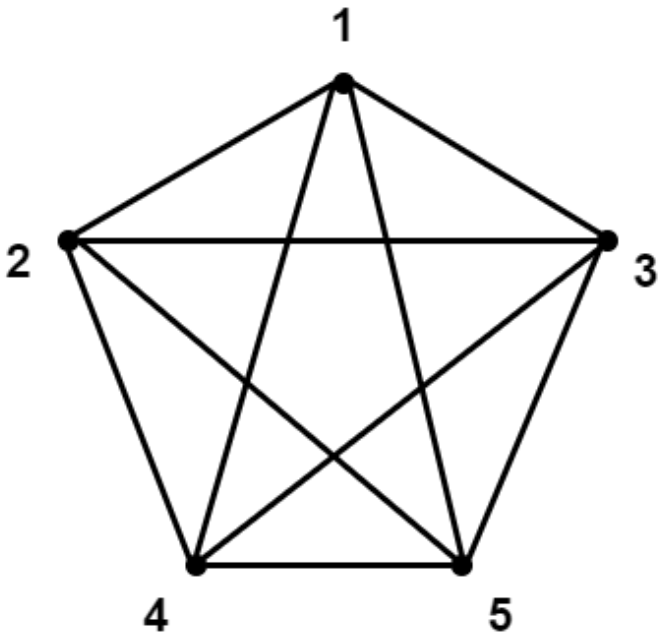
Euler's formula - implication

If a connected simple planar graph G has m edges and $n \geq 3$ vertices, then: $m \leq 3n - 6$

K_5 is nonplanar

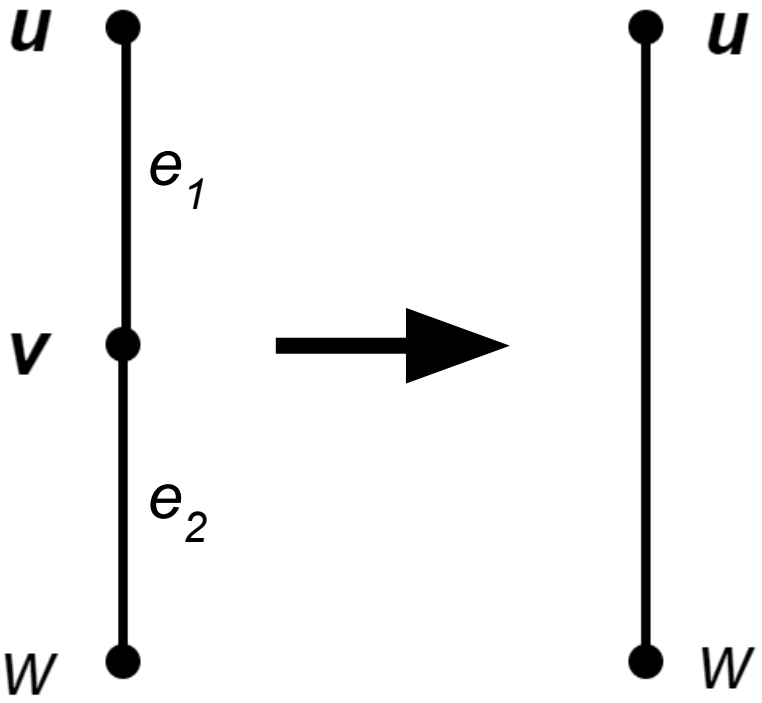
$n = 5$ and $m = 10$

$m = 10 \leq 3n - 6 = 9$



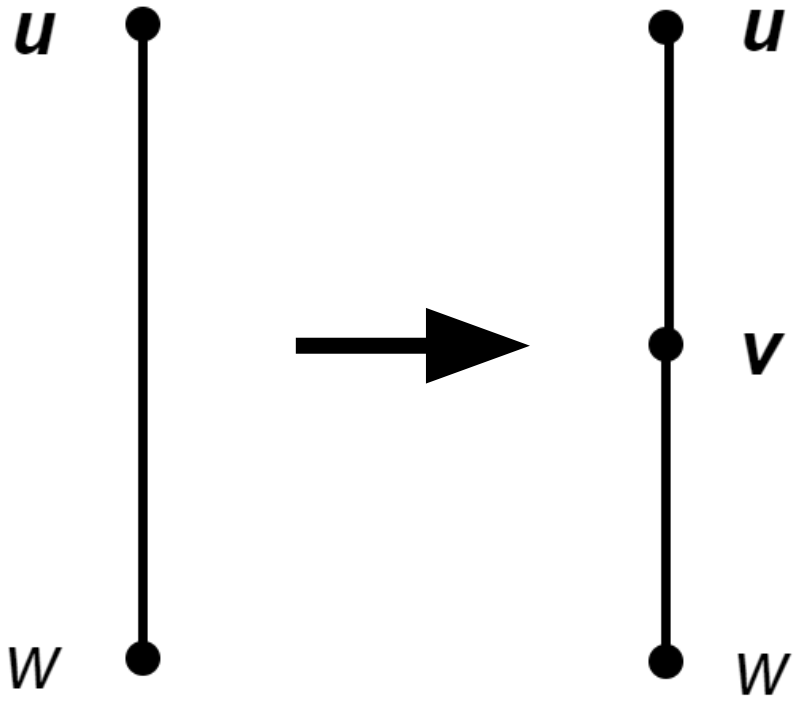
Series merger

Removal of degree 2 vertex v and replacing e_1 and e_2 by a simple edge (u, w) is called series merger.



