# Part II.
# Lexical Analysis:
# Models

# Regular Expressions (RE): Definition

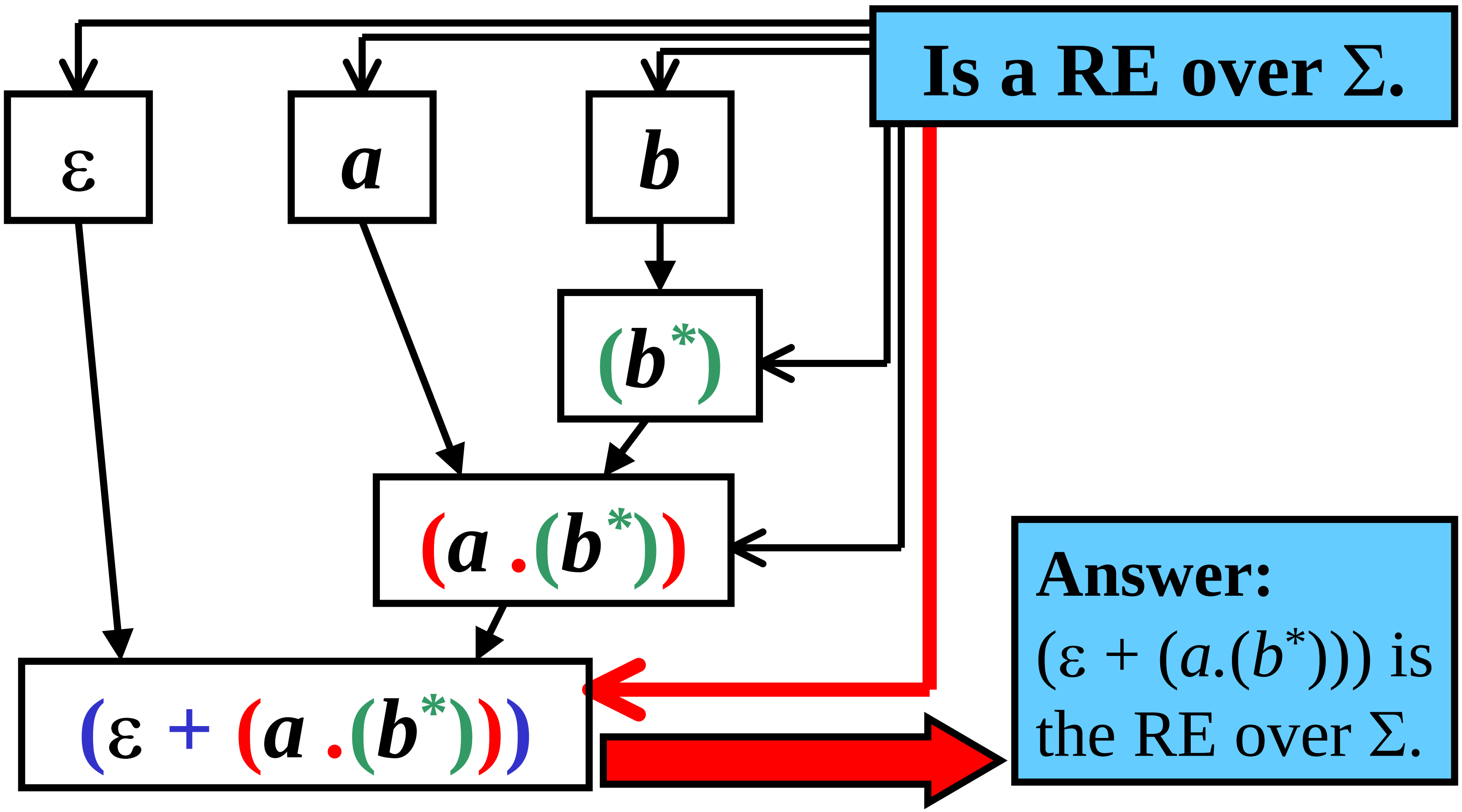
**Gist: Expressions with operators ., +, and * that denote concatenation, union, and iteration, respectively.**

**Definition:** Let $\Sigma$ be an alphabet. The ***regular expressions*** over $\Sigma$ and the ***languages they denote*** are defined as follows:

- $\varnothing$ is a RE denoting the empty set
- $\varepsilon$ is a RE denoting $\{\varepsilon\}$
- $a$, where $a \in \Sigma$, is a RE denoting $\{a\}$
- Let $r$ and $s$ be regular expressions denoting the languages $L_r$ and $L_s$, respectively; then
  - $(r.s)$ is a RE denoting $L = L_r L_s$
  - $(r + s)$ is a RE denoting $L = L_r \cup L_s$
  - $(r^*)$ is a RE denoting $L = L_r^*$

# Regular Expressions: Example

**Question:** Is $(\varepsilon + (a.(b^*)))$ the regular expression over $\Sigma = \{a, b\}$ **?**




**Answer:**
$(\varepsilon + (a.(b^*)))$ is the RE over $\Sigma$.

# Simplification

1) Reduction of the number of parentheses by

**Precedences: $^{*}$ > . > +**

2) Expression $r.s$ is simplified to $rs$

3) Expression $rr^{*}$ or $r^{*}r$ is simplified to $r^{+}$

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**Example:**

$((\boldsymbol{a}.(\boldsymbol{a}^{*})) + ((\boldsymbol{b}^{*}).\boldsymbol{b}))$ can be written as $\underbrace{\boldsymbol{a}\,.\boldsymbol{a}^{*} + \boldsymbol{b}^{*}.\boldsymbol{b}}$ ,

and $\boldsymbol{a}\,.\boldsymbol{a}^{*} + \boldsymbol{b}^{*}.\boldsymbol{b}$ can be written as $\boldsymbol{a}^{+} + \boldsymbol{b}^{+}$

# Regular Language (RL)

**Gist: Every RE denotes a regular language**

**Definition:** Let $L$ be a language. $L$ is a *regular language* (RL) if there exists a regular expression $r$ that denotes $L$.

