Unlimitedly Deep Pushdown Automata and Their Computational Completeness

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- State grammar:

$$G = (V, W, T, P, S)$$

$$(p, A) \rightarrow (q, w)$$

- Example:

$$G = (\{S, A, B, C, a, b, c\}, \{s, p, q, f\}, \{a, b, c\}, P, S)$$

$$P = \{(s, S) \rightarrow (p, AC), (p, A) \rightarrow (q, aAb),$$
$$(p, A) \rightarrow (f, \epsilon), (q, C) \rightarrow (p, Cc), (f, C) \rightarrow (f, \epsilon)\}$$

$$(s, S) \Rightarrow (p, AC) \Rightarrow (q, aAbC) \Rightarrow$$
$$(p, aAbCc) \Rightarrow (f, abCc) \Rightarrow (f, abc)$$

- Deep pushdown automaton:

$$M = (Q, T, N, R, s, S, F)$$

$$mpA \rightarrow qw, m \in \mathbb{N}$$

- Example:

$$M = (\{s, q, p, f\}, \{a, b, c\}, \{S, A, \#\}, R, s, S, \{f\})$$

$$R = \{1sS \rightarrow qAA, 1qA \rightarrow paAb, 1qA \rightarrow fab,$$
$$2pA \rightarrow qAc, 1fA \rightarrow fc\}$$

$$(s, abc, S\#) \vdash (q, abc, AA\#) \vdash (p, abc, aAbA\#) \vdash$$
$$(q, abc, aAbAc\#) \vdash (f, abc, abAc\#) \vdash (f, abc, abc\#)$$

Definitions III

- Unlimitedly deep pushdown automaton

$$M = (Q, T, N, R, s, S, F)$$

$$pA \to qw$$

- Example:

$$M = (\{s, q, p, f\}, \{a, b, c\}, \{S, A, C\#\}, R, s, S, \{f\})$$

$$R = \{sS \to qAC, qA \to paAb, qA \to fab, \\ pC \to qCc, fC \to fc\}$$

$$(s, abc, S\#) \vdash (q, abc, AC\#) \vdash (p, abc, aAbC\#) \vdash \\ (q, abc, aAbCc\#) \vdash (f, abc, abCc\#) \vdash (f, abc, abc\#)$$

Idea

- State grammar / Unlimitedly deep pushdown automaton
- Construction
- Proof of same acceptance power

- Input: State Grammar $G = (V, W, T, P, S)$
- Output: Unlimitedly Deep PDA $M = (Q, T, N, R, s, S, W)$
- Method:

1. $Q = W \cup \{s\}, s \notin W$;
2. $\forall (p, S) \to (q, x) \in P, p, q \in W$, add $s\# \to pS\#$ to R;
3. $\forall (p, A) \to (q, x) \in P, p, q \in W, A \in N$, add $pA \to qx$ to R.

$$P = \{(p, S) \to (q, x), (r, S) \to (t, y), \ldots\}$$

$$R = \{s\# \to pS\#, s\# \to rS\#, pS \to qx, rS \to ty, \ldots\}$$

Unlimitedly Deep PDA $\rightarrow$ State Grammar

- Input: Unlimitedly Deep PDA $M = (Q, T, N, R, s, S, F)$
- Output: State Grammar $G = (V, W, T, P, S)$
- Method:

1. $W = Q \cup \{s'\}, s' \notin Q$;
2. $\forall sA \rightarrow qx \in R, q \in Q$, add $(s', S) \rightarrow (s, A)$ to P;
3. $\forall pA \rightarrow qx \in R, p, q \in Q, A \in N$, add $(p, A) \rightarrow (q, x)$ to P.

$$R = \{sA \rightarrow qx, \quad sB \rightarrow ty, \dots\}$$

$$P = \{(s', S) \rightarrow (s, A), (s', S) \rightarrow (s, B), (s, A) \rightarrow (q, x), (s, B) \rightarrow (p, y), \dots\}$$

Theorem

For every state grammar G, there exists an unlimitedly deep pushdown automaton M such that $L(G) = L(M)$.

Lemma

For every state grammar G, there exists an unlimitedly deep pushdown automaton M such that $L(G) \subseteq L(M)$.

Claim (1)

Let $(p, S) \Rightarrow^j (q, xz)$ in G, where $p, q \in W$, $x \in T^*$, and $z \in (NV^*)^*$. Then, $(p, xw, S\#) \vdash^* (q, w, z\#)$ in M, where $p, q \in Q$ and $w \in T^*$.

Basis. Let $j = 0$, $(p, S) \Rightarrow^0 (p, S)$, from the construction, we obtain $(p, w, S\#) \vdash^0 (p, w, S\#)$ in M.

Induction Hypothesis. Assume there is $i \geq 0$ such that Claim (1) holds true for all $0 \leq j \leq i$.

Induction Step.

$$(p, S) \Rightarrow^i (h, xuAv) \Rightarrow (q, xu\alpha v) \quad [(h, A) \rightarrow (q, \alpha)]$$

$$(p, xyw, S\#) \vdash^* (h, yw, uAv\#) \vdash (q, w, z\#) \quad [hA \rightarrow q\alpha]$$

$$y = \textit{max-prefix}(u\alpha v, T^*), z = \textit{max-suffix}(u\alpha v, NV^*)$$

Claim (2)

Let $(p, xw, S\#) \vdash^j (q, w, z\#)$ *in* M*, where* $p, q \in Q$*,* $x, w \in T^*$ *and* $z \in (NV^*)^*$*. Then,* $(p, S) \Rightarrow^* (q, xz)$ *in* G*, where* $p, q \in W$*.*

Basis. Let $j = 0$, $(p, w, S\#) \vdash^0 (p, w, S\#)$ in M, from the construction, we obtain $(p, S) \Rightarrow^0 (p, S)$ in G.

Induction Hypothesis. Assume there is $i \geq 0$ such that Claim (2) holds true for all $0 \leq j \leq i$.

Induction Step.

$$(p, xyw, S\#) \vdash^i (h, yw, uAv\#) \vdash (q, w, z\#) \quad [hA \rightarrow q\alpha]$$

$$(p, S) \Rightarrow^* (h, xuAv) \Rightarrow (q, xu\alpha v) \quad [(h, A) \rightarrow (q, \alpha)]$$

$$y = \textit{max-prefix}(u\alpha v, T^*),\ z = \textit{max-suffix}(u\alpha v, NV^*)$$

Lemma

For every unlimitedly deep pushdown automaton M, there exists a state grammar G such that $L(M) \subseteq L(G)$.

Proven the unlimitedly deep pushdown automata are computationally complete.

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