Advanced Lectures: L-systems

0L System

History: Astrid Lindenmayer, 1968

Inspiration: the growth process in living organisms

Classical approach:

Only a certain *part* of a word is rewritten at one moment.

Lindenmayer approach:

All letters of a word must be rewritten at the same time.

0L Definition

Zero-sided left-side handle | Lindenmayer system

Gist: ***no nonterminals, parallel derivation step***

Definition: *0L system* is a triple $H = (V, P, w)$:

1) V is a finite *alphabet* of symbols

2) P is a finite set of *rules* of the form: $a \rightarrow x$, where $a \in V$ and $x \in V^*$

3) $w \in V^+$ is the *starting string* (axiom).

Derivation

Gist: parallel rewriting of all symbols

Definition: *Direct derivation* ($\Rightarrow$):

$a_1 a_2 \ldots a_n \Rightarrow x_1 x_2 \ldots x_n$, $n \geq 1$, if $a_i \rightarrow x_i \in P$ for all $i = 1,\ldots,n$.

Derivation ($\Rightarrow^*$): reflexive and transitive closure of $\Rightarrow$.

Example:

$$u = a_1 \; a_2 \; \ldots \; a_n$$

$$\Downarrow \quad \Downarrow \qquad \Downarrow$$

$$v = x_1 \; x_2 \; \ldots \; x_n$$

and we write: $\boldsymbol{u \Rightarrow v}$

Language & Example

Gist: all reachable sentential forms in $L(H)$

Definition: Language $L(H) = \{x : w \Rightarrow^{*} x\}$.

Example: $H = (\{a\}, \{a \to aa\}, a)$

$a = a^{2^0} = a$

$\downarrow$

$a\ a = a^{2^1} = a^2$

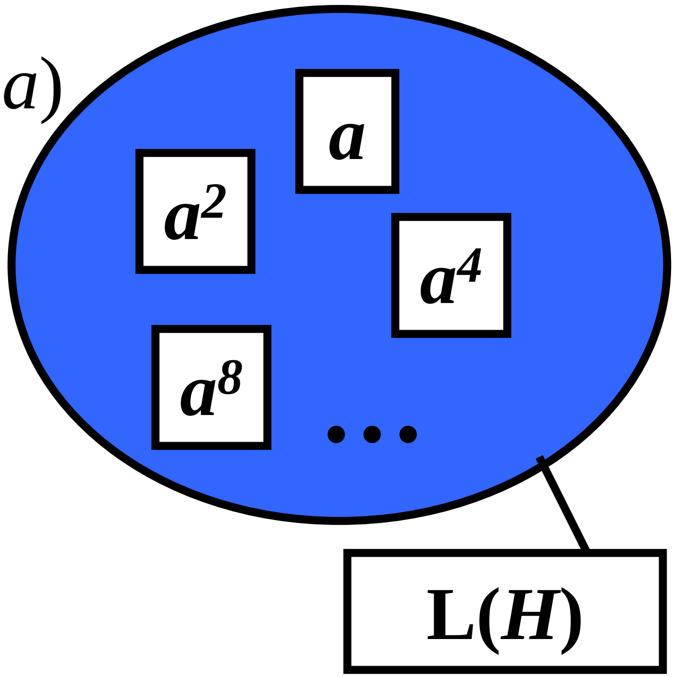
$\swarrow \quad \searrow$

$a\ a\ a\ a = a^{2^2} = a^4$

$\swarrow \quad \swarrow \quad \searrow \quad \searrow$

$a\ a\ a\ a\ a\ a\ a\ a = a^{2^3} = a^8$

$\vdots$



$L(H) = \{a^{2^n} : n \geq 0\}$

Variants of L-systems

Gist: determinism and/or no erasing rules

Let $H = (V, P, w)$ is a 0L system.

Definition: *Deterministic* 0*L system* (D0L):
For each $a \in V$ there is exactly one rule
$a \rightarrow x \in P$.

Definition: *Propagating* 0*L system* (P0L):
For each $a \rightarrow x \in P$ holds $x \neq \varepsilon$.

Note: PD0L is a deterministic propagating 0L system.

Note 2: The previous example is a PD0L system.