**Denotation**: $L(r)$ means the language denoted by $r$.

**Examples:**

$r_1 = ab + ba$ denotes $L_1 = \{ab, ba\}$

$r_2 = a^+b^*$ denotes $L_2 = \{a^nb^m: n \geq 1, m \geq 0\}$

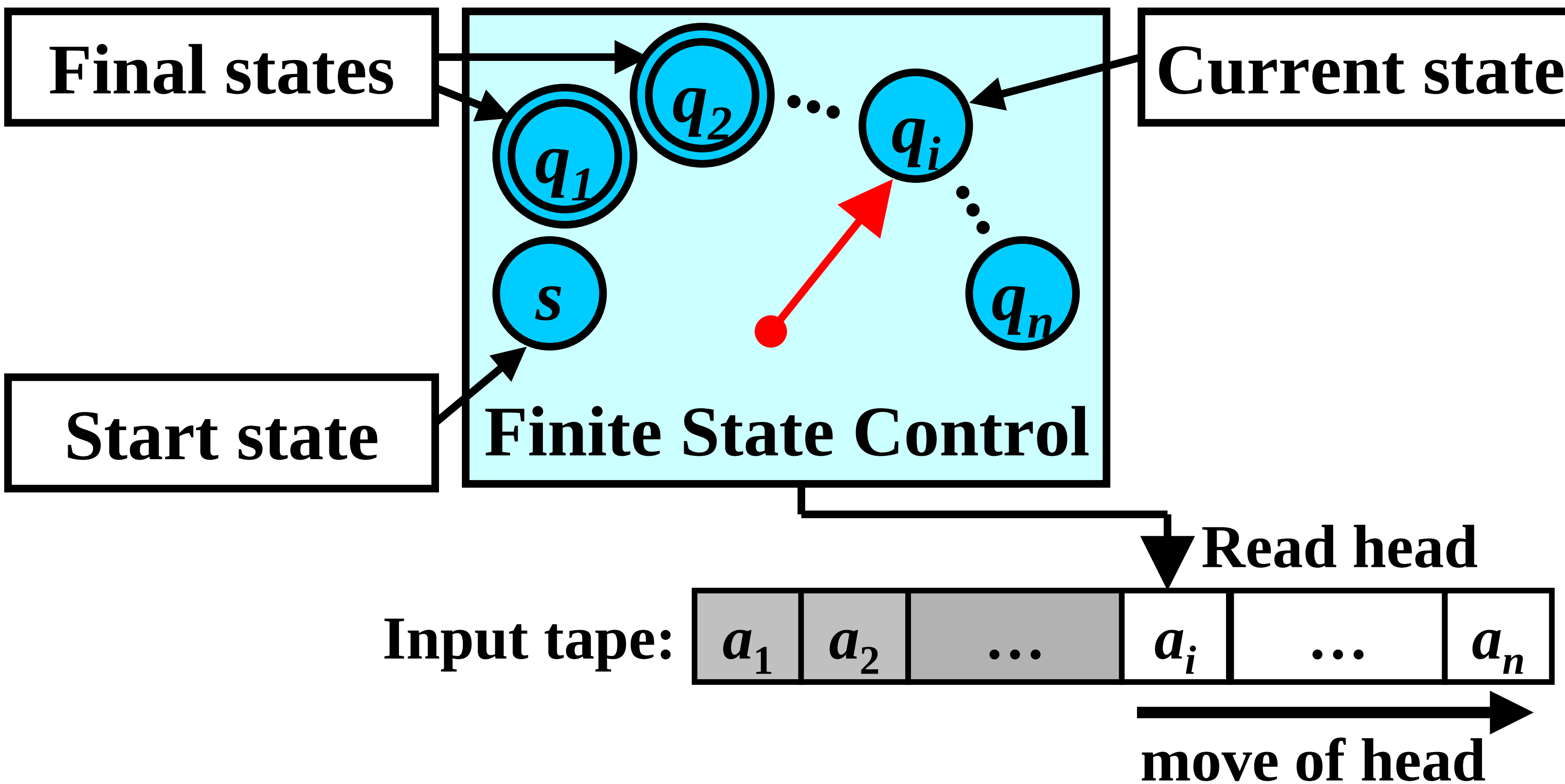
$r_3 = ab(a + b)^*$ denotes $L_3 = \{x: ab$ is prefix of $x\}$

$r_4 = (a + b)^*ab(a + b)^*$ denotes $L_4 = \{x: ab$ is substring of $x\}$

**$L_1$, $L_2$, $L_3$ , $L_4$ are regular languages over $\Sigma$**

# Finite Automata (FA)

**Gist: The simplest model of computation based on a finite set of states and computational rules.**



# Finite Automata: Definition

**Definition:** *A finite automaton* (FA) is a 5-tuple:

$M = (Q, \Sigma, R, s, F)$, where

- $Q$ is a *finite set of states*
- $\Sigma$ is an *input alphabet*
- $R$ is a *finite set of rules* of the form: $pa \rightarrow q$,
  where $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$
- $s \in Q$ is the *start state*
- $F \subseteq Q$ is a set of *final states*

**Mathematical note on rules:**

- Strictly mathematically, $R$ is a relation from $Q \times (\Sigma \cup \{\varepsilon\})$ to $Q$
- Instead of $(pa, q)$, however, we write the rule as $pa \rightarrow q$

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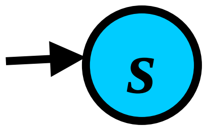
- $pa \rightarrow q$ means that with $a$, $M$ can move from $p$ to $q$
- if $a = \varepsilon$, no symbol is read

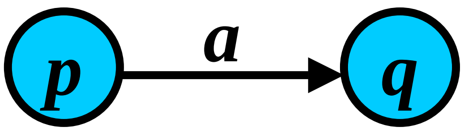
# Graphical Representation

denotes a state $q \in Q$



denotes the start state $s \in Q$

denotes a final state $f \in F$

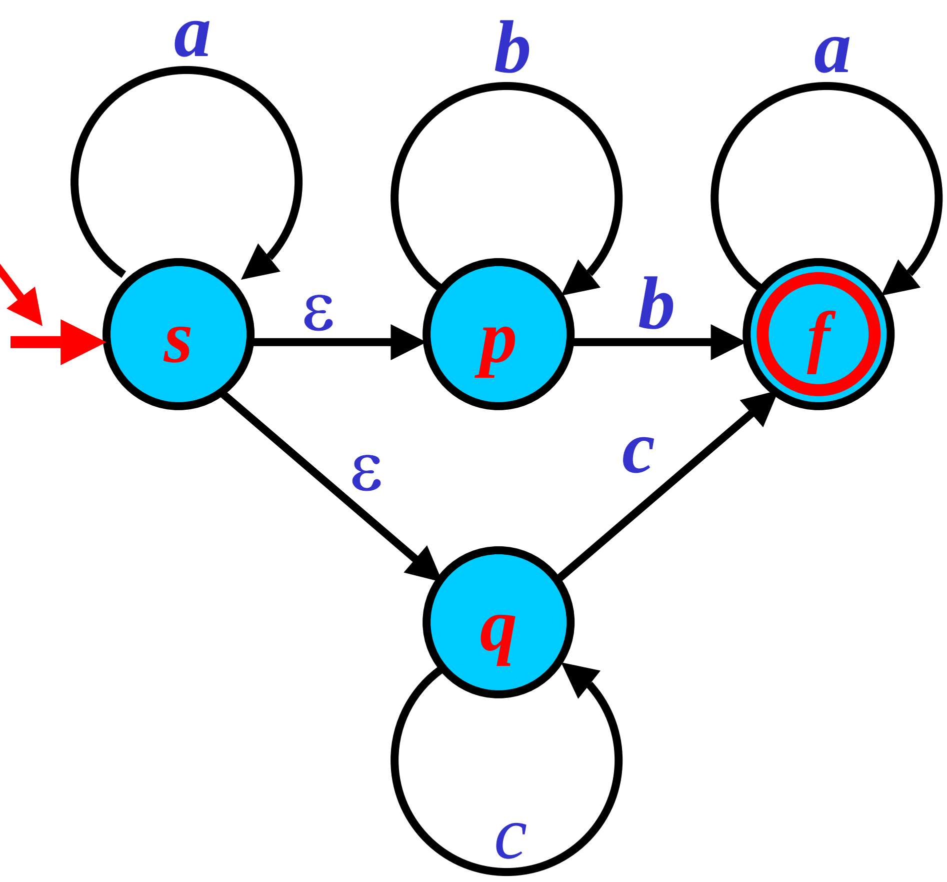


denotes $pa \rightarrow q \in R$

# Graphical Representation: Example

$M = (Q, \Sigma, R, s, F)$,
where:

- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b, c\}$;
- $R = \{sa \rightarrow s,$
  $s \rightarrow p,$
  $pb \rightarrow p,$
  $pb \rightarrow f,$
  $s \rightarrow q,$
  $qc \rightarrow q,$
  $qc \rightarrow f,$
  $fa \rightarrow f\}$;
- $F = \{f\}$