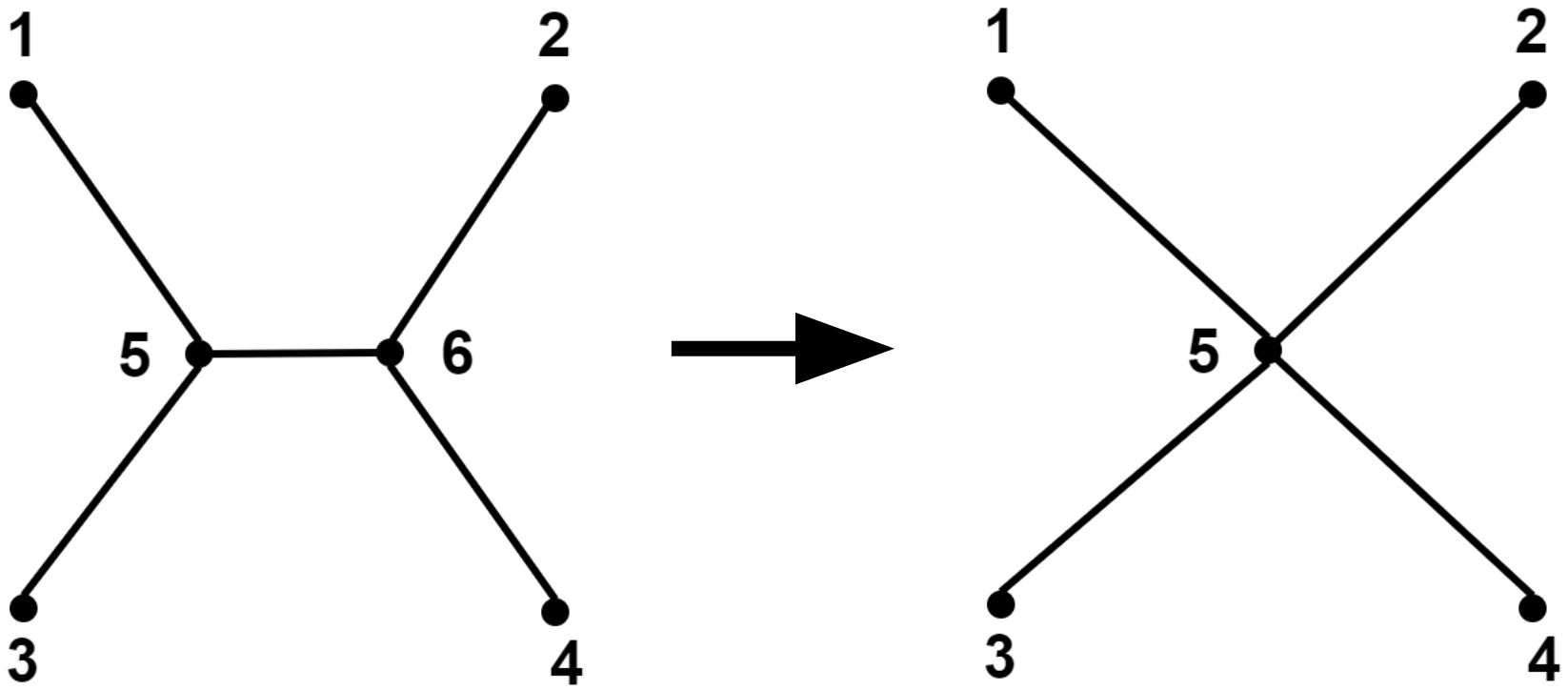
Series insertion

Adding a new vertex v on an edge (u, w) thereby creating the edges (u, v) and (v, w), is called series insertion.



Edge contraction

Edge contraction is an operation which removes an edge from a graph while simultaneously merging the two vertices.



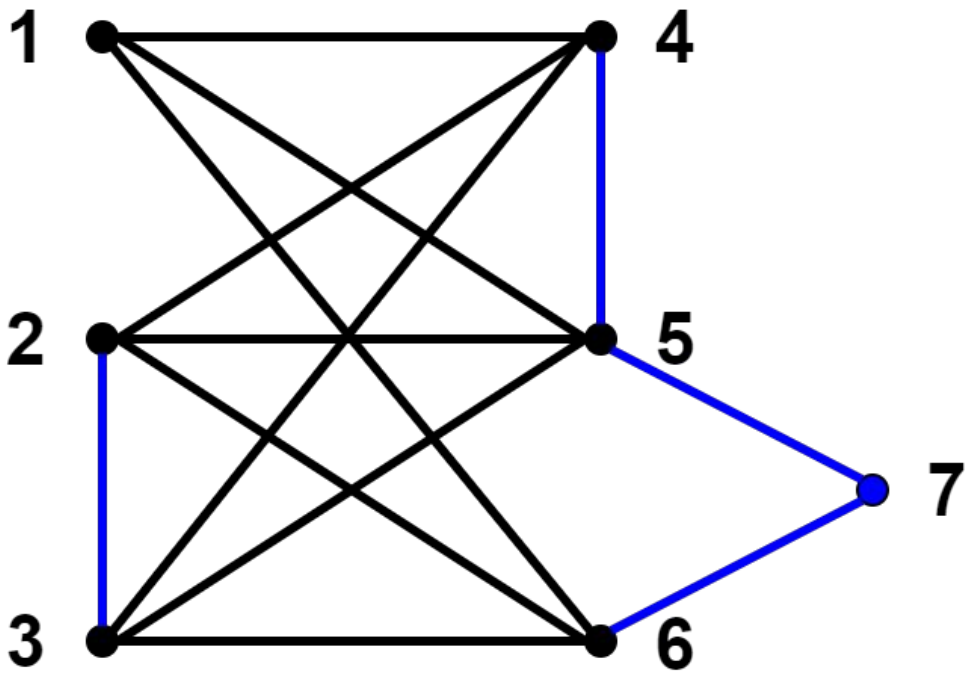
Planarity and Duality

Homeomorphic graphs:

- Two graphs are said to be homeomorphic if they are isomorphic or can be made isomorphic by repeated series insertions and/or mergers.
- If a graph G is planar, then any graph homeomorphic to G is also planar, that is, planarity of a graph is not affected by series insertions or mergers.

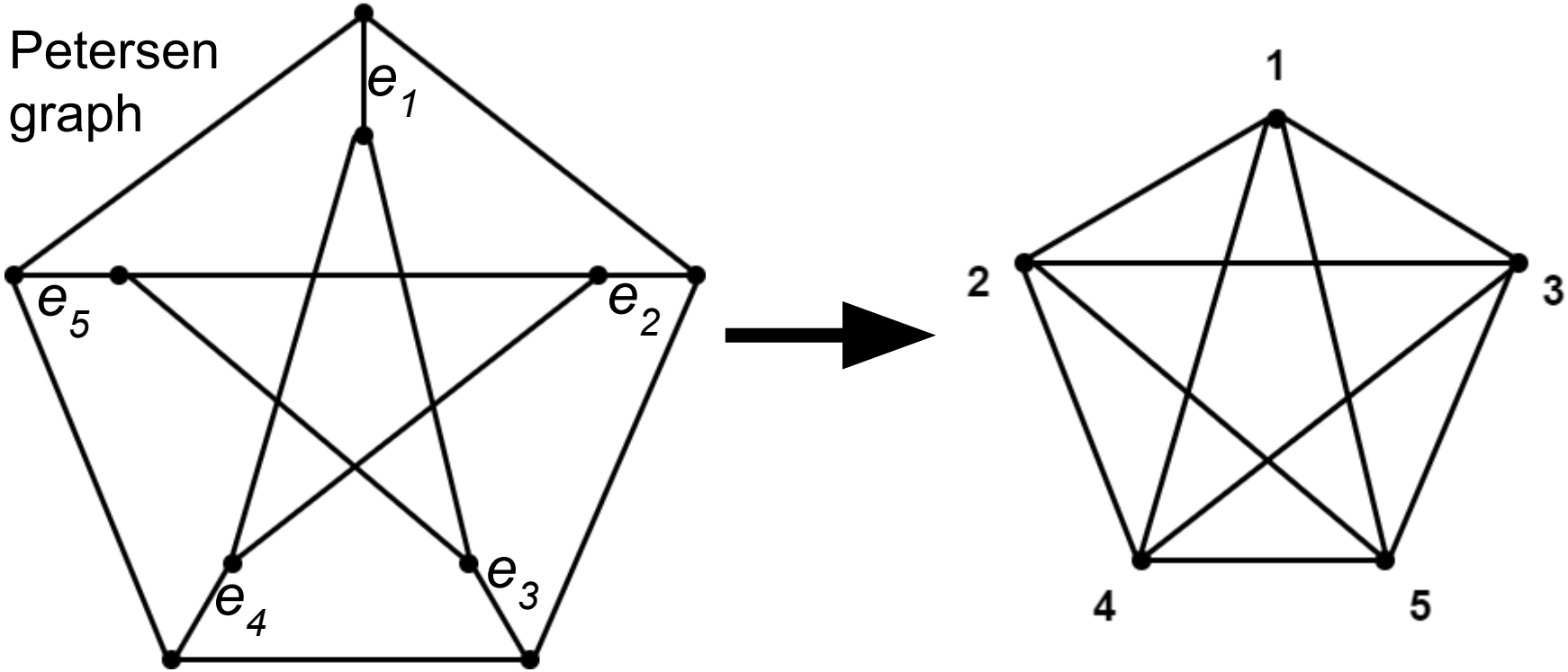
Harary's theorem:

A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.