Example – Red Alga Simulation

Model of red alga growth by PD0L system:

Starting string: 1 + **Rules:**

$1 \rightarrow 23$
$2 \rightarrow 2$
$3 \rightarrow 24$
$4 \rightarrow 54$
$[\rightarrow [$
$5 \rightarrow 6$
$6 \rightarrow 7$
$7 \rightarrow 8[1]$
$8 \rightarrow 8$
$] \rightarrow]$

Red Alga Simulation 1/11

Derivation:

$1 \Rightarrow 23$

$1 \rightarrow 23$
$2 \rightarrow 2$
$3 \rightarrow 24$
$4 \rightarrow 54$
$[\rightarrow [$

$5 \rightarrow 6$
$6 \rightarrow 7$
$7 \rightarrow 8[1]$
$8 \rightarrow 8$
$] \rightarrow]$

Red Alga Simulation 2/11

Derivation:

$1 \Rightarrow 23 \Rightarrow 224$

$1 \rightarrow 23$
$2 \rightarrow 2$
$3 \rightarrow 24$
$4 \rightarrow 54$
$[\rightarrow [$

$5 \rightarrow 6$
$6 \rightarrow 7$
$7 \rightarrow 8[1]$
$8 \rightarrow 8$
$] \rightarrow]$

Red Alga Simulation 3/11

Derivation:

$1 \Rightarrow 23 \Rightarrow 224 \Rightarrow 2254$

1	$\rightarrow$	23
2	$\rightarrow$	2
3	$\rightarrow$	24
4	$\rightarrow$	54
[	$\rightarrow$	[
5	$\rightarrow$	6
6	$\rightarrow$	7
7	$\rightarrow$	8[1]
8	$\rightarrow$	8
]	$\rightarrow$	]

Red Alga Simulation 4/11

Derivation:

1 $\Rightarrow$ 23 $\Rightarrow$ 224 $\Rightarrow$ 2254 $\Rightarrow$ 22654

2	2	6	5	4

$1 \rightarrow 23$
$2 \rightarrow 2$
$3 \rightarrow 24$
$4 \rightarrow 54$
$[\rightarrow [$

$5 \rightarrow 6$
$6 \rightarrow 7$
$7 \rightarrow 8[1]$
$8 \rightarrow 8$
$] \rightarrow]$

Red Alga Simulation 5/11

Derivation:

$1 \Rightarrow 23 \Rightarrow 224 \Rightarrow 2254 \Rightarrow 22654 \Rightarrow 227654$

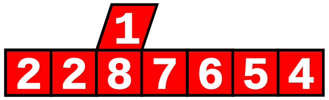
2	2	7	6	5	4

1	$\rightarrow$	23
2	$\rightarrow$	2
3	$\rightarrow$	24
4	$\rightarrow$	54
[	$\rightarrow$	[
5	$\rightarrow$	6
6	$\rightarrow$	7
7	$\rightarrow$	8[1]
8	$\rightarrow$	8
]	$\rightarrow$	]

Red Alga Simulation 6/11

Derivation:

1 ⇒ 23 ⇒ 224 ⇒ 2254 ⇒ 22654 ⇒ 227654 ⇒ 228[1]7654

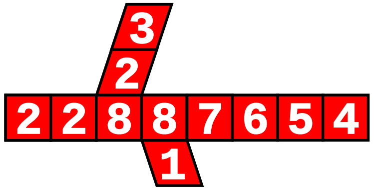


1	→	23
2	→	2
3	→	24
4	→	54
[	→	[
5	→	6
6	→	7
7	→	8[1]
8	→	8
]	→	]

Red Alga Simulation 7/11

Derivation:

1 ⇒ ... ⇒ 228[1]7654 ⇒ 228[23]8[1]7654

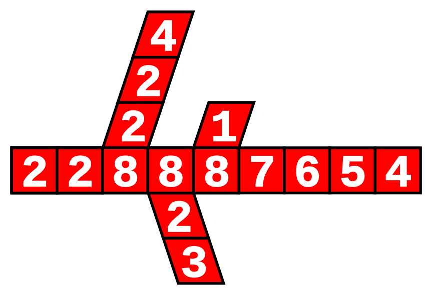


1 → 23
2 → 2
3 → 24
4 → 54
[→ [
5 → 6
6 → 7
7 → 8[1]
8 → 8
] →]

Red Alga Simulation 8/11

Derivation:

1 ⇒ ... ⇒ 228[23]8[1]7654 ⇒ 228[224]8[23]8[1]7654

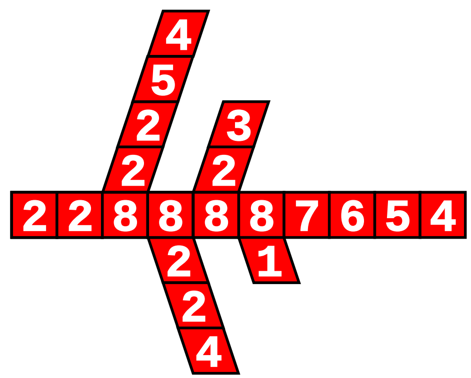


1	→	23
2	→	2
3	→	24
4	→	54
[	→	[
5	→	6
6	→	7
7	→	8[1]
8	→	8
]	→	]

Red Alga Simulation 9/11

Derivation:

$1 \Rightarrow^*$ 228[2254]8[224]8[23]8[1]7654



$1 \rightarrow 23$

$2 \rightarrow 2$

$3 \rightarrow 24$

$4 \rightarrow 54$

$[\rightarrow [$

$5 \rightarrow 6$

$6 \rightarrow 7$

$7 \rightarrow 8[1]$

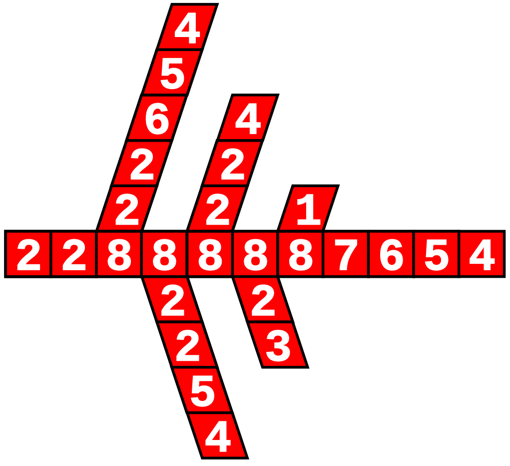
$8 \rightarrow 8$

$] \rightarrow]$

Red Alga Simulation 10/11

Derivation:

$1 \Rightarrow^* 228[22654]8[2254]8[224]8[23]8[1]7654$

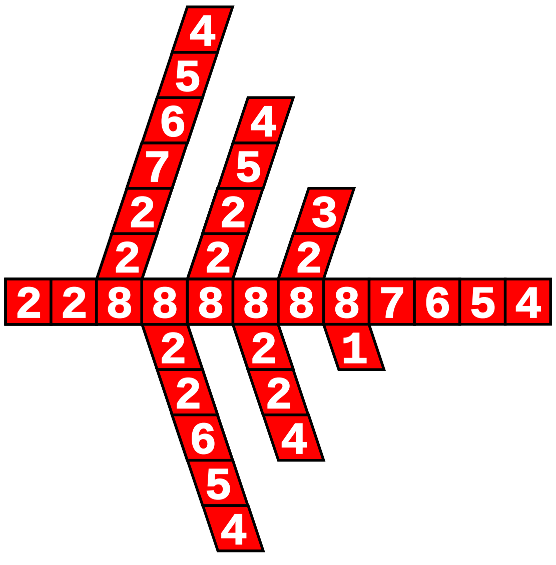


$1 \rightarrow 23$
$2 \rightarrow 2$
$3 \rightarrow 24$
$4 \rightarrow 54$
$[\rightarrow [$
$5 \rightarrow 6$
$6 \rightarrow 7$
$7 \rightarrow 8[1]$
$8 \rightarrow 8$
$] \rightarrow]$

Red Alga Simulation 11/11

Derivation:

1 $\Rightarrow^*$ 228[227654]8[22654]8[2254]8[224]8[23]8[1]7654



Operations over 0L languages

Gist: properties different from CFLs

Not closed under operations:

- union of 0L languages
- intersection of 0L languages
- complementation of 0L language
- concatenation of 0L languages
- positive closure ($^+$) of 0L language

Closed under operations:

- reversal of 0L language

E0L Definition

Gist: improved variant of 0L system

Definition: *Extended 0L system* is a quadruple $G = (V, T, P, w)$ where
V, P, w have the same meaning as in 0L system,
$T \subseteq V$ is set of terminals

Note: $\Rightarrow$, $\Rightarrow^*$ – by analogy with 0L systems.

Definition: *Language* generated by E0L system G is $\mathbf{L}(G) = \{ x : w \Rightarrow^* x, x \in T^* \}$.

Note: If $V = T$ in E0L system then it is 0L system.

E0L system – Example

Gist: E0L are more powerful than 0L;

starting string has not to be inluded in L(G).