# Tabular Representation

- **Columns:** Member of $\Sigma \cup \{\varepsilon\}$
- **Rows:** States of $Q$
- **First row:** The start state
- **Underscored:** Final states

| | ... | ***a*** | ... | $\varepsilon$ |
|---|---|---|---|---|
| ***s*** | | | | |
| ... | | | | |
| ***p*** | | $t(p, a)$ | | |
| ... | | | | |
| $\underline{\boldsymbol{f}}$ | | | | |

$$t(p, a) = \{q : pa \rightarrow q \in R\}$$

# Tabular Representation: Example

$M = (Q, \Sigma, R, s, F)$, where:

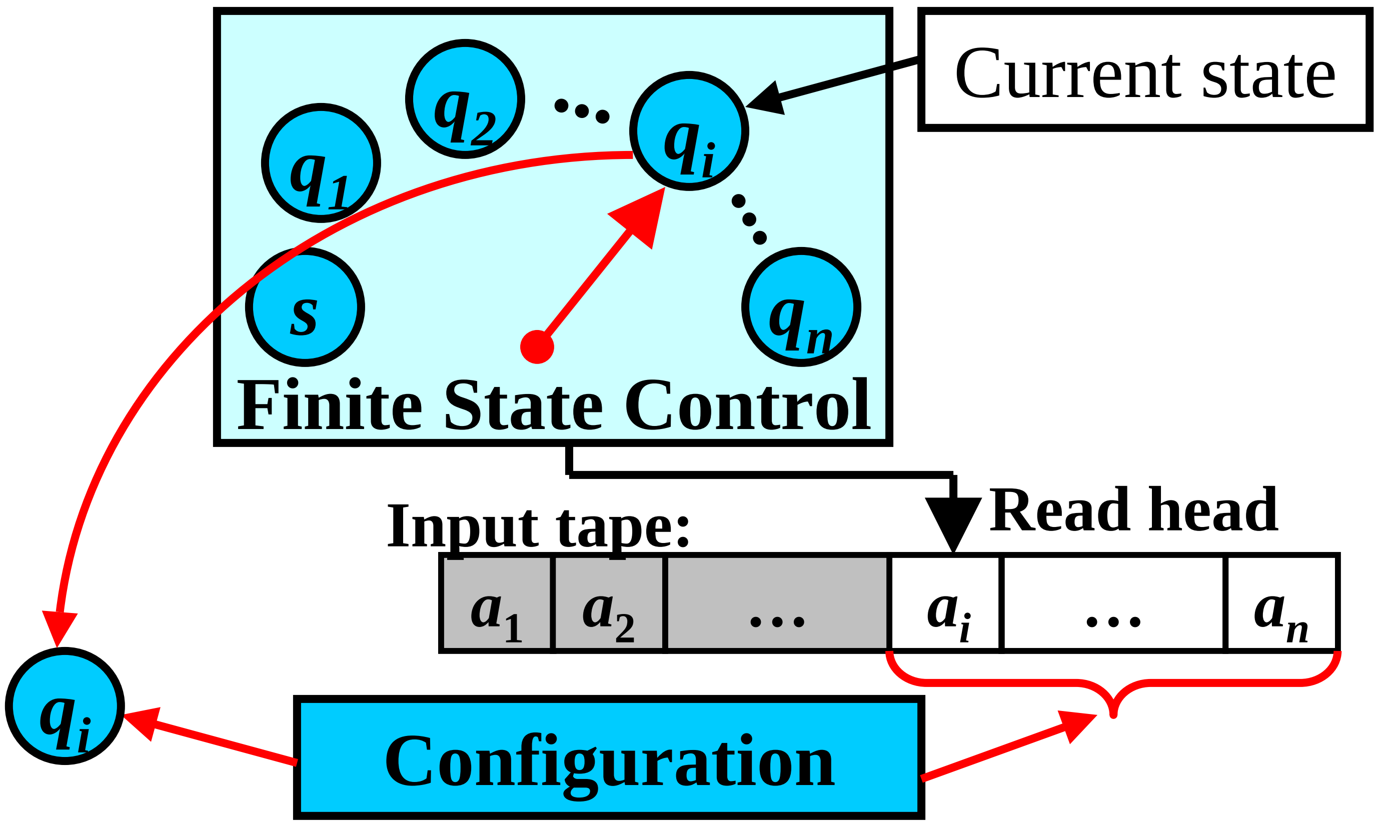
- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b, c\}$;
- $R = \{sa \rightarrow s,$
  $s \rightarrow p,$
  $pb \rightarrow p,$
  $pb \rightarrow f,$
  $s \rightarrow q,$
  $qc \rightarrow q,$
  $qc \rightarrow f,$
  $fa \rightarrow f\}$;
- $F = \{f\}$

| | $a$ | $b$ | $c$ | $\varepsilon$ |
|---|---|---|---|---|
| $s$ | $\{s\}$ | $\varnothing$ | $\varnothing$ | $\{p, q\}$ |
| $p$ | $\varnothing$ | $\{p, f\}$ | $\varnothing$ | $\varnothing$ |
| $q$ | $\varnothing$ | $\varnothing$ | $\{q, f\}$ | $\varnothing$ |
| $\underline{f}$ | $\{f\}$ | $\varnothing$ | $\varnothing$ | $\varnothing$ |

# Configuration

**Gist: Instance description of FA**

**Definition:** Let $M = (Q, \Sigma, R, s, F)$ be a FA.
*A configuration* of $M$ is a string $\chi \in Q\Sigma^*$



# Move

**Gist: Computational step of FA**

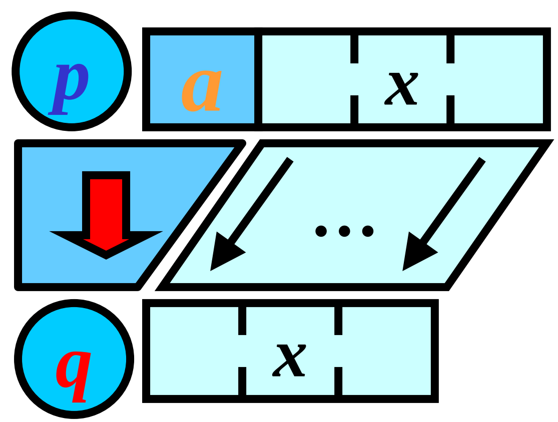
**Definition:** Let $pax$ and $qx$ be two configurations of $M$, where $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $x \in \Sigma^*$. Let $r = pa \rightarrow q \in R$ be a rule. Then $M$ makes a *move* from $pax$ to $qx$ according to $r$, written as $pax \vdash qx$ [$r$] or, simply, $pax \vdash qx$

**Note:** if $a = \varepsilon$, no input symbol is read

**Configuration:**

**Rule:** $pa \rightarrow q$

**New configuration:**



# Sequence of Moves 1/2

**Gist: Several consecutive computational steps**

**Definition:** Let $\chi$ be a configuration. $M$ makes *zero moves* from $\chi$ to $\chi$; in symbols,

$$\chi \vdash^{0} \chi\ [\varepsilon] \text{ or, simply, } \chi \vdash^{0} \chi$$

**Definition:** Let $\chi_0, \chi_1, \ldots, \chi_n$ be a sequence of configurations, $n \geq 1$, and $\chi_{i-1} \vdash \chi_i\ [r_i]$, $r_i \in R$, for all $i = 1, \ldots, n$; that is,

$$\chi_0 \vdash \chi_1\ [r_1] \vdash \chi_2\ [r_2] \ldots \vdash \chi_n\ [r_n]$$

Then $M$ makes $n$ *moves* from $\chi_0$ to $\chi_n$:

$$\chi_0 \vdash^{n} \chi_n\ [r_1 \ldots r_n] \text{ or, simply, } \chi_0 \vdash^{n} \chi_n$$

# Sequence of Moves 2/2

If $\chi_0 \ \vdash^n \chi_n \ [\rho]$ for some $n \geq 1$, then

$$\chi_0 \ \vdash^+ \chi_n \ [\rho].$$

If $\chi_0 \ \vdash^n \chi_n \ [\rho]$ for some $n \geq 0$, then

$$\chi_0 \ \vdash^* \chi_n \ [\rho].$$

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**Example:** Consider

$\boldsymbol{p}\boldsymbol{a}bc \vdash \boldsymbol{q}bc$ [**1**: $\boldsymbol{pa} \rightarrow \boldsymbol{q}$], and $\boldsymbol{q}\boldsymbol{b}c \vdash \boldsymbol{r}c$ [**2**: $\boldsymbol{qb} \rightarrow \boldsymbol{r}$].