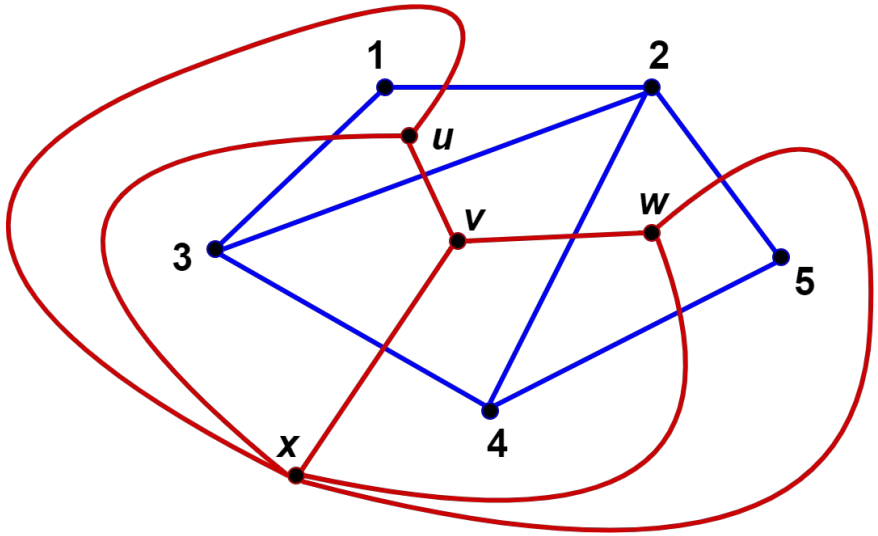
Wagner, Harary and Tutte:

A graph is planar if and only if it does not contain a subgraph contractible to K_5 or $K_{3,3}$.



Dual graph

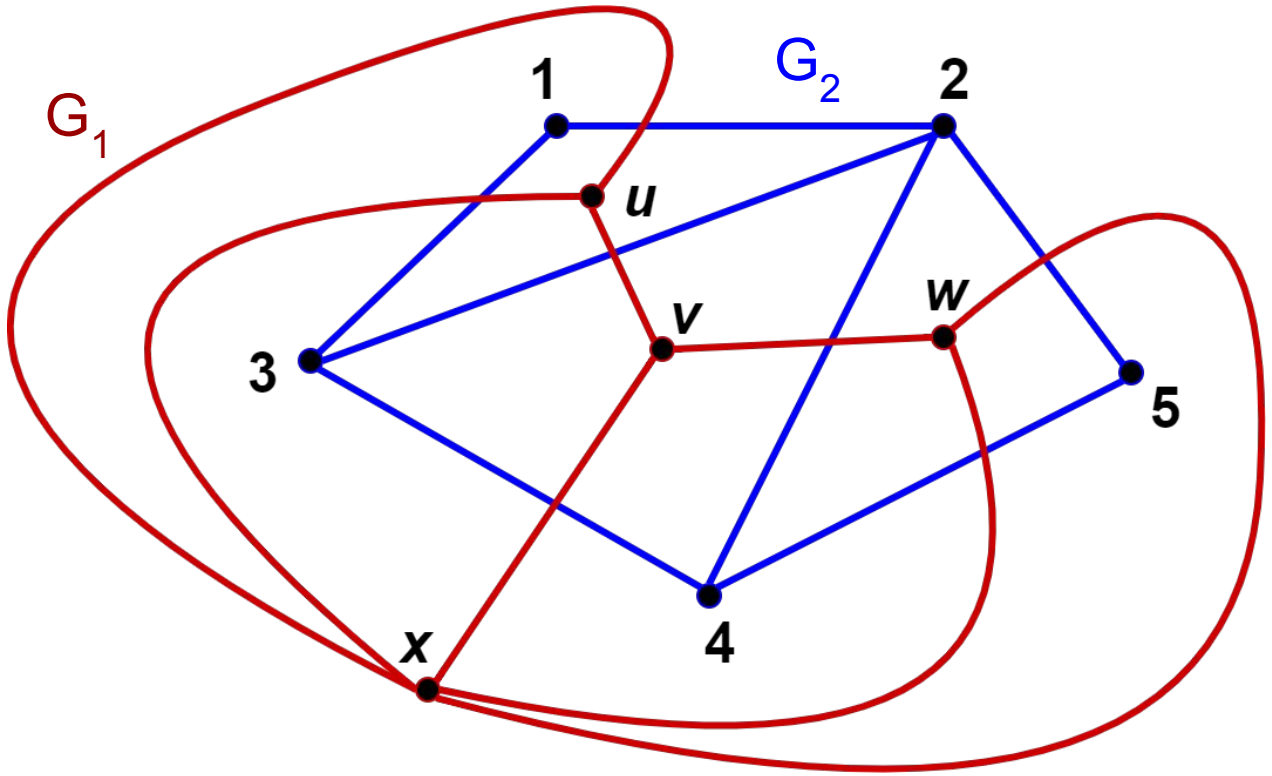
Dual graph of a plane graph G is a graph that has a vertex for each face of G. The dual graph has an edge whenever two faces of G are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge.



Planarity and Duality

Dual graphs theorem:

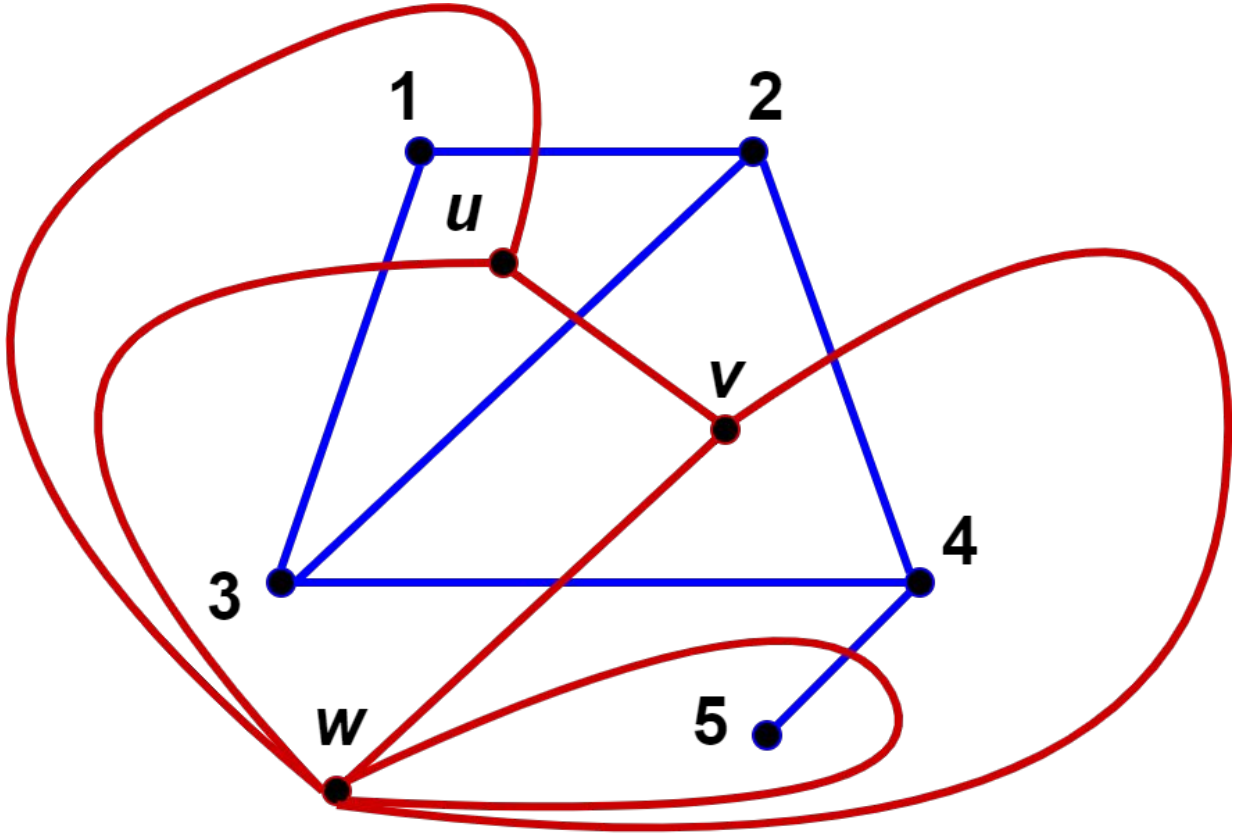
If G_1 is a dual of G_2, then G_2 is a dual of G_1.