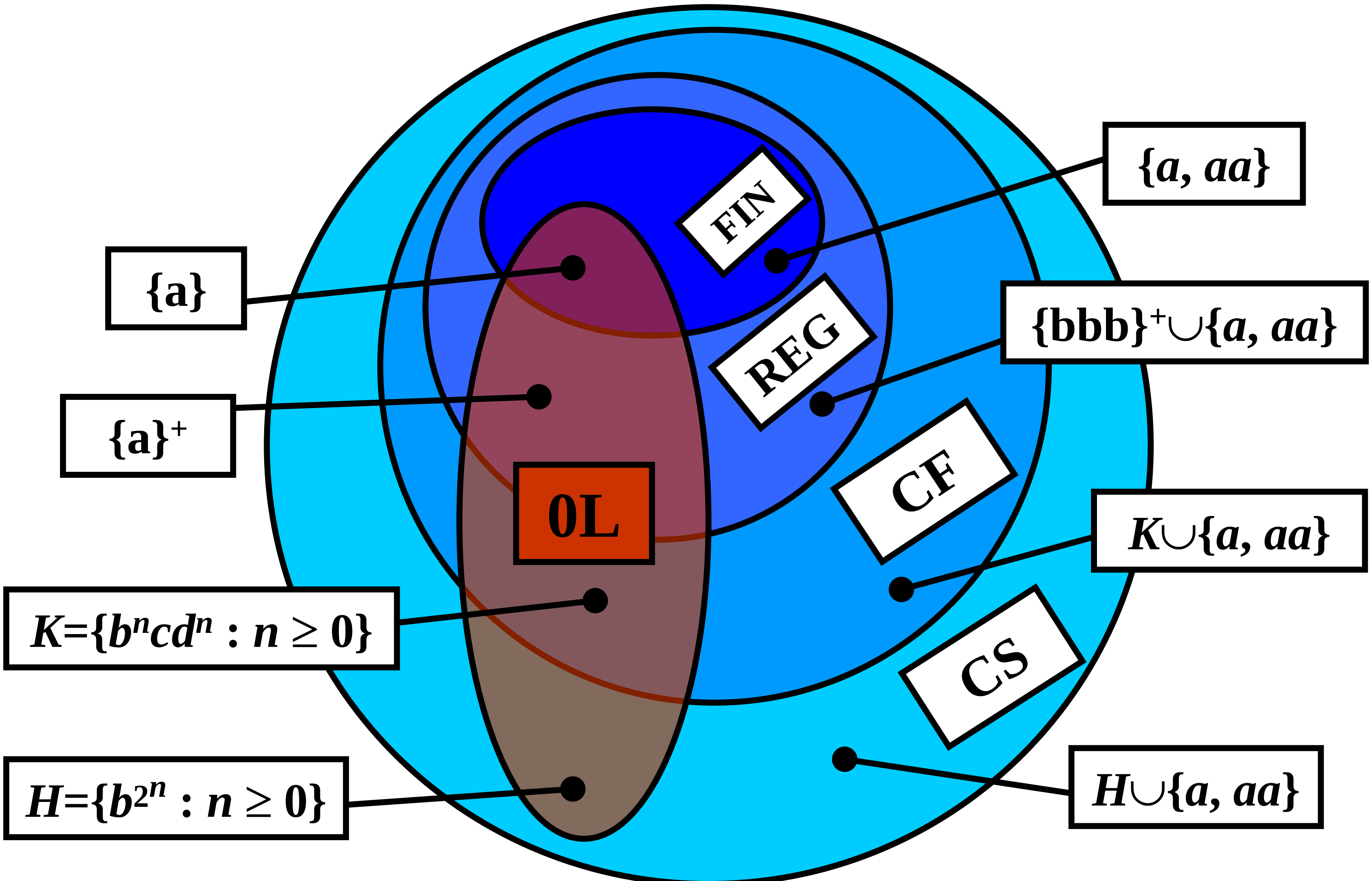
Example: $G = (\{S, a, b\}, \{a, b\}, P, S)$

$P = \{ S \to a,\ S \to b,\ a \to aa,\ b \to bb \}$

$L(G) = \{a^{2^n} : n \geq 0\} \cup \{b^{2^n} : n \geq 0\}$

$L(G) \in \mathbf{L(E0L) - L(0L)} \Rightarrow \mathbf{L(0L) \subset L(E0L)}.$

0L & Chomsky hierarchy



L systems & Chomsky hierarchy