Then,

$\boldsymbol{pab}c \vdash^2 \boldsymbol{r}c$ [**1 2**],

$\boldsymbol{pab}c \vdash^+ \boldsymbol{r}c$ [**1 2**],

$\boldsymbol{pab}c \vdash^* \boldsymbol{r}c$ [**1 2**]

# Accepted Language

**Gist: *M* accepts *w* if it can completely read *w* by a sequence of moves from *s* to a final state**

**Definition:** Let $M = (Q, \Sigma, R, s, F)$ be a FA. The *language accepted by M*, $L(M)$, is defined as:

$$L(M) = \{w: w \in \Sigma^*, sw \vdash^* f, f \in F\}$$

$M = (Q, \Sigma, R, s, F)$:

if $q_n \in F$ then $w \in L(M)$;
otherwise, $w \notin L(M)$

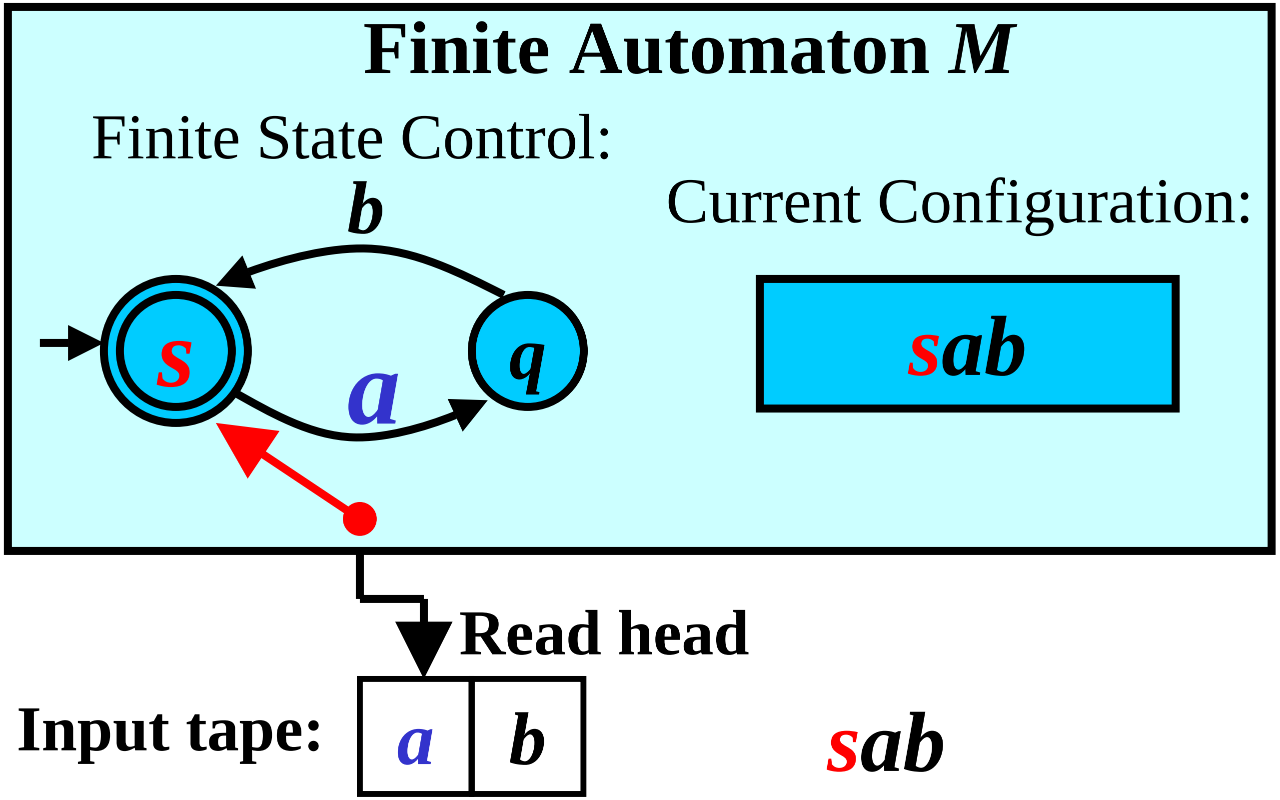
$$s\underbrace{a_1a_2\ldots a_n}_{w} \vdash q_1a_2\ldots a_n \vdash \ldots \vdash q_{n-1}a_n \vdash q_n$$

# FA: Example 1/3

$M = (Q, \Sigma, R, s, F)$, where:

$Q = \{s, q\}$, $\Sigma = \{a, b\}$, $R = \{sa \rightarrow q, qb \rightarrow s\}$, $F = \{s\}$