Planarity and Duality

Theorem:

Every planar graph has a dual.

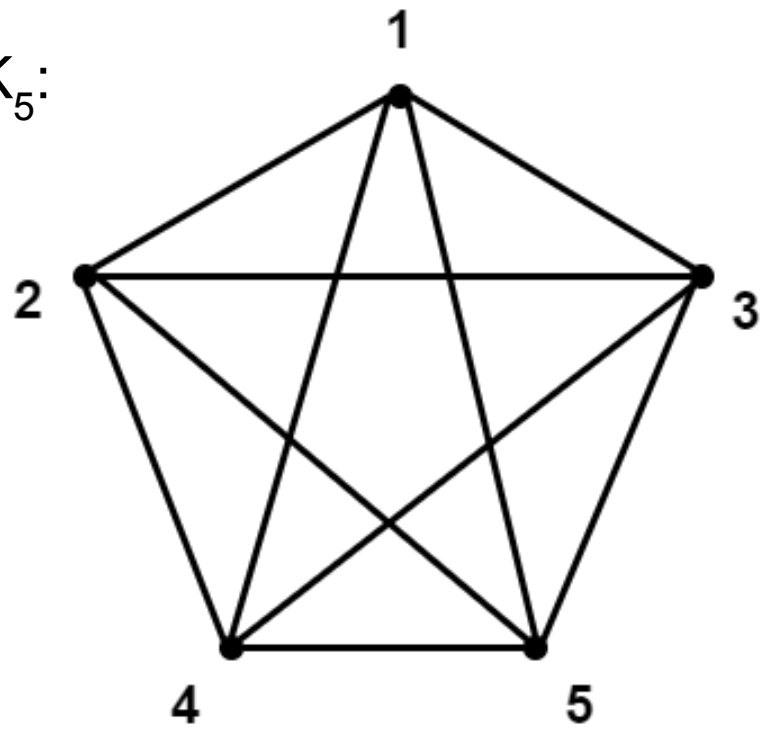


Planarity and Duality

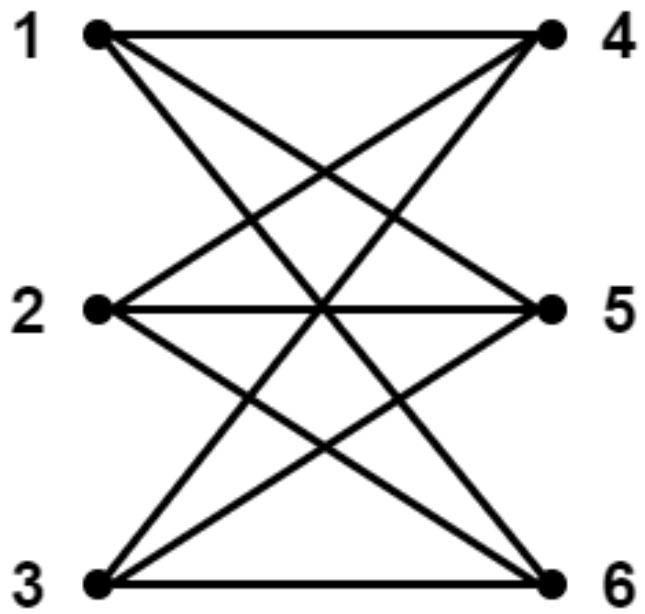
Lemma

K_5 and $K_{3,3}$ have no duals

K_5:



$K_{3,3}$:



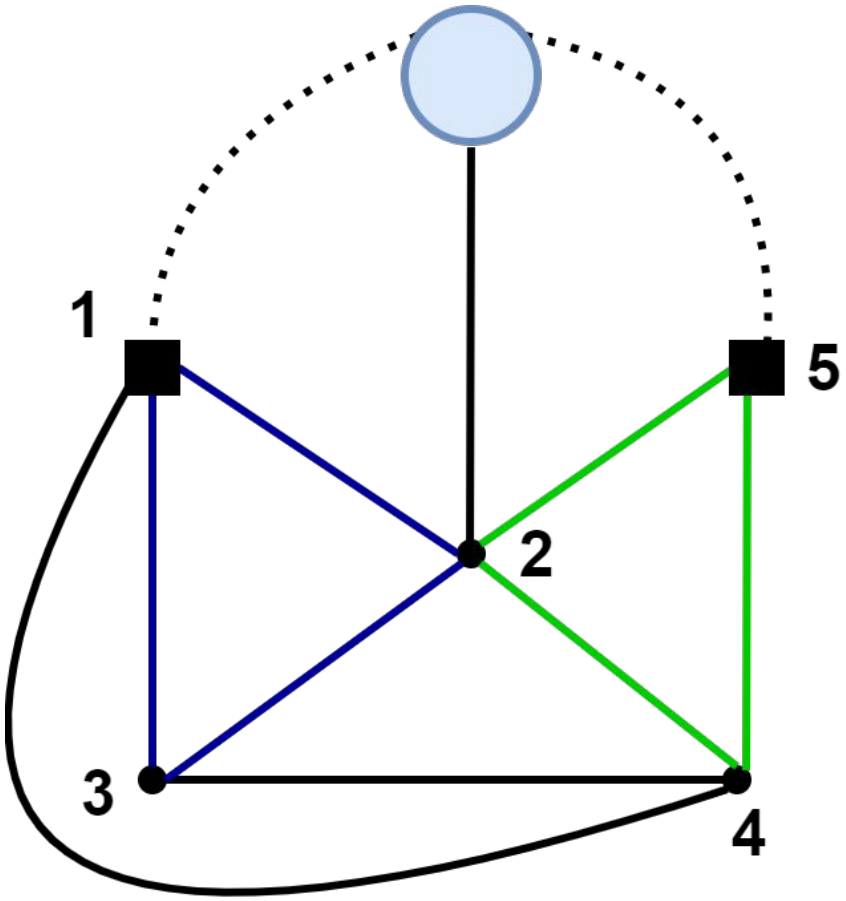
Edge Addition Planarity Testing Algorithm