**Question:** $ab \in L(M)$ **?**



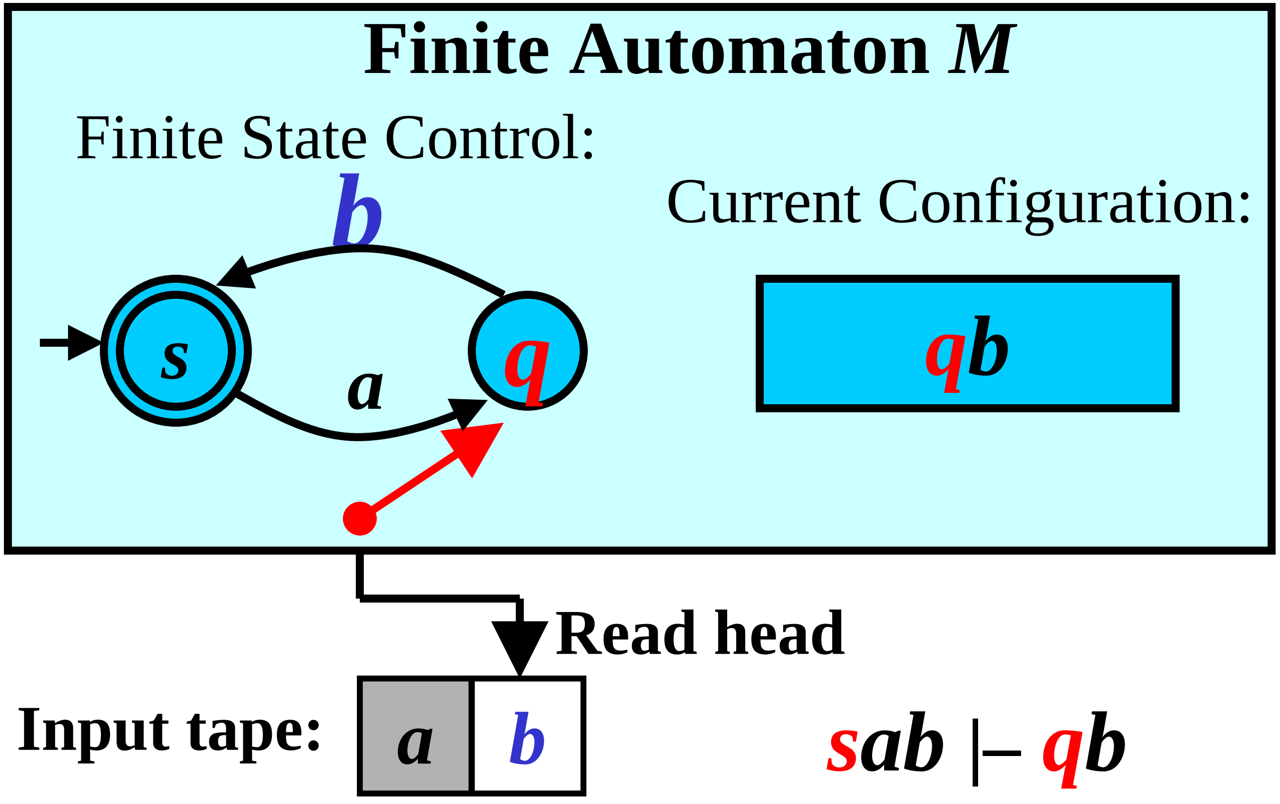

***sab***

# FA: Example 2/3

$M = (Q, \Sigma, R, s, F)$, where:

$Q = \{s, q\}$, $\Sigma = \{a, b\}$, $R = \{sa \to q, qb \to s\}$, $F = \{s\}$

**Question:** $ab \in L(M)$ **?**



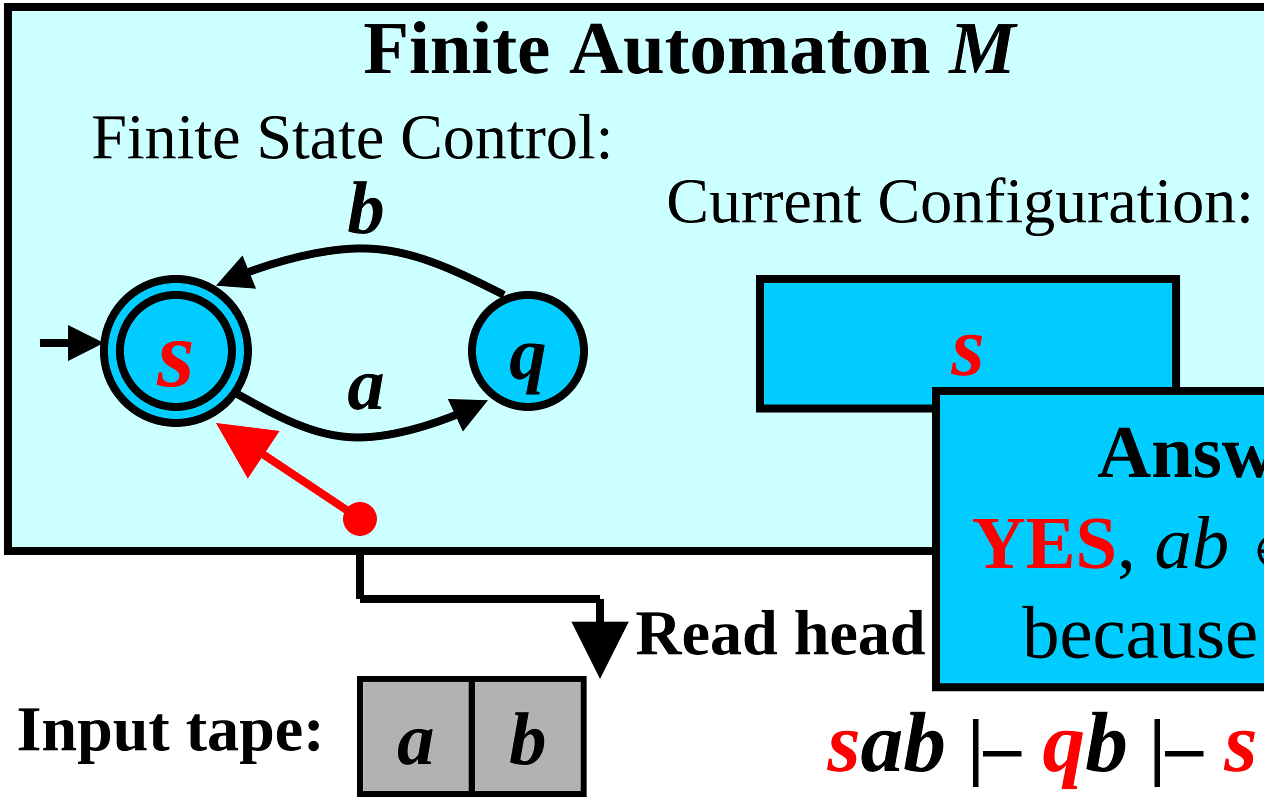

***sab* |– *qb***

# FA: Example 3/3

$M = (Q, \Sigma, R, s, F)$, where:

$Q = \{s, q\}$, $\Sigma = \{a, b\}$, $R = \{sa \rightarrow q, qb \rightarrow s\}$, $F = \{s\}$

**Question:** $ab \in L(M)$ **?**




**Answer:**
**YES**, $ab \in L(M)$, because $s \in F$

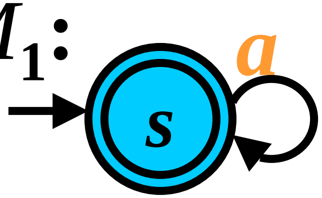
$sab \vdash qb \vdash s$

# Equivalent Models

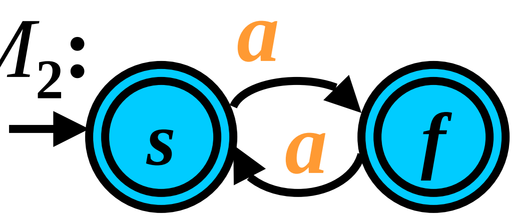
**Definition:** Two models for languages, such as FAs, are equivalent if they both specify the same language.

**Example:**

$M_1$:




$M_2$:




**Question:** Is $M_1$ equivalent to $M_2$ ?

**Answer:** $M_1$ and $M_2$ **are equivalent** because

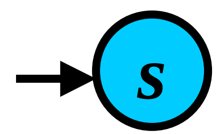
$$L(M_1) = L(M_2) = \{a^n : n \geq 0\}$$

# Conversion of RE to FA: Basics 1/5

**Gist: Algorithm that converts any RE to an equivalent FA (lex in UNIX).**

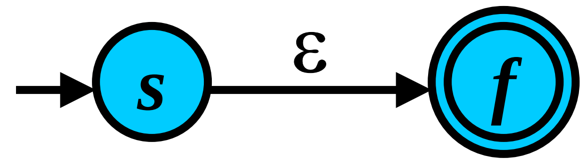
- For a RE $r = \varnothing$, there is an equivalent FA $M_{\varnothing}$.

**Proof:** $M_{\varnothing}$ :



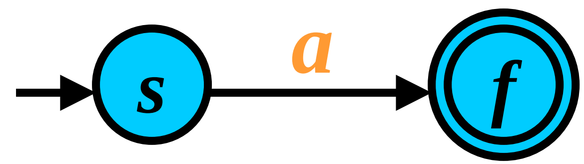

- For a RE $r = \varepsilon$, there is an equivalent FA $M_{\varepsilon}$ .

**Proof:** $M_{\varepsilon}$ :




- For a RE $r = a$, $a \in \Sigma$, there is an equivalent FA $M_a$ .