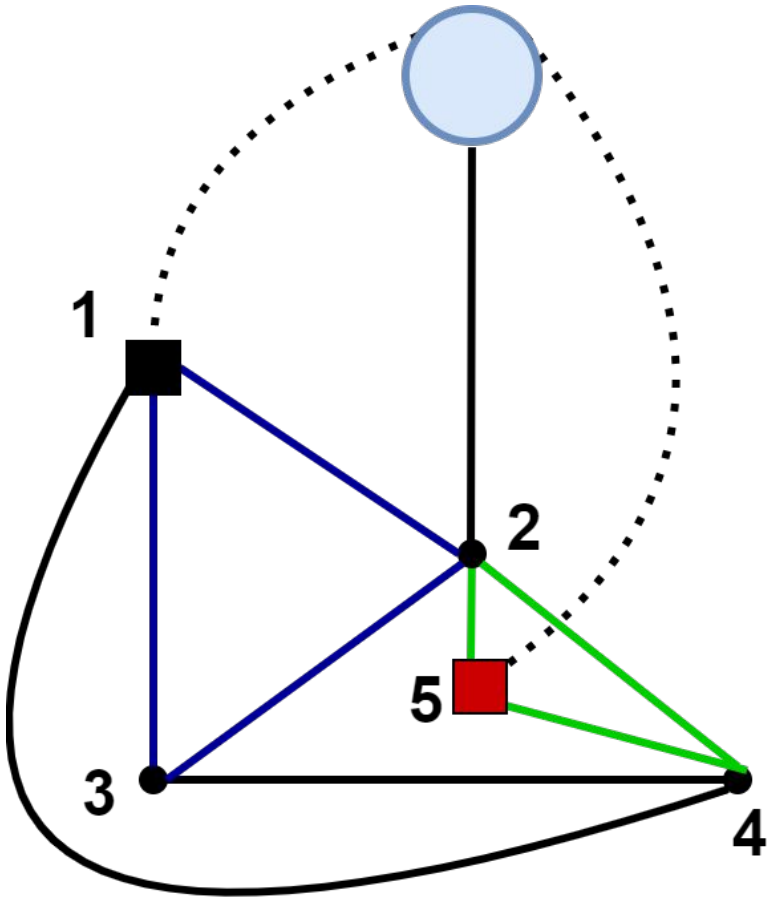
Algorithm: Edge Addition Planarity

1: Initialize embedding E based on input graph G

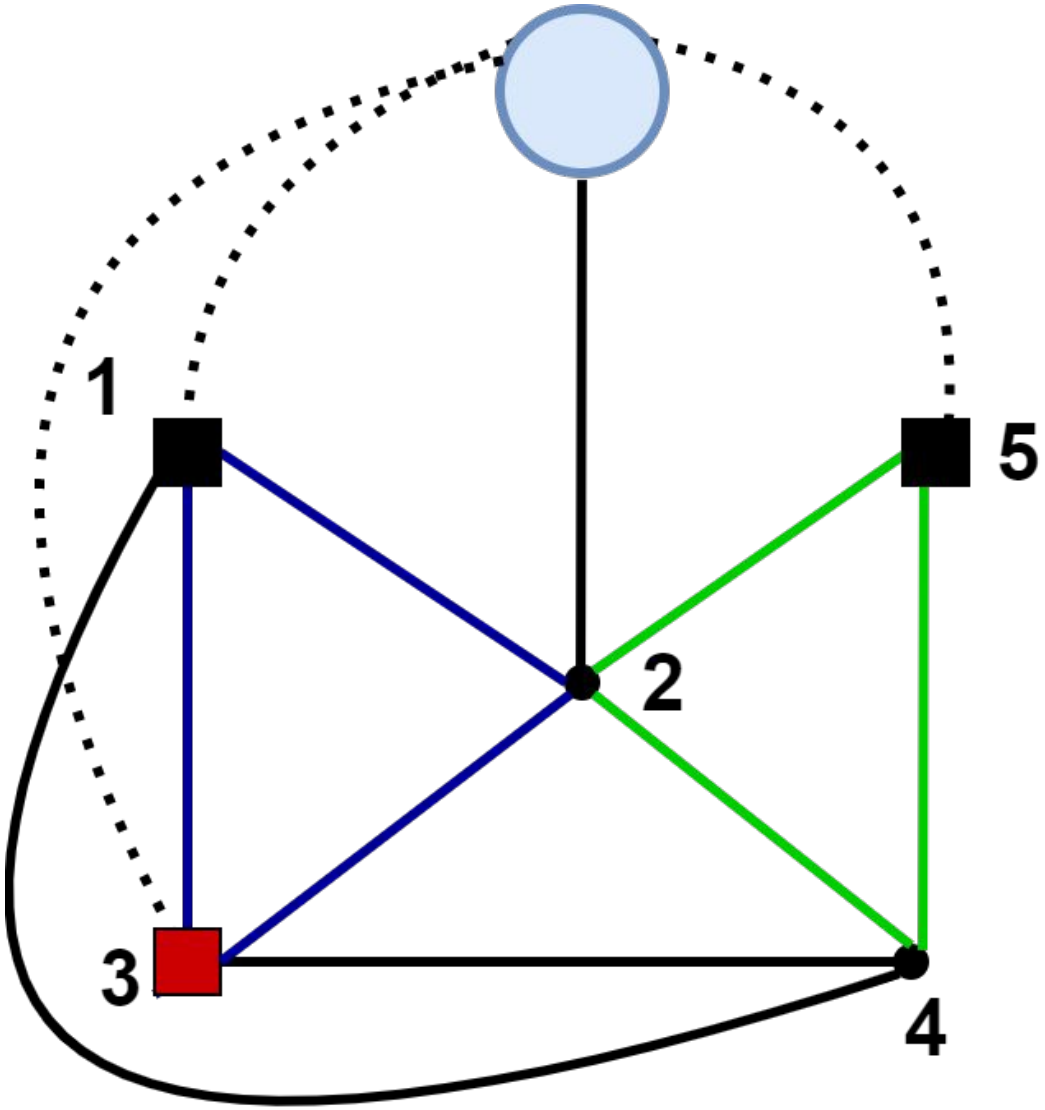
2: For each vertex v from $n - 1$ down to 0
3: Establish pertinence for step v within E
4: For each successive DFS child c of v
5: Embed the tree edge (v, c) as a singleton biconnected component (vc, c)
6: Perform Walkdown to embed back edges from vc to descendants of c
7: if any back edge from v to a descendant of c was not embedded
8: Isolate planarity obstruction and return NONPLANAR
9: Postprocess planar embedding and return PLANAR

Edge Addition Planarity Testing Algorithm





Edge Addition Planarity Testing Algorithm

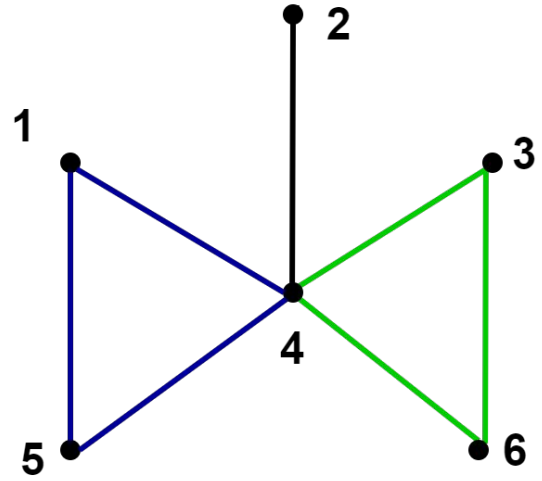


Planarity Testing Based on PC-Trees

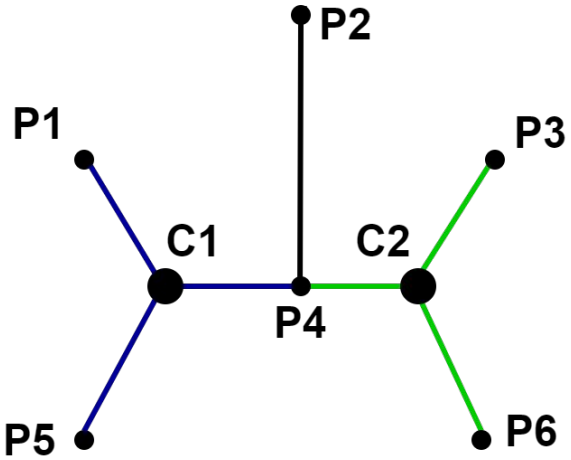
Shih and Hsu algorithm

- Improved vertex addition approach
- Runs in linear time
- Uses structure called PC-tree (contains P and C nodes)
- P-node is an original node of the graph
- C-node represents a biconnected component

Graph:



PC-Tree:



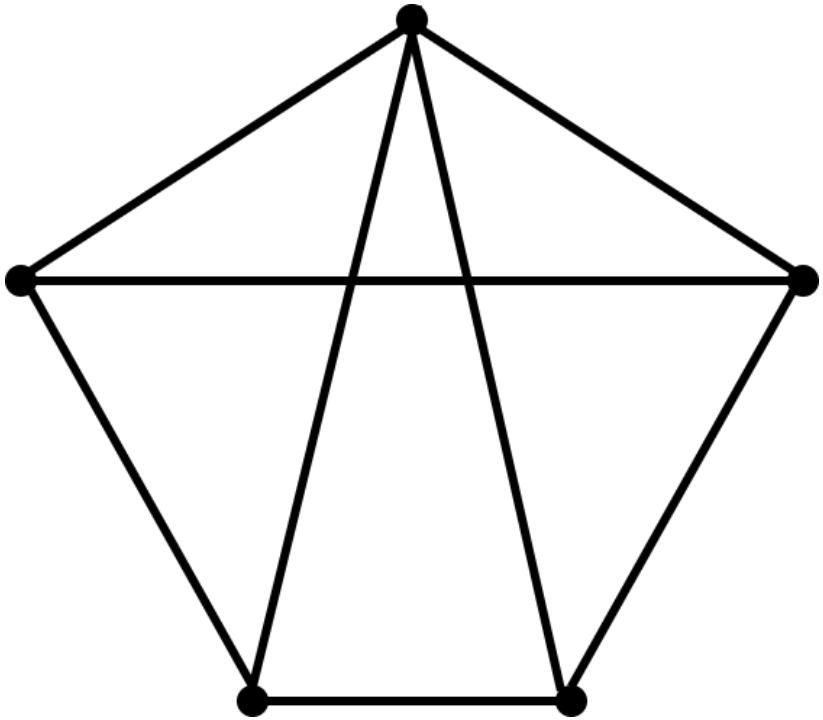
Graph drawing

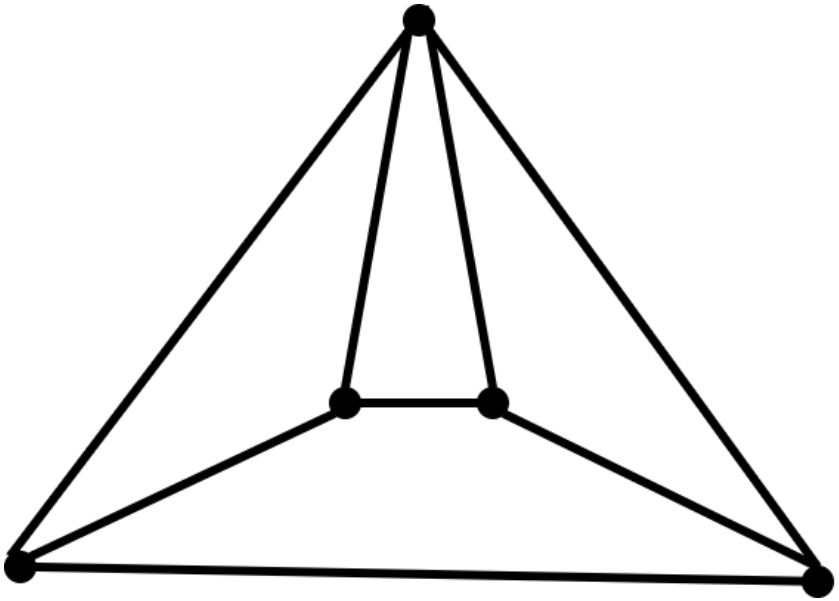
A drawing of a graph or network diagram is a pictorial representation of the vertices and edges of a graph. In other words it is a diagram consisting of a collection of objects corresponding to the vertices of the graph together with some line segments corresponding to the edges connecting the objects.

- Visualization of information represented by the graph
- Used from ancient time to represent abstract things like ideas and concepts as well as concrete things like maps
- Represents information modeled as objects and relationships

Quality measures

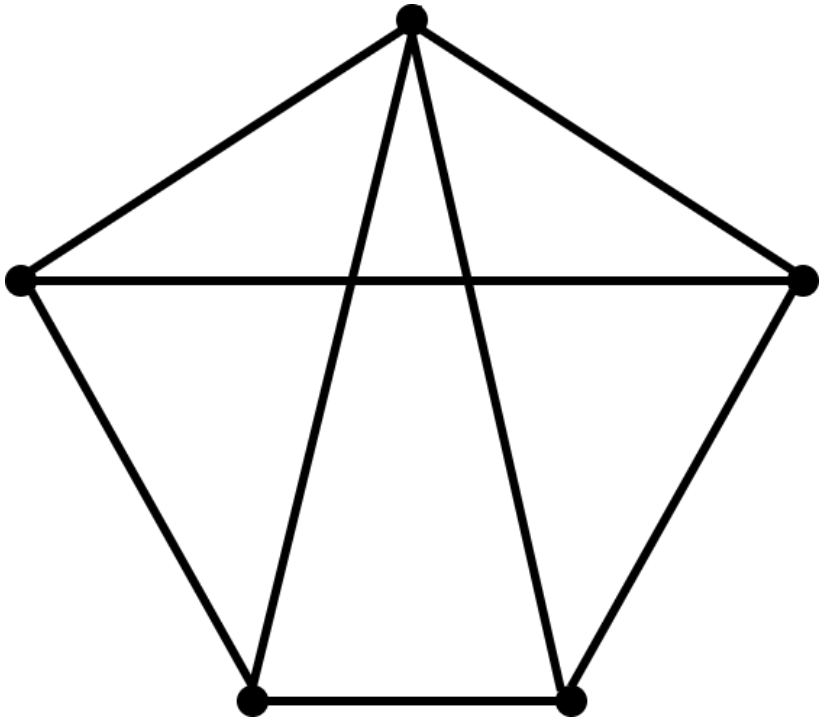
The **crossing number** of a drawing is the number of pairs of edges that cross each other.

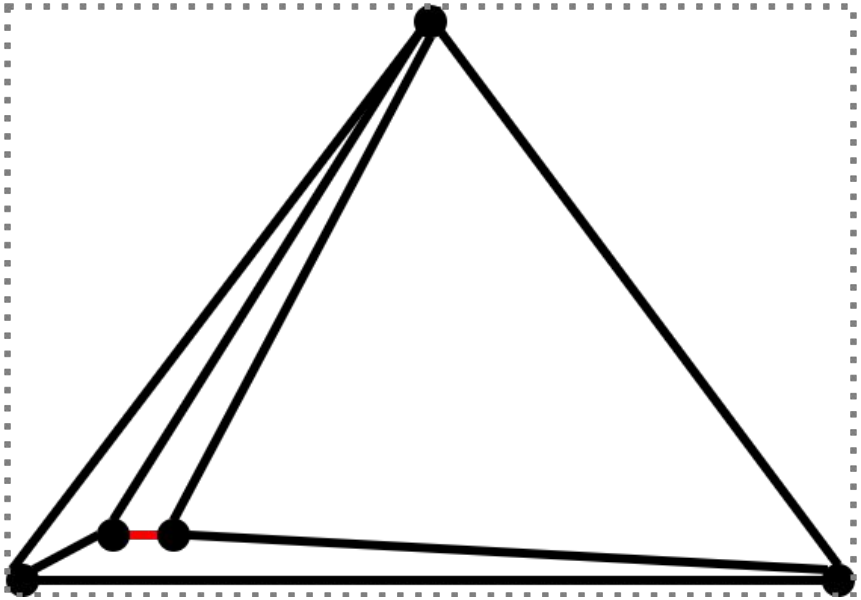




Quality measures

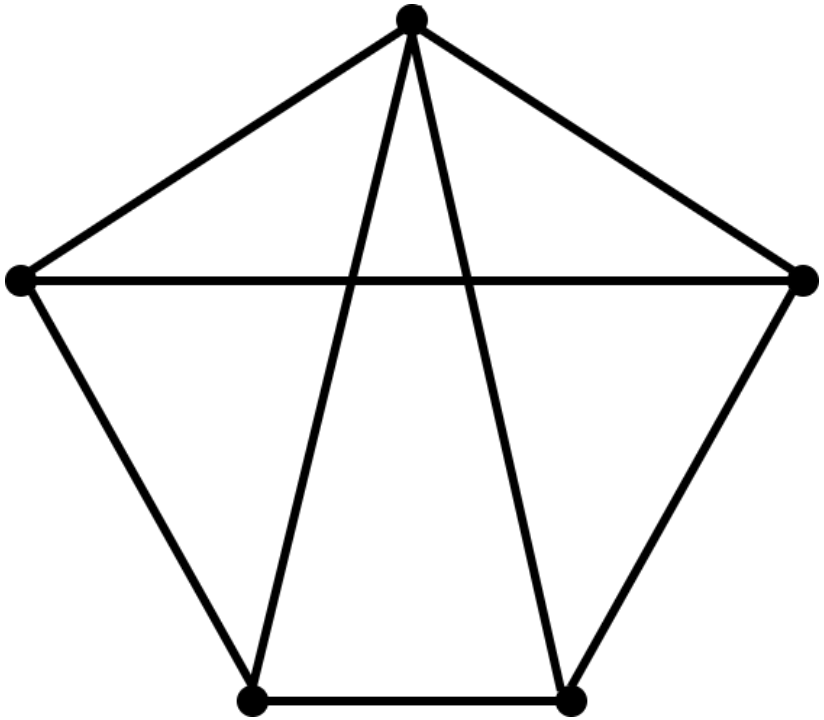
The **drawing area** is the size of its smallest bounding box, relative to the closest distance between any two vertices.