**Proof:** $M_a$ :



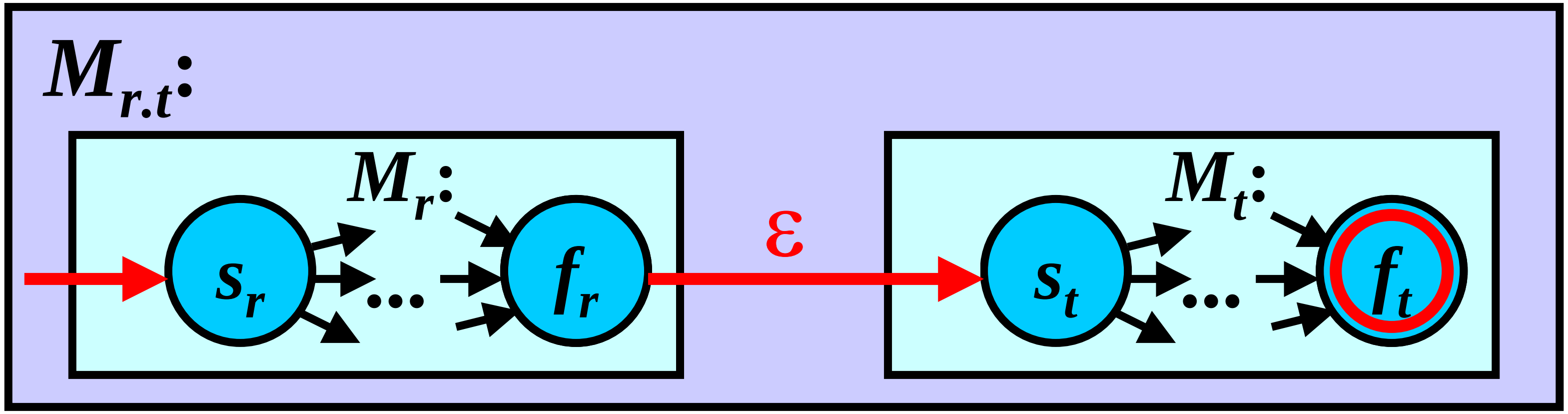
# RE to FA: Concatenation 2/5

- Let $\boldsymbol{r}$ be a RE over $\Sigma$ and $\boldsymbol{M_r} = (Q_r, \Sigma, R_r, s_r, \{f_r\})$ be an FA such that $L(M_r) = L(r)$.
- Let $\boldsymbol{t}$ be a RE over $\Sigma$ and $\boldsymbol{M_t} = (Q_t, \Sigma, R_t, s_t, \{f_t\})$ be an FA such that $L(M_t) = L(t)$.
- Then, for the RE $\boldsymbol{r.t}$, there exists an equivalent FA $\boldsymbol{M_{r.t}}$

**Proof:** Let $Q_r \cap Q_t = \emptyset$.

**Construction:**

$$M_{r.t} = (Q_r \cup Q_t, \Sigma, R_r \cup R_t \cup \{f_r \rightarrow s_t\}, s_r, \{f_t\})$$



# RE to FA: Union 3/5

- Let $\boldsymbol{r}$ be a RE over $\Sigma$ and $\boldsymbol{M_r} = (Q_r, \Sigma, R_r, s_r, \{f_r\})$ be an FA such that $L(M_r) = L(r)$.
- Let $\boldsymbol{t}$ be RE over $\Sigma$ and $\boldsymbol{M_t} = (Q_t, \Sigma, R_t, s_t, \{f_t\})$ be an FA such that $L(M_t) = L(t)$.
- For a RE $\boldsymbol{r + t}$, there exists an equivalent FA $\boldsymbol{M_{r+t}}$

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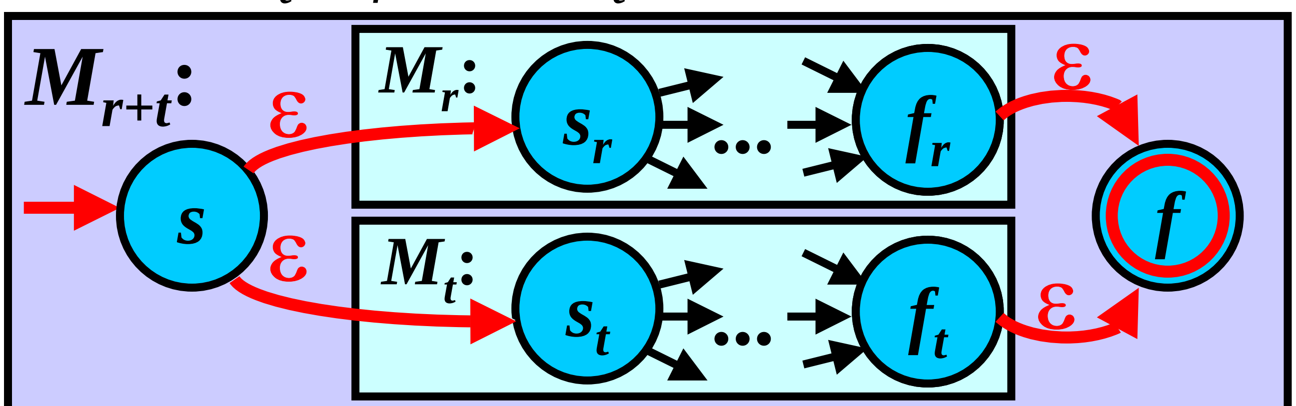
**Proof:** Let $Q_r \cap Q_t = \emptyset$, $s, f \notin Q_r \cup Q_t$.

**Construction**

$M_{r+t}$ = $(Q_r \cup Q_t \cup \{s, f\}, \Sigma, R_r \cup R_t \cup \{s \to s_r, s \to s_t, f_r \to f, f_t \to f\}, s, \{f\})$



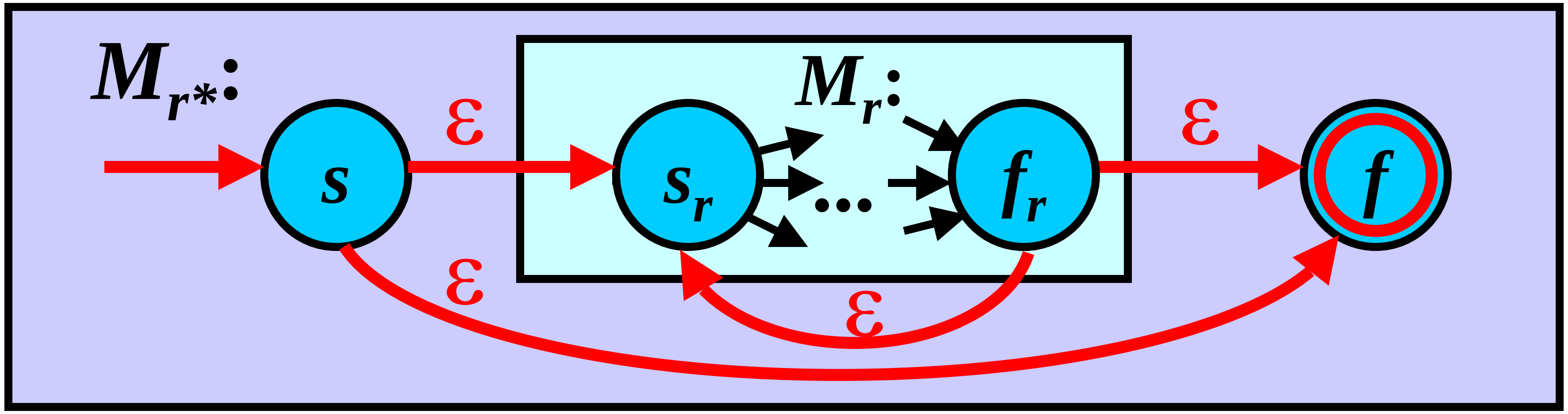
# RE to FA: Iteration 4/5

- Let $r$ be a RE over $\Sigma$ and $M_r = (Q_r, \Sigma, R_r, s_r, \{f_r\})$ be an FA such that $L(M_r) = L(r)$.
- For the RE $r^*$, there exists an equivalent FA $M_{r^*}$