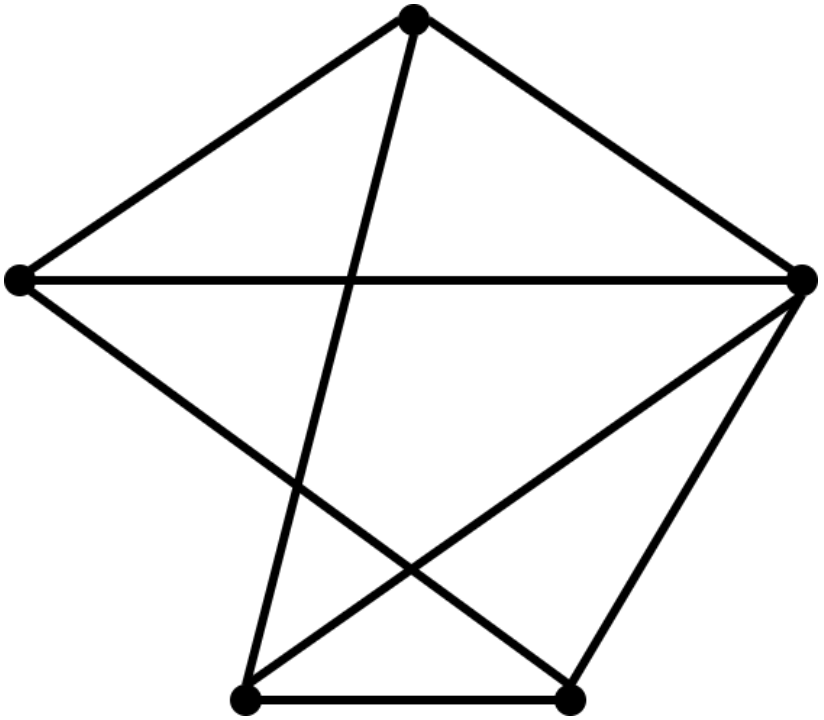




Quality measures

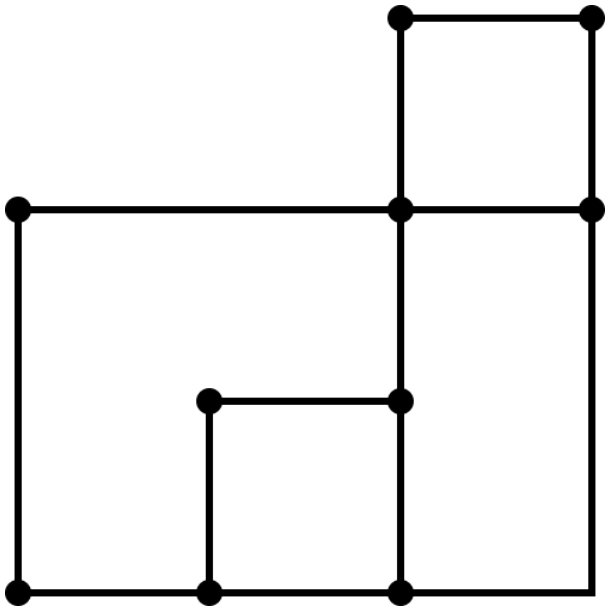
The **drawing symmetry** is the problem of finding symmetry groups within a given graph and drawing it.





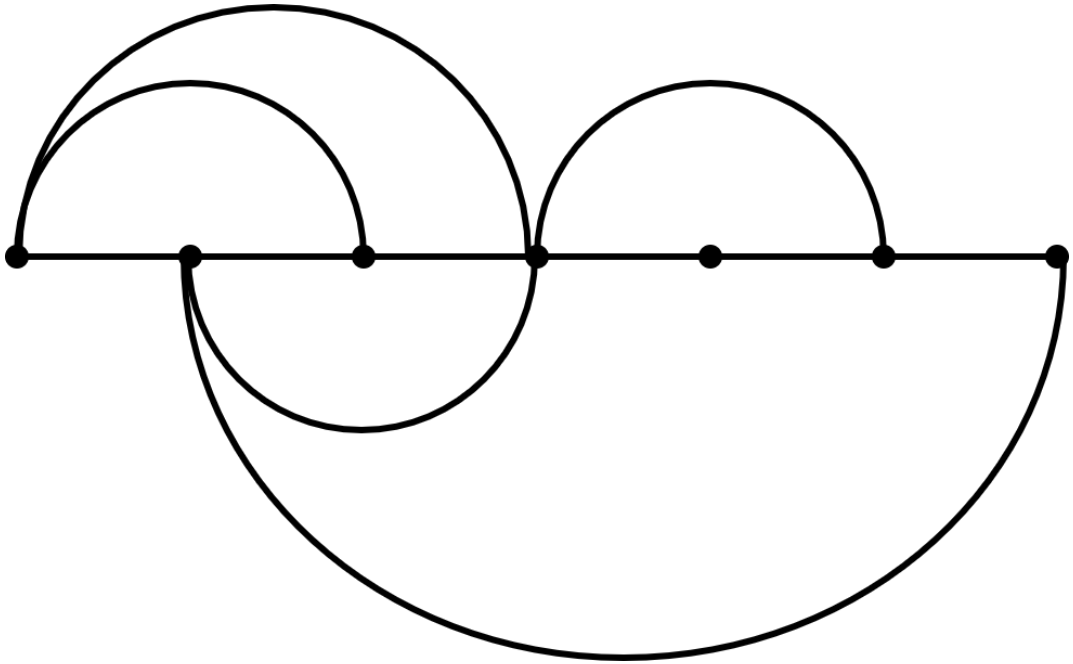
Drawing Styles

An **orthogonal drawing** is a drawing of a plane graph in which each edge is drawn as a chain of horizontal and vertical line segments.



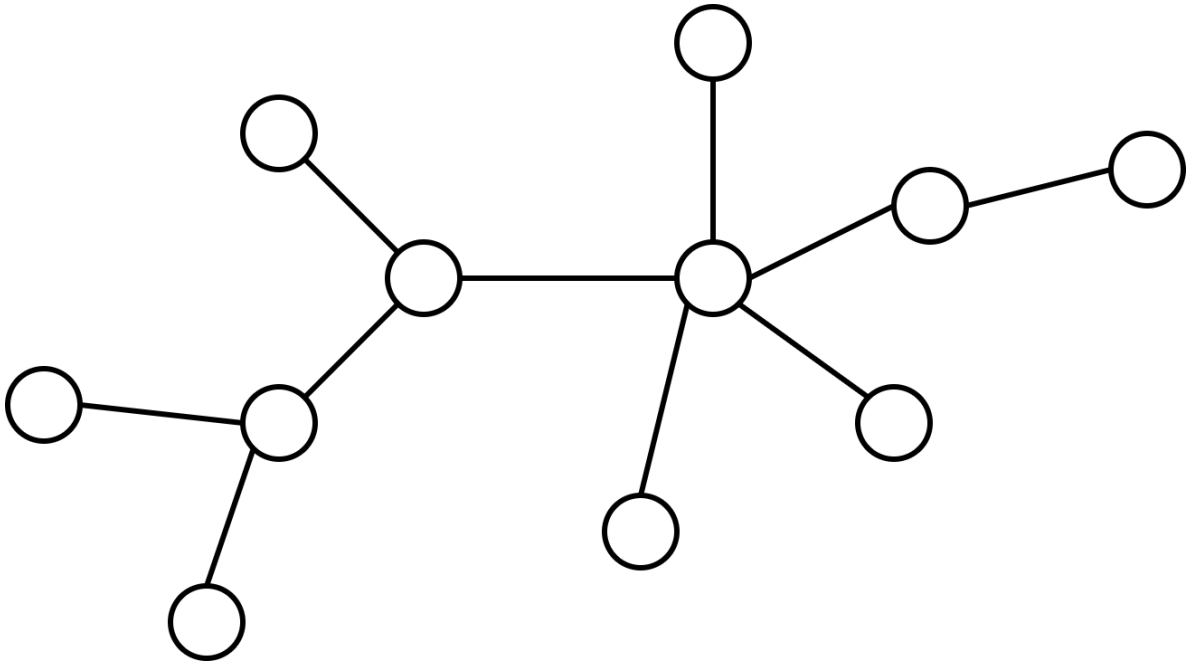
Drawing Styles

An arc diagram is graph drawing, in which the vertices of a graph are placed along a line in the Euclidean plane, with edges being drawn as semicircles.



Drawing Styles

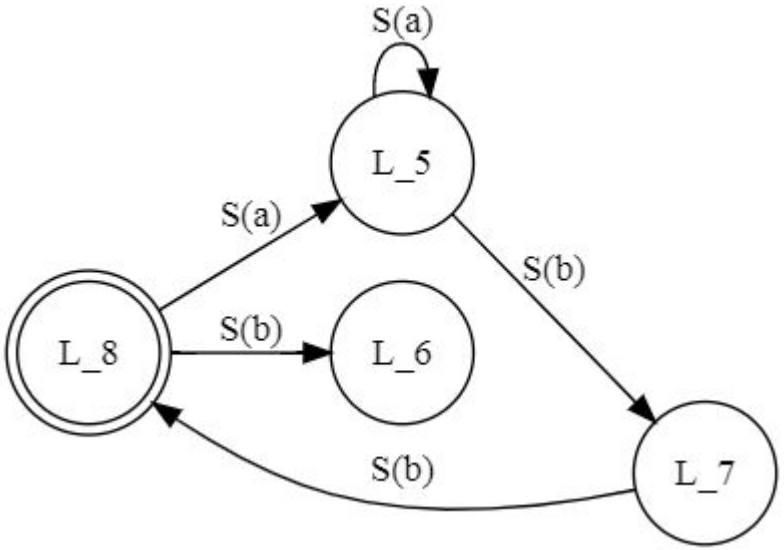
In **force-based layout** systems, the graph drawing software modifies an initial vertex placement by continuously moving the vertices according to a system of forces.



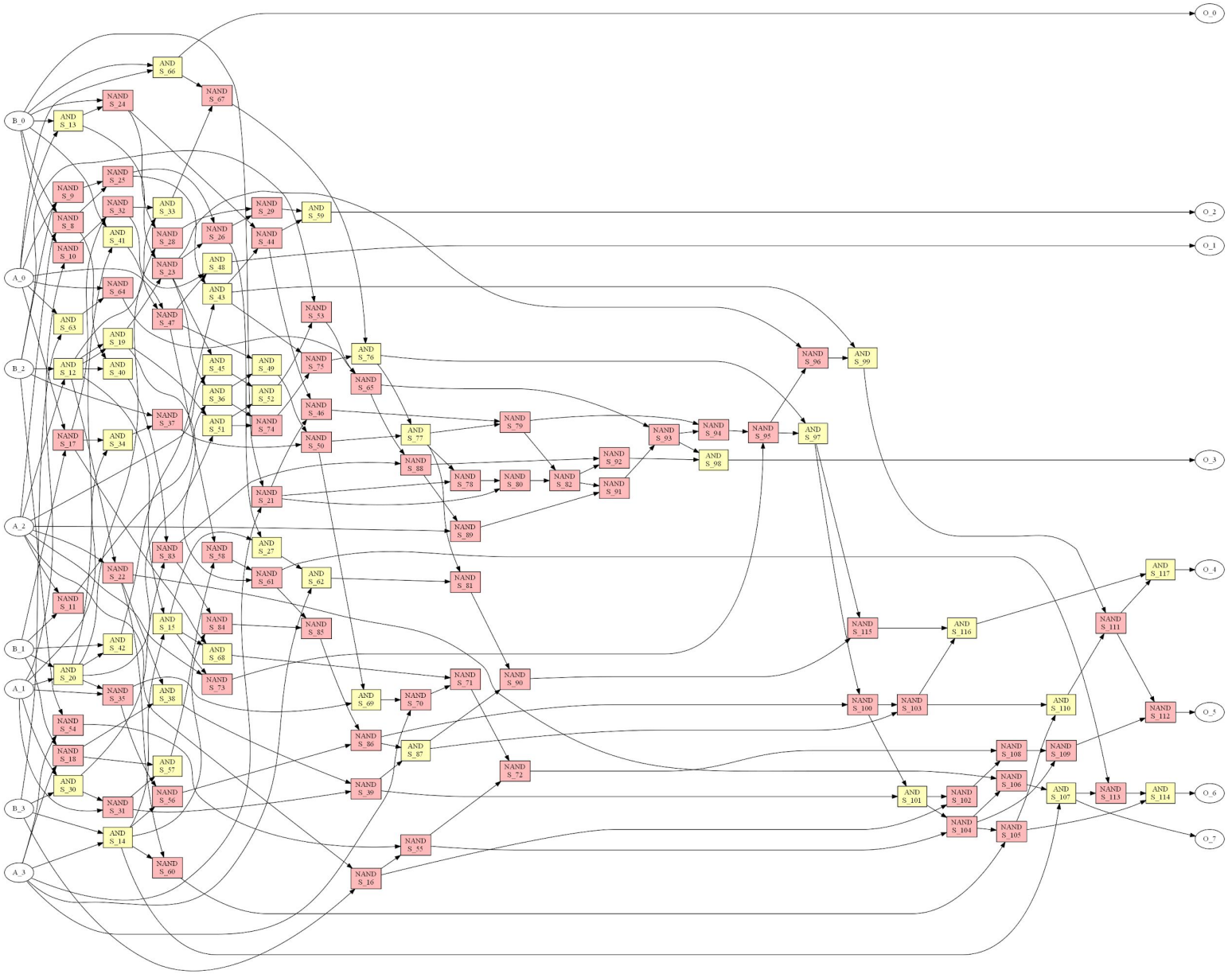
Graph visualization tool: Graphviz

Graphviz is open source graph visualization software. It has several main graph layout programs, that take descriptions of graphs in a simple text language, and make diagrams in several useful formats

```
digraph finite_state_machine {
        rankdir=LR;
        size="8,5"
        node [shape = doublecircle]; L_8;
        node [shape = circle];
        L_5 -> L_7 [ label = "S(b)" ];
        L_5 -> L_5 [ label = "S(a)" ];
        L_7 -> L_8 [ label = "S(b)" ];
        L_8 -> L_6 [ label = "S(b)" ];
        L_8 -> L_5 [ label = "S(a)" ];
}
```



Graph Drawing



Thank you for your attention!