**Proof:** Let $s, f \notin Q_r$.

**Construction:**

$$M_{r^*} = (Q_r \cup \{s, f\}, \Sigma, R_r \cup \{s \rightarrow s_r, f_r \rightarrow f, f_r \rightarrow s_r, s \rightarrow f\}, s, \{f\})$$



# RE to FA: Completion 5/5

- **Input:** RE $r$ over $\Sigma$
- **Output:** FA $M$ such that $L(r) = L(M)$

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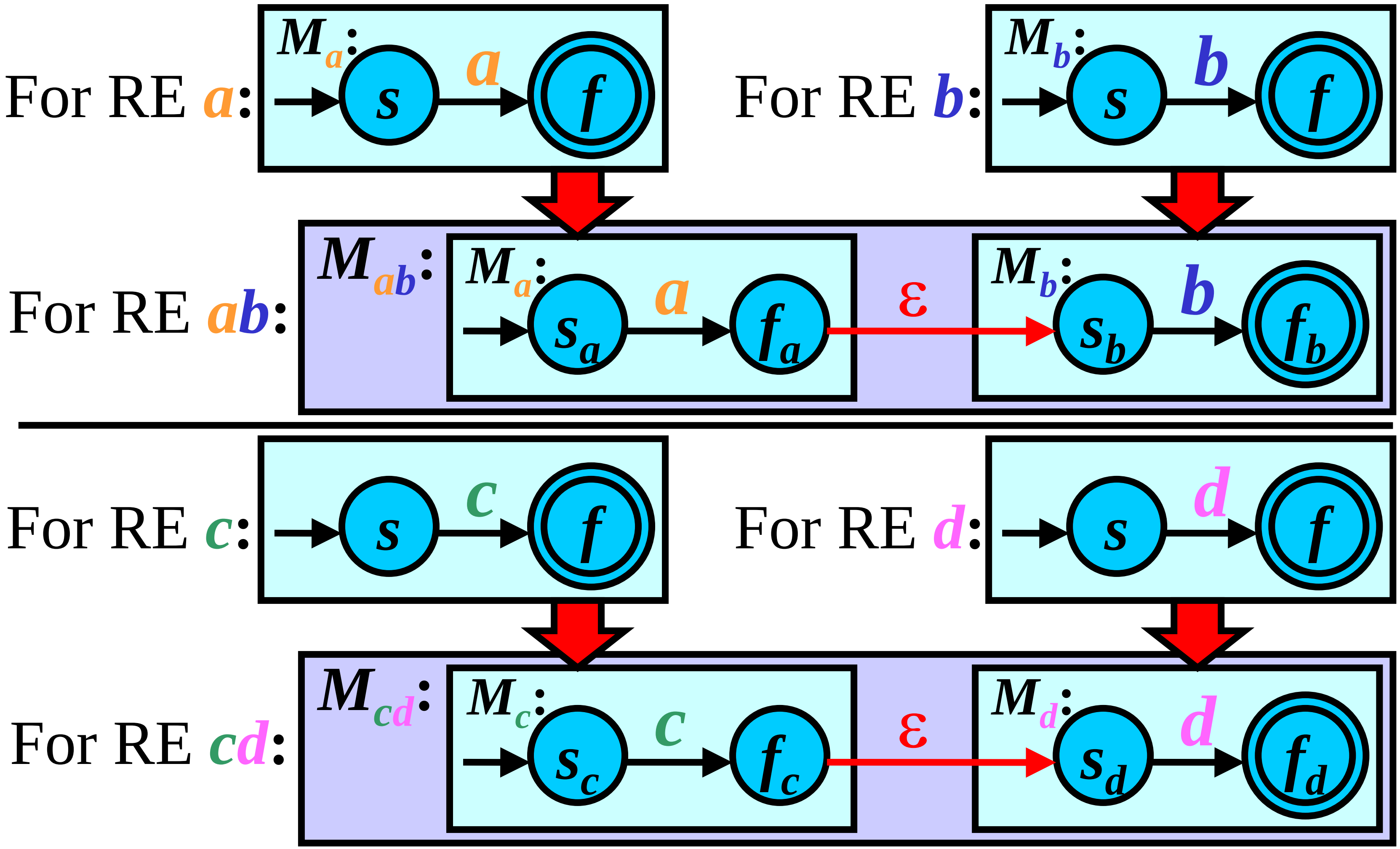
- **Method:**
- **From "inside" of $r$, repeatedly use the next rules to construct $M$:**
  - for RE $\emptyset$, construct FA $M_{\emptyset}$
  - for RE $\varepsilon$, construct FA $M_{\varepsilon}$
  - for RE $a \in \Sigma$, construct FA $M_a$

  (see 1/5)
  - **let** for REs $r$ and $t$, there already exist FAs $M_r$ and $M_t$, respectively; **then**,
    - for RE $r.t$, construct FA $M_{r.t}$ (see 2/5)
    - for RE $r + t$, construct FA $M_{r+t}$ (see 3/5)
    - for RE $r^*$ construct FA $M_{r^*}$ (see 4/5)

# RE to FA: Example 1/3

Transform RE $r = ((ab) + (cd))^*$ to an equivalent FA $M$

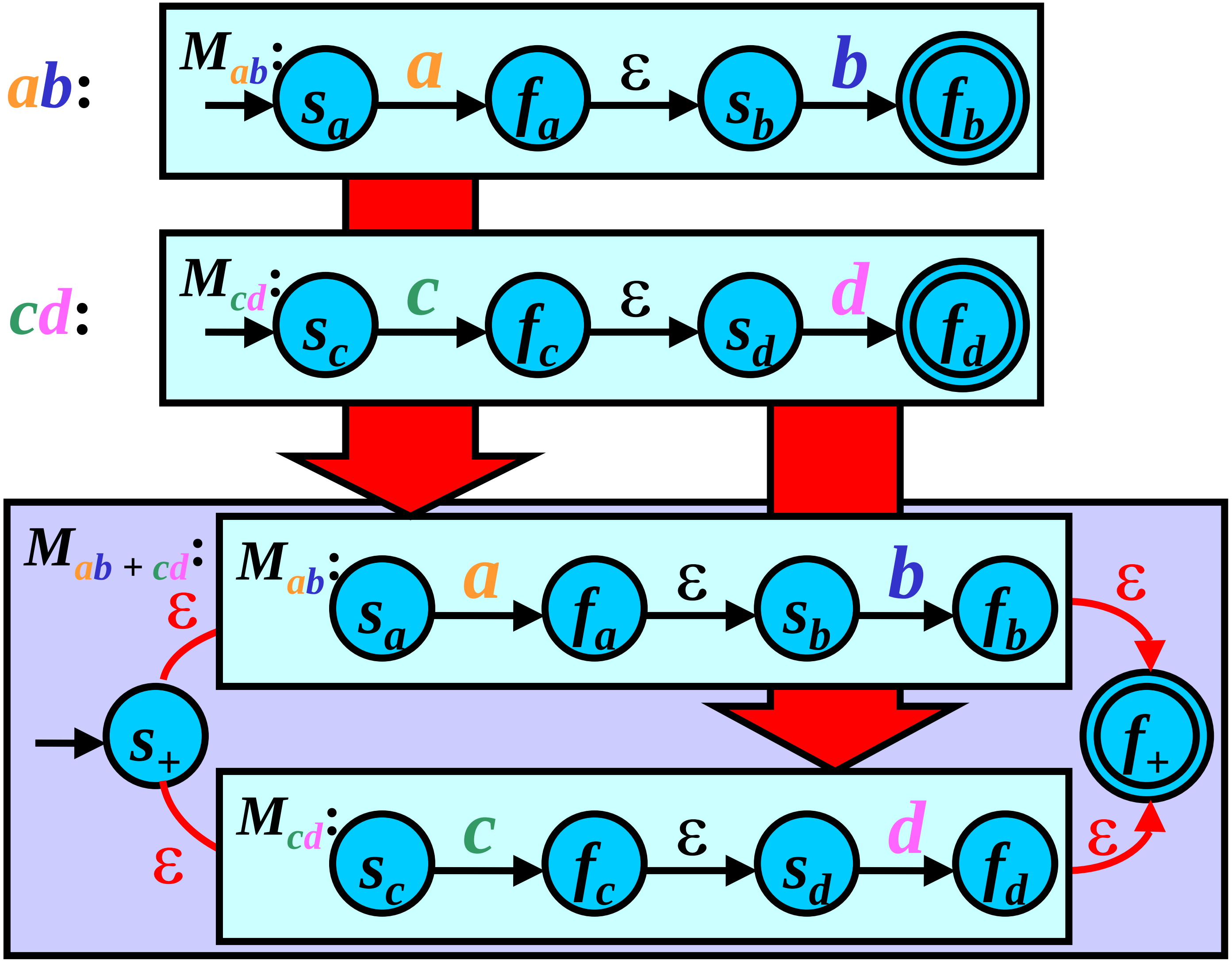


# RE to FA: Example 2/3

For RE *ab*:

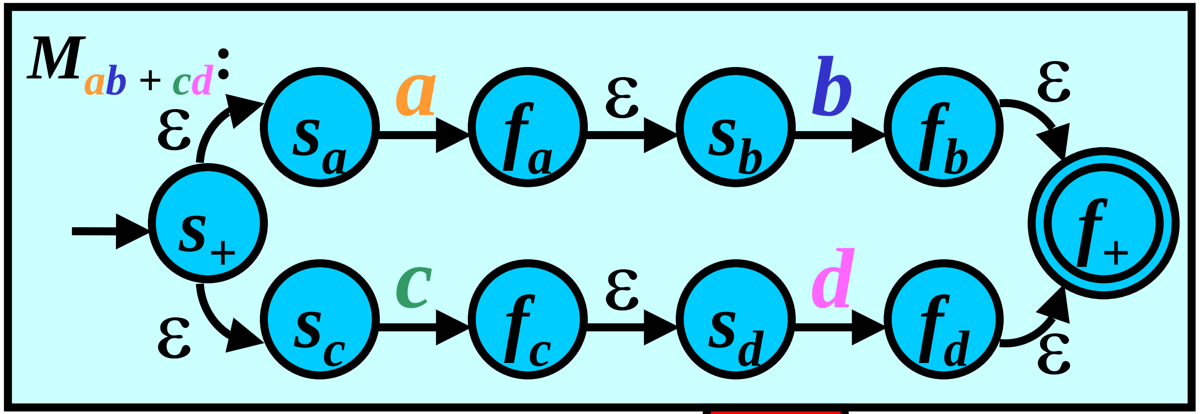
For RE *cd*:

For RE *ab* + *cd*:

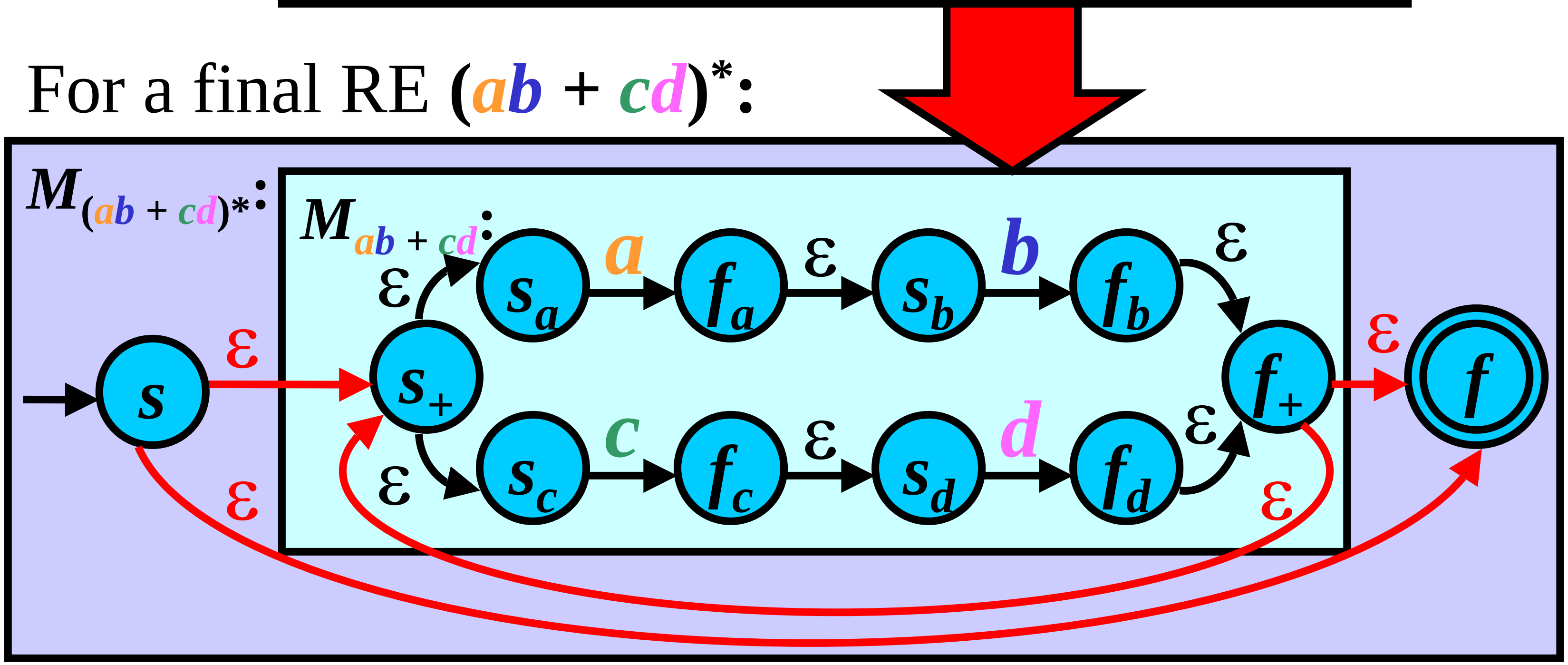


# RE to FA: Example 3/3

For RE
*ab* + *cd*:




For a final RE (*ab* + *cd*)*:



# Models for Regular Languages

**Theorem:** For every RE $r$, there is an FA $M$ such that $L(r) = L(M)$.

**Proof** is based on the previous algorithm.

**Theorem:** For every FA $M$, there is an RE $r$ such that $L(M) = L(r)$.

**Proof:** See page 210 in [Meduna: Automata and Languages]

**Conclusion:** The fundamental models for regular languages are

**1) Regular expressions** **2) Finite Automata**